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Sets of n Squares of Which Any $n - 1$ Have Their Sum Square

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Abstract. A systematic method is given for calculating sets of n squares of which any $n - 1$ have their sum square. A particular method is developed for $n = 4$. Tables give the smallest solution for each $n \leqslant 8$ and other small solutions for $n \leqslant 5$.

1. Introduction. We give numerical solutions in positive integers of the equations

$$x_1^2 + y_1^2 = \cdots = x_n^2 + y_n^2 = x_1^2 + \cdots + x_n^2 \qquad (n \geqslant 3), \tag{1}$$

with $x_i \neq x_j$ for $i \neq j$. The cases $n = 3, 4$ have been studied by many authors; references are given in [1, Chapter XIX].

For general n, Gill [2] gave in 1848 a method for finding solutions of (1), but his method, based on complicated trigonometrical calculations, is impractical for finding actual solutions for $n \geqslant 5$.

We give a simple method for finding explicit solutions for $n \geqslant 5$.

2. Method. We study the more general equations

$$\alpha x_1^2 + y_1^2 = \cdots = \alpha x_n^2 + y_n^2 = \beta(x_1^2 + \cdots + x_n^2), \tag{2}$$

where α and β are given integers. From a known solution (x_i, y_i) we construct another solution (x_i', y_i'). Setting

$$S = \sum_{i=1}^{n} x_i^2, \qquad P = \sum_{i=1}^{n} x_i y_i,$$

we seek λ, μ such that

$$\begin{cases} x_i' = \lambda S x_i - \mu P y_i, \\ y_i' = \alpha\mu P x_i + \lambda S y_i \end{cases}$$

is another solution. We easily find

$$\alpha x_i'^2 + y_i'^2 = \beta S(\lambda^2 S^2 + \alpha\mu^2 P^2),$$

$$\sum_{i=1}^{n} x_i'^2 = S\left[\lambda^2 S^2 + (\mu^2(n\beta - \alpha) - 2\lambda\mu)P^2\right],$$

whence $2\lambda = \mu(n\beta - 2\alpha)$. The solution sought is

$$\begin{cases} x_i' = (n\beta - 2\alpha)Sx_i - 2Py_i, \\ y_i' = 2\alpha P x_i + (n\beta - 2\alpha)Sy_i. \end{cases} \tag{3}$$

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Iteration of the formulae (3) leads back to the original solution. However, we obtain a different solution if we first change the sign of one or more of the x_i. We can thus construct solutions of the equations (2) provided that we know a particular solution, which may be trivial. For the equations (1) the formulae (3) become

$$\begin{cases} x_i' = (n-2)Sx_i - 2Py_i, \\ y_i' = 2Px_i + (n-2)Sy_i, \end{cases} \tag{4}$$

and we have a trivial solution

$$x_1 = \cdots = x_{n-2} = 0, \quad x_{n-1} = a, \quad x_n = b,$$

where a and b are integers satisfying $a^2 + b^2 = c^2$.

3. Small Values of n**.** (a) $n = 3$. The solution 0, a, b, with $a^2 + b^2 = c^2$, is not wholly trivial, as it satisfies $x_i \neq x_j$ for $i \neq j$, but it is of little interest. An application of the formulae (4) gives

$$x_1 = 4abc, \quad x_2 = a(c^2 - 4b^2), \quad x_3 = b(c^2 - 4a^2),$$
$$y_1 = c^3, \quad y_2 = b(c^2 + 4a^2), \quad y_3 = a(c^2 + 4b^2).$$

We thus obtain the Euler cuboid (rectangular parallelepiped with integer edges x_1, x_2, x_3 and integer face diagonals y_1, y_2, y_3; see [4], for example). From $a = 3$, $b = 4$, $c = 5$ we obtain the solution

$$44, \quad 117, \quad 240.$$

(b) $n = 4$. The same method gives the "semitrivial" solution

$$x_1 = x_2 = 2abc, \quad x_3 = a(b^2 - a^2), \quad x_4 = b(a^2 - b^2),$$
$$y_1 = y_2 = c^3, \quad y_3 = b(2a^2 + c^2), \quad y_4 = a(2b^2 + c^2).$$

Changing the sign of x_2 (to ensure a new solution) and x_4 (to simplify), we apply (4) to obtain

$$x_1 = 2abc(4b^4 - 3c^4), \quad x_2 = 2abc(4a^4 - 3c^4),$$
$$x_3 = a(b^2 - a^2)(4a^4 - 3c^4), \quad x_4 = b(b^2 - a^2)(4b^4 - 3c^4).$$

From $a = 3$, $b = 4$, $c = 5$ we obtain the solution

$$23828, \quad 32571, \quad 102120, \quad 186120.$$

(c) $n = 5$. We give only a numerical solution. Beginning with a trivial solution having $x_1 = x_2 = x_3 = x_4$, we apply the formulae (4) to

$$x_1 = x_2 = -x_3 = -x_4 = 4, \quad x_5 = 1,$$
$$y_1 = y_2 = y_3 = y_4 = 7, \quad y_5 = 8.$$

This gives

$$x_1 = x_2 = 668, \quad x_3 = x_4 = 892, \quad x_5 = 67,$$
$$y_1 = y_2 = 1429, \quad y_3 = y_4 = 1301, \quad y_5 = 1576.$$

Changing the sign of x_2 and x_4 and applying (4) again, we obtain the solution

$$1673\,15281, \quad 46847\,01124, \quad 52882\,64996, \quad 63838\,46756, \quad 69333\,47524.$$

(d) $n = 6$. We apply (4) to the trivial solution

$$x_1 = x_2 = x_3 = x_4 = 0, \quad x_5 = 3, \quad x_6 = 4,$$
$$y_1 = y_2 = y_3 = y_4 = 5, \quad y_5 = 4, \quad y_6 = 3,$$

and obtain

$$x_1 = x_2 = x_3 = x_4 = 60, \quad x_5 = 27, \quad x_6 = 64,$$
$$y_1 = y_2 = y_3 = y_4 = 125, \quad y_5 = 136, \quad y_6 = 123.$$

Changing the sign of x_3 and x_4 and applying (4) again, we obtain

$$x_1 = x_2 = 56440, \quad x_3 = x_4 = 35640, \quad x_5 = 32187, \quad x_6 = 38884,$$
$$y_1 = y_2 = 91085, \quad y_3 = y_4 = 101165, \quad y_5 = 102316, \quad y_6 = 99963.$$

Change of sign of x_2 and x_4 and a third application of (4) gives the solution

303 99288 95652, 320 53666 06047, 334 13500 01384,
352 04352 90636, 499 66347 59436, 542 92638 80052.

4. $n = 4$ **Reconsidered.** Tebay [9] gives the simple solution

$$x_1 = (s^2 - 1)(s^2 - 9)(s^2 + 3), \quad x_3 = 4s(s + 1)(s - 3)(s^2 + 3),$$
$$x_2 = 4s(s - 1)(s + 3)(s^2 + 3), \quad x_4 = 2s(s^2 - 1)(s^2 - 9).$$

With changes of sign and sequence, $s = 2$ gives the solution 60, 105, 168, 280. He obtains this parametric solution by imposing special conditions, the first being $x_1 x_2 + x_2 x_3 + x_3 x_1 = 0$ (with change of sign of x_3).

Martin [6] examines Tebay's method and corrects some mistakes. He remarks that Euler had given an equivalent solution without derivation [1, p. 503]. We now give a method for constructing numerous solutions for $n = 4$, the foregoing parametric solution appearing as a special case. Consider the equation

$$u_1^4 + u_2^4 + u_3^4 + u_4^4 = 2\left(u_1^2u_2^2 + u_1^2u_3^2 + u_1^2u_4^2 + u_2^2u_3^2 + u_2^2u_4^2 + u_3^2u_4^2\right),$$

which we abbreviate as

$$\sum u_i^4 = 2\sum u_i^2 u_j^2. \tag{5}$$

Numerical solutions of this equation are easily found by computer search. The following equations are equivalent:

$$4\left(u_3^2u_4^2 + u_4^2u_2^2 + u_2^2u_3^2\right) = \left(u_2^2 + u_3^2 + u_4^2 - u_1^2\right)^2, \tag{6}$$

$$4\left(u_1^2u_2^2 + u_3^2u_4^2\right) = \left(u_1^2 + u_2^2 - u_3^2 - u_4^2\right)^2, \tag{7}$$

$$\left(\sum u_i^2\right)^2 = 4\sum u_i^2 u_j^2, \tag{8}$$

$$\left(u_1^2 + u_2^2 - u_3^2 - u_4^2\right)\left(u_1^2 + u_3^2 - u_4^2 - u_2^2\right)\left(u_1^2 + u_4^2 - u_2^2 - u_3^2\right) = 8\sum u_i^2 u_j^2 u_k^2. \tag{9}$$

Set

$$x_1 = u_2u_3u_4, \quad x_2 = u_1u_3u_4, \quad x_3 = u_1u_2u_4, \quad x_4 = u_1u_2u_3.$$

Then Eq. (6) shows that we have a solution of the equations (1). This solution has some interesting properties.

Setting

$$A^2 = x_1^2x_2^2 + x_3^2x_4^2, \quad B^2 = x_1^2x_3^2 + x_4^2x_2^2, \quad C^2 = x_1^2x_4^2 + x_2^2x_3^2,$$

we see from (7) that A, B, C are integers. Setting $E^2 = A^2 + B^2 + C^2$, we see from (8) that E is an integer. Finally, Eq. (9) shows that

$$S = x_1^2 + x_2^2 + x_3^2 + x_4^2 = ABC/x_1x_2x_3x_4.$$

These relations are homogeneous and so are valid whether or not the solution x_1, x_2, x_3, x_4 is primitive. The following result is valid only for a primitive solution. Set

$$D = \mathrm{GCD}(x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4),$$
$$\Delta = \mathrm{GCD}(A, B, C).$$

Then we have

$$x_1x_2x_3x_4 = D^2/\Delta,$$

as is easily verified by calculating the p-adic values of D, Δ, $x_1x_2x_3x_4$. For p prime we may suppose that

$$v_p(u_1) = 0, \quad v_p(u_2) = \alpha, \quad v_p(u_3) = \beta, \quad v_p(u_4) = \gamma,$$

with $0 \leqslant \alpha \leqslant \beta \leqslant \gamma$. For the corresponding primitive solution we then have

$$v_p(x_1) = \gamma, \quad v_p(x_2) = \gamma - \alpha, \quad v_p(x_3) = \gamma - \beta, \quad v_p(x_4) = 0,$$

and we easily obtain

$$v_p(D) = 2\gamma - \alpha - \beta, \qquad v_p(\Delta) = \gamma - \alpha - \beta,$$
$$v_p(x_1x_2x_3x_4) = 3\gamma - \alpha - \beta,$$

from which the result follows.

A parametric solution to Eq. (5) is obtained by the following method. The identity

$$(p+q+r)(p-q-r)(q-r-p)(r-p-q)$$
$$= p^4 + q^4 + r^4 - 2(q^2r^2 + r^2p^2 + p^2q^2)$$

shows that

(10) $\quad p + q + r = 0$ implies $p^4 + q^4 + r^4 = 2(q^2r^2 + r^2p^2 + p^2q^2)$.

We rewrite (5) in the form

$$u_4^4 - 2u_4^2(u_1^2 + u_2^2 + u_3^2) + u_1^4 + u_2^4 + u_3^4 - 2(u_2^2u_3^2 + u_3^2u_1^2 + u_1^2u_2^2) = 0.$$

Setting $u_1 + u_2 + u_3 = 0$, we have from (10)

$$u_4^2 = 2(u_1^2 + u_2^2 + u_3^2).$$

To make u_4 rational, we set

$$u_1 = v_2^2 - v_3^2, \quad u_2 = v_3^2 - v_1^2, \quad u_3 = v_1^2 - v_2^2 \quad \text{with } v_1 + v_2 + v_3 = 0.$$

In effect we have from (10)

$$2(u_1^2 + u_2^2 + u_3^2) = (v_1^2 + v_2^2 + v_3^2)^2,$$

whence $u_4 = v_1^2 + v_2^2 + v_3^2$. We thus obtain

$$x_1 = (v_3^2 - v_1^2)(v_1^2 - v_2^2)(v_1^2 + v_2^2 + v_3^2),$$
$$x_2 = (v_1^2 - v_2^2)(v_2^2 - v_3^2)(v_1^2 + v_2^2 + v_3^2),$$
$$x_3 = (v_2^2 - v_3^2)(v_3^2 - v_1^2)(v_1^2 + v_2^2 + v_3^2),$$
$$x_4 = (v_2^2 - v_3^2)(v_3^2 - v_1^2)(v_1^2 - v_2^2),$$

with $v_1 + v_2 + v_3 = 0$. This is equivalent to Tebay's solution, which is obtained by setting $v_2 = 2$ (abandoning homogeneity) and $v_1 = s - 1$, whence $v_3 = -(s + 1)$.

We note that Euler made several studies of (5) [1, p. 661]; however, there is no mention of the relation between Eqs. (1) and (5).

5. Tables. In Table 1 we give the smallest solution (that with minimum S) for $3 \leqslant n \leqslant 8$, and in Tables 2–4 we give all solutions for $3 \leqslant n \leqslant 5$ having $S \leqslant 10^9$. For $n = 3$ tables have been given by Lal and Blundon [3], Leech [5] and Spohn [8]. The present computations were done on the IBM 370 computer at C.I.R.C.E. Each S is expressed as the sum of two squares $x_i^2 + y_i^2$ in all possible ways by the method of Nicolas [7]. We retain only those S which are expressible in at least n ways; we then have to test whether any n of these satisfy

$$\sum_{i=1}^{n} x_i^2 = S.$$

It may be remarked that it is never necessary to test whether an integer is a perfect square.

TABLE 1
The smallest solutions

n	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	S
3	44	117	240						73225
4	60	105	168	280					121249
5	28	64	259	392	680				688025
6	1332	1539	1756	3012	6348	7104			107062345
7	936	3840	5904	7332	7683	10400	11160		395971225
8	79	112	404	632	896	916	1828	2092	9941345

TABLE 2
$n = 3$

	x_1	x_2	x_3	S		x_1	x_2	x_3	S
1	44	117	240	73225	18	495	4888	8160	90723169
2	240	252	275	196729	19	2925	3536	11220	146947321
3	85	132	720	543049	20	1008	1100	12075	148031689
4	160	231	792	706225	21	2964	9152	9405	180998425
5	140	480	693	730249	22	1080	1881	14560	216698161
6	1008	1100	1155	3560089	23	4368	4901	13860	235198825
7	187	1020	1584	3584425	24	7840	9828	10725	273080809
8	429	880	2340	6434041	25	7579	8820	17472	440504425
9	832	855	2640	8392849	26	8789	10560	17748	503751625
10	828	2035	3120	14561209	27	10296	11753	16800	526380625
11	780	2475	2992	15686089	28	6072	16929	18560	667933825
12	195	748	6336	40742425	29	5643	14160	21476	693567625
13	1560	2295	5984	43508881	30	14112	15400	19305	808991569
14	1755	4576	6732	69339625	31	4900	17157	23760	882910249
15	528	5796	6325	73878025	32	4599	18368	23760	923071825
16	1155	6300	6688	85753369	33	935	17472	25704	966840625
17	1575	1672	9120	88450609					

TABLE 3
$n = 4$

	x_1	x_2	x_3	x_4	S	u_1	u_2	u_3	u_4
1	60	105	168	280	121249	3	5	8	14
2	420	728	1365	1560	5003209	7	8	15	26
3	385	792	840	1980	5401489	14	33	35	72
4	672	1120	1980	3465	17632609	32	56	99	165
5	585	1008	1456	5460	33289825	12	45	65	112
6	840	1520	1995	6384	47751481	5	16	21	38
7	880	1155	5040	5544	58245961	10	11	48	63
8	624	2625	3220	6432	59019025				
9	1848	3575	4620	7800	98380129	77	130	168	325
10	2508	5544	5985	8360	142735825	63	88	95	210
11	2295	3808	7344	10080	175308625	51	70	135	224
12	1232	8316	9141	10368	261726985				
13	3276	5005	11880	16632	453540025	65	91	216	330
14	2040	2520	11781	26180	834696361	18	40	187	231
15	4620	8184	11935	26040	908848081	11	24	35	62

Where a solution can be obtained by the method of Section 4, the values of u_i are given.

TABLE 4
$n = 5$

	x_1	x_2	x_3	x_4	x_5	S
1	28	64	259	392	680	688025
2	1112	1225	1876	3184	5768	49664225
3	2105	2648	2980	3736	4720	56559425
4	203	2240	3920	4240	6104	75661625
5	696	1200	3475	4980	6360	79250041
6	56	208	1400	4060	9065	100664225
7	557	1747	4141	5219	8285	116389325
8	427	3164	3980	6220	7420	119778425
9	1183	1300	2240	7280	8080	126391889
10	1095	3063	4119	5527	10329	164783125
11	1952	2360	5020	6089	10520	182326625
12	595	3549	5235	9555	10893	250310125
13	2328	5824	7368	9975	14196	394653025
14	2207	4417	5215	12479	14161	407836325
15	483	5328	6356	15000	17304	593448025
16	49	2152	5600	16076	18088	621607025
17	3799	9560	11384	13732	16112	683585825
18	2425	3020	8596	19628	20020	874951025

Remark. In the solutions 7, 10, 12 and 14, all the x_i are odd.

6. Concluding Remarks. (a) Examination of the tables suggests that there may be simple parametric solutions for $n \geqslant 5$, but we have not found them by the present method.

(b) There exist values of α, β for which Eq. (2) has trivial solutions; these can then be transformed into nontrivial solutions. This is the case when we replace the sums of $n - 1$ squares by their arithmetic means.

(c) I shall return later to the case of $n = 3$ with general α, β. Several of the systems of equations studied in [1, Chapter XIX], are effectively of this type. They are, however, treated by methods specific to each problem; we can now treat them by a uniform method.

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