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Chebyshev Quadrature Rules for a New Class of Weight Functions

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Abstract. Proof is given that the weight functions $w(x, p) = 1/[\pi(p + x)\sqrt{x(1 - x)}]$ on $(0, 1)$ admit Chebyshev quadratures for any fixed $p \geqslant 1$, and every N. For the particular cases when $p = 1$ and $p = 2$, the nodes are tabulated to ten decimal places for N-point rules with $N = 2, 4, 6, 8$, and 12. Numerical tables are also given for a coefficient in the expression of the error term.

1. Introduction. With a specified nonnegative weight function w, the problem of constructing the sequence of Chebyshev quadrature formulas

$$(1) \qquad \int_a^b w(x)f(x)\,dx \simeq H_N \sum_{k=1}^{N} f(x_{kN}) \qquad (N = 1,2,\ldots),$$

consists in determining the common coefficient H_N and the unique N nodes x_{kN} so that (1) is exact if f is any polynomial not exceeding degree N. All the nodes must be distinct and located in the interval $[a, b]$ for each value of N; otherwise, the desired formula does not exist. Uniformity in the coefficients maximizes numerical stability and minimizes computational work. The question of the possibility of the construction leads to the main task of investigating the zeros of the polynomial that provides the nodes. Such an investigation, however, may be very difficult, particularly if the given weight function contains parameters.

Some broad results regarding the existence of Chebyshev rules over infinite intervals of integration are found in Kahaner and Ullman [1], Wilf [2], and Ullman [3]. Several results (of a negative and positive nature) are also known for some specific weight functions on a finite interval. Bernstein [4], for example, proved that quadrature formulas (1) are impossible if $w(x) \equiv 1$ on $[-1, 1]$ and $N = 8$ or $N \geqslant 10$. Hermite showed that for every value of N, the weight function $w(x) = 1/\pi\sqrt{1 - x^2}$ on $(-1, 1)$ allows these integration rules (as well as having the Gaussian degree of precision $2N - 1$). More recently, Ullman [5] proved that Chebyshev quadrature formulas (1) are admissible for the weight function

$$(2) \qquad w(x, a) = \frac{1 + 2ax}{\pi(1 + 4a^2 + 4ax)\sqrt{1 - x^2}}$$

on $(-1, 1)$ for $N \geqslant 1$, when $|a| \leqslant 1/4$. (This represents an infinite one-parameter family, and it yields the familiar Hermite weight function when $a = 0$.)

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There appears to be no other concrete example in the literature where Chebyshev quadrature is possible for every N on a finite interval of integration. The problem of characterizing all weight functions, such that formulas (1) exist for all N, remains open. Hopefully, exhibiting further specific examples may help elucidate a solution to the problem.

In this paper, we shall show that functions of the form

$$w(x, p) = \frac{1}{\pi(2p + 1 + x)\sqrt{1 - x^2}} \qquad (-1 < x < 1), \tag{3}$$

also furnish an infinite one-parameter class of weight functions that admits Chebyshev quadrature for every N and any fixed $p \geqslant 1$. Note that none of these functions can be obtained from Ullman's weight function by assigning any given value to his parameter a, in Eq. (2). Instead of (3) on $(-1, 1)$, however, we actually take the equivalent weight functions

$$w(x, p) = \frac{\sqrt{p(p + 1)}}{\pi} \frac{1}{(p + x)\sqrt{x(1 - x)}} \qquad (p \geqslant 1), \tag{4}$$

on the interval $0 < x < 1$, the constant factor $\sqrt{p(p + 1)}$ being introduced for the convenience of making the zero moment equal to unity. It is seen that when $p = \infty$, the family has $1/\pi\sqrt{x(1 - x)}$ as a member, which is equivalent to $1/\pi\sqrt{1 - x^2}$ on $(-1, 1)$. After establishing the existence of Chebyshev quadrature for the class (4), we shall give some numerical results for the particular cases

$$w(x, 1) = \sqrt{2}/\left[\pi(1 + x)\sqrt{x(1 - x)}\right]$$

and

$$w(x, 2) = \sqrt{6}/\left[\pi(2 + x)\sqrt{x(1 - x)}\right].$$

2. Construction of the Chebyshev Formulas. In order to develop N-point Chebyshev quadrature rules

$$\begin{aligned} C_N(f) &= H_N(p) \sum_{k=1}^{N} f(x_{kN}) \\ &= \frac{\sqrt{p(p + 1)}}{\pi} \int_0^1 \frac{f(x)\,dx}{(p + x)\sqrt{x(1 - x)}} - E_N(f) \qquad (N \geqslant 1), \end{aligned} \tag{5}$$

for our weight function (4), we must find $N + 1$ parameters, the N abscissas x_{kN} and the coefficient H_N, such that the error $E_N(f) = 0$ whenever f is an arbitrary polynomial of degree $n \leqslant N$. The common coefficient is simply

$$H_N = 1/N, \tag{6}$$

determined by the requirement that $E_N(f) = 0$ when $f(x) \equiv 1$. There are two main approaches (viz., algebraic and analytic) to the determination of the remaining unknown parameters.

To get the abscissas x_{kN} algebraically, one may calculate the moments

$$M_n = \int_0^1 x^n w(x, p)\,dx, \tag{7}$$

and obtain the simultaneous nonlinear system

$$(8) \qquad \sum_{k=1}^{N} x_{kN}^n = NM_n = \beta_{nN} \qquad (n = 1, 2, \ldots, N),$$

which can be resolved by use of Newton's identities. The unique N nodes x_{kN} are then found to be zeros of the Nth-degree monic polynomial

$$(9) \qquad Y_N(x, p) = \frac{1}{N!}\begin{vmatrix} x^N & x^{N-1} & x^{N-2} & x^{N-3} & \cdots & x & 1 \\ \beta_{1N} & 1 & 0 & 0 & \cdots & 0 & 0 \\ \beta_{2N} & \beta_{1N} & 2 & 0 & \cdots & 0 & 0 \\ \beta_{3N} & \beta_{2N} & \beta_{1N} & 3 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \beta_{NN} & \beta_{N-1,N} & \beta_{N-2,N} & \beta_{N-3,N} & \cdots & \beta_{1N} & N \end{vmatrix}$$

involving the power sums

$$(10) \qquad \beta_{nN}(\alpha) = NM_n = \frac{N}{4^n}\left[\binom{2n}{n} + 2\sum_{j=1}^{n}(-1)^j\alpha^j\binom{2n}{n-j}\right],$$

with

$$(11) \qquad \alpha = 2p + 1 - 2\sqrt{p(p+1)}, \quad \text{or} \quad p = (1-\alpha)^2/4\alpha.$$

Except for small values of N, this method is unwieldy.

An alternate, analytical approach, due to Chebyshev himself [6], provides us with a better expression for the desired polynomial. This requires the use of the familiar form (e.g., see Krylov [7, p. 183])

$$(12) \qquad U(z, p) = G \cdot \exp\left[\frac{1}{H_N}\int_0^1 w(x, p)\ln(z - x)\, dx\right], \qquad z > 1,$$

where only the polynomial part of its expansion is to be considered, with the constant G being chosen so that the coefficient of z^N is unity. Equation (12) can be manipulated to produce

$$(13) \quad \int_0^1 w(x, p)\ln(z - x)\, dx = \ln\frac{1}{4r} - 2\sum_{m=1}^{\infty}\frac{(-1)^m}{m}(\alpha r)^m = \ln\left[\frac{(1 + \alpha r)^2}{4r}\right],$$

where α is given by (11), and

$$(14) \qquad r = 2z - 1 + 2\sqrt{z(z-1)} = (\sqrt{z} + \sqrt{z-1})^2.$$

Therefore, with (6) and (13), Eq. (12) leads to

$$(15) \qquad U(z, p) = \frac{G}{(4r)^N}(1 + \alpha r)^{2N} = \frac{G}{4^N}\sum_{j=0}^{2N}\binom{2N}{j}\alpha^j r^{j-N},$$

which must be truncated in order to get the so-called "proper terms" of the series. Noting that the polynomial part of $1/r^{N-j}$ is the *shifted* Chebyshev polynomial $T^*_{N-j}(z)$, taking $G = 2$, and setting $z = x$, we obtain the final explicit form

$$(16) \qquad Y_N(x, p) \equiv \phi_N(x, \alpha) = \text{polynomial part of } U(x, p)$$
$$= \frac{2}{4^N}{\sum_{j=0}^{N}}'\binom{2N}{j}\alpha^j T^*_{N-j}(x),$$

where the prime on the summation sign means that the term when $j = N$ is to be halved. If $\alpha = 0$, i.e., when $p = \infty$, this reduces immediately to

$$Y_N(x, \infty) \equiv \phi_N(x, 0) = \frac{2}{4^N} T_N^*(x), \tag{17}$$

whose zeros are the abscissas for the Chebyshev quadrature involving the well-known function $1/\pi\sqrt{x(1-x)}$.

3. Establishment of Existence. To prove that the N zeros of (16), which provide the nodes x_{kN} for formulas (5), are all distinct and located in the interval [0, 1] for every fixed $p \geqslant 1$ and each $N \geqslant 1$, we now show that $Y_N(x, p)$ will change signs N times on [0, 1] if $0 < \alpha \leqslant 3 - 2\sqrt{2}$, i.e., for all $p \geqslant 1$. The analysis is straightforward but involves some manipulative details that, for brevity, are not included.

Replacing $T_{N-j}^*(x)$ in (16) by $T_{N-j}^*(x) = \frac{1}{2}[R^{N-j} + R^{-N+j}]$, where

$$R = 2x - 1 + 2\sqrt{x(x-1)}, \tag{18}$$

gives

$$\begin{aligned} Y_N(x, p) &\equiv \phi_N(x, \alpha) \\ &= -\frac{\alpha^N}{4^N}\binom{2N}{N} + \frac{R^{-N}}{4^N}\sum_{j=0}^{N}\binom{2N}{j}\alpha^j R^{2N-j} + \frac{R^N}{4^N}\sum_{j=0}^{N}\binom{2N}{j}\alpha^j R^{-2N+j}. \end{aligned} \tag{19}$$

Since the *truncated* binomial series

$$\sum_{j=0}^{N}\binom{2N}{j}\alpha^j R^{2N-j} = (\alpha + R)^{2N} - N\binom{2N}{N}\int_0^\alpha (\alpha - t)^N (R + t)^{N-1}\,dt, \tag{20}$$

Eq. (19) becomes

$$\begin{aligned} \phi_N(x, \alpha) = &-\frac{\alpha^N}{4^N}\binom{2N}{N} + \frac{1}{4^N}\left[R^{-N}(\alpha + R)^{2N} + R^N(\alpha + R^{-1})^{2N}\right] \\ &-\frac{N}{4^N}\binom{2N}{N}\int_0^\alpha (\alpha - t)^N\left[R^{-N}(R + t)^{N-1} + R^N(R^{-1} + t)^{N-1}\right] dt. \end{aligned} \tag{21}$$

It can be shown that

$$\begin{aligned} &\frac{1}{4^N}\left[R^{-N}(\alpha + R)^{2N} + R^N(\alpha + R^{-1})^{2N}\right] \\ &\qquad = \frac{2}{4^N}\left[(1-\alpha)^2 + 4\alpha x\right]^N T_N^*\left[\frac{x(1+\alpha)^2}{(1-\alpha)^2 + 4\alpha x}\right], \end{aligned} \tag{22}$$

and that

$$\begin{aligned} &-\frac{N}{4^N}\binom{2N}{N}\int_0^\alpha (\alpha - t)^N\left[R^{-N}(R^{-N} + t)^{N-1} + R^N(R^{-1} + t)^{N-1}\right] dt \\ &\qquad = \frac{\alpha^N}{4^N}\binom{2N}{N} - \alpha^{2N}\phi_N(x, 1/\alpha). \end{aligned} \tag{23}$$

Hence Eq. (16) may be written in the interesting form

$$\phi_N(x, \alpha) = \frac{2}{4^N}\left[(1-\alpha)^2 + 4\alpha x\right]^N T_N^*\left[\frac{x(1+\alpha)^2}{(1-\alpha)^2 + 4\alpha x}\right] - \alpha^{2N}\phi_N(x, 1/\alpha), \tag{24}$$

or as

$$\phi_N(x,\alpha) = \frac{2}{4^N}\left[(1-\alpha)^2 + 4\alpha x\right]^N \cdot \cos(NA) - \alpha^{2N}\phi_N(x, 1/\alpha), \tag{25}$$

where

$$\cos A = \frac{2x(1+\alpha^2) - (1-\alpha)^2}{(1-\alpha)^2 + 4\alpha x}. \tag{26}$$

We note from (26) that if $0 \leqslant x \leqslant 1$, then $\pi \geqslant A \geqslant 0$. Noticing also that when $A = A_m = m\pi/N$, i.e., if

$$\begin{aligned} x = t_{mN} &= \frac{(1-\alpha)^2(1+\cos A_m)}{2(1+\alpha^2 - 2\alpha\cos A_m)} \\ &= \frac{p\cos^2(m\pi/2N)}{p + \sin^2(m\pi/2N)} \qquad (m = 0, 1, \ldots, N), \end{aligned} \tag{27}$$

we then have $\cos(NA_m) = (-1)^m$. Thus the left-hand side of Eq. (25) will change signs exactly N times whenever

$$\frac{2}{4^N}\left[(1-\alpha)^2 + 4\alpha x\right]^N > \alpha^{2N}|\phi_N(x, 1/\alpha)| > 0 \tag{28}$$

for $0 \leqslant x \leqslant 1$. If $0 < \alpha < 1$, however, we have, from (16),

$$\alpha^{2N}|\phi_N(x,1/\alpha)| = \frac{2\alpha^N}{4^N}\sum_{j=0}^{N}{}'\binom{2N}{j}\alpha^{N-j}|T^*_{N-j}(x)| \leqslant \frac{2\alpha^N}{4^N}\sum_{j=0}^{N}{}'\binom{2N}{j} = \alpha^N; \tag{29}$$

so, instead of (28), it is sufficient to consider the inequality

$$\frac{2}{4^N}\left[(1-\alpha)^2 + 4\alpha x\right]^N > \alpha^N > 0, \quad \text{or} \quad p + x > (2)^{-1/N}. \tag{30}$$

Now this is clearly true for each N and any x in [0, 1] if $p \geqslant 1$, i.e., for all α in $0 < \alpha \leqslant 3 - 2\sqrt{2}$. But it will not hold for a fixed p and every N and x if $p < 1$, i.e., when $\alpha > 3 - 2\sqrt{2}$. The case for $\alpha = 0$ ($p = \infty$) is given by (17).

Since the polynomial $\phi_N(x,\alpha)$, or its equivalent $Y_N(x, p)$ in (16), has N variations of sign in the interval [0, 1] for every value of N whenever $p \geqslant 1$, it will indeed furnish the N distinct real nodes x_{kN} for the quadratures (5), with the common coefficient being $H_N = 1/N$. The functions given by (4) therefore represent an infinite one-parameter family of Chebyshev weight functions on (0, 1). One can show that the sequence $C_N(f)$ of approximate integration formulas (5) will converge to the true value for any continuous function f.

4. Remark. It can be seen that our weight function (4), as well as Ullman's (2), is a product

$$w(x,p) = q(x,p)h(x), \tag{31}$$

in which $q(x, p)$ is a certain rational function containing a parameter, and where $h(x)$ is the classical Hermite weight function $1/\pi\sqrt{x(1-x)}$ which already admits Chebyshev quadrature on (0, 1). The form (31) may thus be regarded as a special modification of the weight function $h(x)$. Since Ullman and the authors have actually found two concrete examples of q of a similar type, where Chebyshev quadrature is still possible with the modified weight function w, this suggests a broader problem of characterizing all nonnegative rational functions q such that the product $q(x)h(x)$ is again a Chebyshev weight function on the same interval.

5. The Error Term. If the function f is other than a polynomial of degree $\leqslant N$, the integration formulas (5) have an error term given (e.g., see Kopal [8, p. 419]) by

$$E_N(f) = \frac{K_N(\alpha)}{(N+1)!} f^{(N+1)}(\varepsilon), \qquad 0 < \varepsilon < 1, \tag{32}$$

where

$$K_N(\alpha) = \frac{1}{N}\left[N M_{N+1} - \sum_{i=1}^{N} (x_{iN})^{N+1} \right], \tag{33}$$

with M_{N+1} given by (7). Knowing the coefficient K_N permits an estimate of the maximum error made by Chebyshev quadrature, of the form

$$|E_N(f)| \leqslant \frac{K_N(\alpha)}{(N+1)!} \max_{0 \leqslant x \leqslant 1} |f^{(N+1)}(x)|. \tag{34}$$

6. Some Computational Results. In order for the quadrature rules (5) to be usable in numerical calculations, the roots x_{iN}, $1 \leqslant i \leqslant N$, of the polynomial (16) must be found, preferably by an on-line, readily user-reproducible technique which incorporates a provision for verifying the accuracy of its results. Such a procedure, based on a single-parameter Newton's method analysis, is used to obtain numerical results for the particular cases when $p = 1$ (for $\alpha = 3 - 2\sqrt{2}$), and when $p = 2$ (for $\alpha = 5 - 2\sqrt{6}$). N-point Chebyshev quadrature rules for these two weight functions are tabulated to ten decimal places for $N = 2, 4, 6, 8,$ and 12. Since the error coefficients K_N in the remainder term (34) may be useful when a bound on the $(N+1)$th derivative can be estimated, we give tabulations of them.

The polynomial $\phi_N(x, \alpha)$ and its first derivative $\phi'_N(x, \alpha)$ may be written as

$$\phi_N(x, \alpha) = \frac{1}{2} V_{0N}(\alpha) + \sum_{j=1}^{N} V_{jN}(\alpha) T_j^*(x),$$

$$\phi'_N(x, \alpha) = \frac{1}{2} P_{0N}(\alpha) + \sum_{j=1}^{N-1} P_{jN}(\alpha) T_j^*(x),$$

where the coefficients $V_{jN}(\alpha)$ and $P_{jN}(\alpha)$ are recursively defined by

$$V_{NN}(\alpha) = \frac{2}{4^N}, \qquad P_{N+1,N}(\alpha) = P_{NN}(\alpha) = 0,$$

$$V_{j-1,N}(\alpha) = \frac{(N+j)\alpha}{(N-j+1)} V_{jN}(\alpha),$$

and

$$P_{j-1,N}(\alpha) = P_{j+1,N}(\alpha) + 4jV_{jN}(\alpha) \qquad (j = 1, 2, \ldots, N).$$

These coefficients depend only on the user-chosen values of N and α, and may therefore be calculated prior to the iterations by Newton's method. The shifted Chebyshev polynomials are evaluated recursively using

$$T_0^*(x) = 1, \quad T_1^*(x) = 2x - 1, \quad T_j^*(x) = 2(2x-1)T_{j-1}^*(x) - T_{j-2}^*(x).$$

The iterative scheme

$$x_{iN}^{(k+1)} = x_{iN}^{(k)} - \frac{\phi_N\left(x_{iN}^{(k)}, \alpha\right)}{\phi_N'\left(x_{iN}^{(k)}, \alpha\right)} \qquad (k = 0, 1, \ldots),$$

was found to be stable and rapidly convergent, because for most N of computational interest, the quantity $|\phi_N'(x, \alpha)|$ tends to maximize near the roots of $\phi_N(x, \alpha)$; this can be seen in the sample plot of $\phi_4(x, \alpha)$ given in Figure 1 for three values of α. Since, as shown from (27), the ith root of ϕ_N lies in the interval $t_{N-i+1,N} < x_{xN} < t_{N-i,N}$ with $0 < x_{1N} < x_{2N} < \cdots < x_{NN} < 1$, we take

$$x_{iN}^{(0)} = \frac{1}{2}\left[t_{N-i,N} + t_{N-i+1,N}\right] \qquad (i = 1, 2, \ldots, N),$$

as the initial guess for each root.

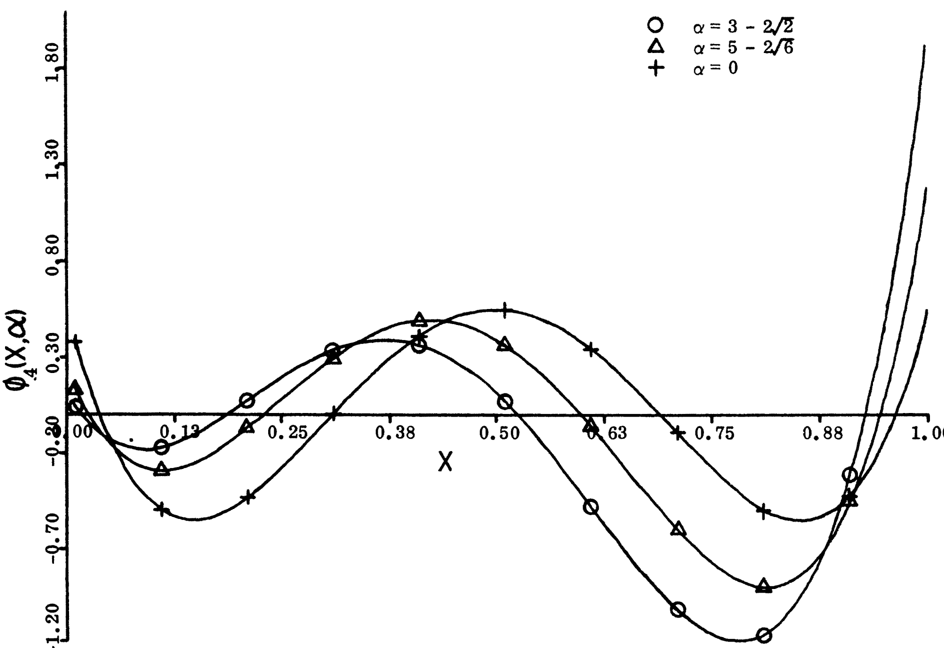

FIGURE 1

Graph illustrating the behavior of $\phi_4(x, \alpha)$ on $[0, 1]$ for 3 particular values of the parameter α within the allowed range of values of α.

Results generated by following this procedure on a CDC 3150 computer with $N = 2, 4, 6, 8$, and 12, and for $\alpha = 0$, $5 - 2\sqrt{6}$, and $3 - 2\sqrt{2}$, are tabulated in Table 1. The extent to which the differences

$$D_{jN}(\alpha) = \beta_{jN}(\alpha) - \sum_{i=1}^{N} x_{iN}^{j} \qquad (j = 1, 2, \ldots, N),$$

deviate from zero provides a measure of the computed accuracy of the N nodes x_{iN}. From all the calculations of the entries in Table 1, we found $D_{jN}(\alpha) \leqslant 10^{-28}$, indicating that our computations were reliable to at least twenty-eight significant figures. For the values of N and α used in Table 1, the error coefficients $K_N(\alpha)$, defined by (34), are tabulated in Table 2.

TABLE 1

Abscissas x_{iN}, $i = 1, 2, \ldots, N$, for our quadrature, generated for 5 particular values of N and 3 values of the parameter α or p. Note that $\alpha = 0$ (or $p = \infty$) corresponds to the familiar Hermite weight function.

	$\alpha = 0$ ($p = \infty$)	$\alpha = 5 - 2\sqrt{6}$ ($p = 2$)	$\alpha = 3 - 2\sqrt{2}$ ($p = 1$)
$N = 2$	0.1464466094	0.0977450101	0.0659028626
	0.8535533906	0.8012344754	0.7625242621
$N = 4$	0.0380602337	0.0254733441	0.0170752109
	0.3086582838	0.2298526735	0.1871077662
	0.6913417162	0.5985128515	0.5249093675
	0.9619397663	0.9441201021	0.9277619049
$N = 6$	0.0170370869	0.0114021218	0.0077412136
	0.1464466094	0.1026906220	0.0808809415
	0.3705904774	0.2818298930	0.2257712812
	0.6294095226	0.5310518400	0.4601690701
	0.8535533906	0.7952976760	0.7441166973
	0.9829629131	0.9746663039	0.9666021705
$N = 8$	0.0096073598	0.0064229328	0.0044099845
	0.0842651938	0.0578066622	0.0449660816
	0.2222148835	0.1599869457	0.1240124328
	0.4024548390	0.3098793161	0.2525702250
	0.5975451610	0.4974414108	0.4257406510
	0.7777851165	0.7000112490	0.6365128547
	0.9157348062	0.8787112362	0.8445150007
	0.9903926402	0.9856581895	0.9809812686
$N = 12$	0.0042775693	0.0028557266	0.0019908973
	0.0380602337	0.0256996862	0.0197860323
	0.1033233299	0.0713390234	0.0540285177
	0.1956192855	0.1395099838	0.1087757624
	0.3086582838	0.2293711548	0.1822621767
	0.4347369039	0.3389414828	0.2778606180
	0.5652630961	0.4643319085	0.3939298750
	0.6913417162	0.5989121599	0.5283045990
	0.8043807145	0.7327141797	0.6727649545
	0.8966766701	0.8526284749	0.8127084158
	0.9619397663	0.9439757914	0.9266692963
	0.9957224307	0.9935973415	0.9914816035

TABLE 2

Error coefficients $K_N(\alpha)$ for the values of N and α used in Table 1.

α	$N = 2$	$N = 4$	$N = 6$	$N = 8$	$N = 12$
0	0.18933D-28	0.37865D-28	0.54694D-28	0.34710D-28	0.65213D-28
$5-2\sqrt{6}$	0.93740D-02	0.69754D-04	0.58728D-06	0.52174D-08	0.44597D-12
$3-2\sqrt{2}$	0.15611D-01	0.33509D-03	0.81380D-05	0.20855D-06	0.14832D-09

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