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The Arithmetic-Harmonic Mean

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Abstract. Consider two sequences generated by

$$a_{n+1} = M(a_n, b_n), \qquad b_{n+1} = M'(a_{n+1}, b_n),$$

where the a_n and b_n are positive and M and M' are means. The paper discusses the nine processes which arise by restricting the choice of M and M' to the arithmetic, geometric and harmonic means, one case being that used by Archimedes to estimate π. Most of the paper is devoted to the arithmetic-harmonic mean, whose limit is expressed as an infinite product and as an infinite series in two ways.

1. Introduction. Recently [3] we have discussed the generalized Archimedean process in which two sequences (a_n) and (b_n) are defined by

$$a_{n+1} = M(a_n, b_n), \tag{1a}$$

$$b_{n+1} = M'(a_{n+1}, b_n), \tag{1b}$$

where $a_0, b_0 \in \mathbf{R}^+$ and M and M' are mappings from $\mathbf{R}^+ \times \mathbf{R}^+$ to $\mathbf{R}^+$ which satisfy the following three properties:

$$a \leqslant b \Rightarrow a \leqslant M(a, b) \leqslant b, \tag{2}$$

$$M(a, b) = M(b, a), \tag{3}$$

$$a = M(a, b) \Rightarrow a = b. \tag{4}$$

We shall refer to such mappings as *means*. In [3] we showed that for all means M and M' the sequences (a_n) and (b_n) converge monotonically to a common limit, which we will denote by $L(a_0, b_0)$, and that the *errors* of both sequences (a_n) and (b_n) tend to zero like $1/4^n$ provided that M and M' possess continuous partial derivatives up to the second order.

Archimedes' process for estimating π (see [4, p. 50]) is a special case (the *original* case) of (1) with $a_0 = 3\sqrt{3}$, $b_0 = \frac{1}{2}3\sqrt{3}$ and M and M', respectively, the harmonic and geometric means. It is well known (see, for example, Phillips [6]) that, for this choice of M and M', there are two cases to consider depending on the initial values a_0 and b_0. First, if $a_0 > b_0 > 0$,

$$a_n = 2^n \frac{a_0 b_0}{\left(a_0^2 - b_0^2\right)^{1/2}} \tan(\theta/2^n), \tag{5}$$

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(6) $$b_n = 2^n \frac{a_0 b_0}{\left(a_0^2 - b_0^2\right)^{1/2}} \sin(\theta/2^n),$$

where $b_0/a_0 = \cos\theta$. In this case we see that

(7) $$L(a_0, b_0) = \frac{a_0 b_0}{\left(a_0^2 - b_0^2\right)^{1/2}} \theta.$$

Second, if $b_0 > a_0 > 0$, we put $b_0/a_0 = \cosh\theta$ and find that a_n and b_n and L are given by (5), (6) and (7) with a_0 and b_0 interchanged in these three formulae and with tan, sin and cos replaced by the corresponding hyperbolic functions. We also note that an alternative formulation of $L(a_0, b_0)$ for this latter case allows us to use the Archimedean process to compute the logarithm function from

(8) $$(t^2 - 1)L\left(1/(t^2 + 1), 1/2t\right) = \log t$$

for $t > 1$. (See, for example, Carlson [2] and Miel [5].)

Thus we have results concerning the convergence and rate of convergence for the general case (1), and we also have a full analysis of Archimedes' special case. This paper is devoted to a study of other special cases of the generalized Archimedean process which are of obvious interest. Specifically, we wish to explore thoroughly the cases where M and M' are drawn from the set $\{A, G, H\}$, where A, G and H denote the arithmetic, geometric and harmonic means, respectively.

2. $M = G$, $M' = H$. The second case which we consider is where $M = G$, $M' = H$, which is the Archimedes process with the two means transposed. It is not difficult to verify that, if $0 < a_0 < b_0$,

(9) $$a_n = 2^{n-1}\alpha \sin(\theta/2^{n-1}),$$

(10) $$b_n = 2^n \alpha \tan(\theta/2^n),$$

where

(11) $$a_0/b_0 = \cos^2\theta \quad \text{and} \quad \alpha = b_0 \Big/ \left(\frac{b_0}{a_0} - 1\right)^{1/2}.$$

It follows that

(12) $$L(a_0, b_0) = \cos^{-1}\left((a_0/b_0)^{1/2}\right) \cdot b_0 \Big/ \left(\frac{b_0}{a_0} - 1\right)^{1/2}.$$

For example, with $a_0 = 3\sqrt{3}/4$ and $b_0 = 3\sqrt{3}$ we have $\theta = \pi/3$; then a_n and b_n correspond respectively to the *areas* of the inscribed and escribed regular polygons of the unit circle with $3 \cdot 2^n$ sides. We recall that, in the Archimedes process proper, a_n and b_n are the *semiperimeters* of these same polygons. Thus we can think of this 'transposed Archimedes' process *as one which Archimedes might have used.* To complete this case we note that, if $0 < b_0 < a_0$, we need to replace sin, tan and cos by the corresponding hyperbolic functions in (9), (10) and (11) and redefine α as $b_0(1 - b_0/a_0)^{-1/2}$.

3. $M = M'$. We now deal with the cases where $M = M' \in \{A, G, H\}$. First we observe that these means may be written in the form

(13) $$M(a, b) = f^{-1}\left(\tfrac{1}{2}(f(a) + f(b))\right),$$

where $f(x) = x$, $\log x$ and $1/x$ gives $M = A, G$ and H, respectively. (We remark in passing that (13) defines a mean in the sense used here for any continuous mapping f from $\mathbf{R}^+$ to $\mathbf{R}^+$ which is strictly monotonic increasing.) Thus the process (1) may be expressed as

$$f(a_{n+1}) = \tfrac{1}{2}(f(a_n) + f(b_n)), \tag{14a}$$

$$f(b_{n+1}) = \tfrac{1}{2}(f(a_{n+1}) + f(b_n)), \tag{14b}$$

and the three cases $M = M' \in \{A, G, H\}$ are reduced to the single case $M = M' = A$. The explicit forms for a_n and b_n in this latter case are easily obtained as

$$a_n = L(a_0, b_0) + \frac{2}{3} \cdot \frac{1}{4^n}(a_0 - b_0), \tag{15}$$

$$b_n = L(a_0, b_0) - \frac{1}{3} \cdot \frac{1}{4^n}(a_0 - b_0), \tag{16}$$

where the common limit is

$$L(a_0, b_0) = \tfrac{1}{3}(a_0 + 2b_0). \tag{17}$$

We note that (15) and (16) show very clearly both the monotonicity and rate of convergence of the errors to which we referred in Section 1 above.

4. $\{M, M'\} = \{A, G\}$. When $M = A$ and $M' = G$ or $M = G$ and $M' = A$, we can reduce the problem to one which we have already considered. For example, if $M = A$ and $M' = G$, (1) becomes

$$a_{n+1} = \tfrac{1}{2}(a_n + b_n), \tag{18a}$$

$$b_{n+1} = (a_{n+1}b_n)^{1/2} \tag{18b}$$

and the substitution $u_n = 1/a_n$, $v_n = 1/b_n$ transforms (18) into the original Archimedean process.

5. The Arithmetic-Harmonic Mean. The final cases which remain to be explored in this paper are when $M = A$ and $M' = H$ and also $M = H$ and $M' = A$. Let us write $L(a_0, b_0)$, as before, to denote the common limit of the sequences defined by

$$a_{n+1} = \tfrac{1}{2}(a_n + b_n), \tag{19a}$$

$$1/b_{n+1} = \tfrac{1}{2}(1/a_{n+1} + 1/b_n). \tag{19b}$$

The other case, with the means A and H interchanged gives the sequences defined by

$$1/a_{n+1} = \tfrac{1}{2}(1/a_n + 1/b_n), \tag{20a}$$

$$b_{n+1} = \tfrac{1}{2}(a_{n+1} + b_n). \tag{20b}$$

If we denote the common limit of the latter pair of sequences by $L'(a_0, b_0)$ it is clear that

$$L'(a_0, b_0) = 1/L(1/a_0, 1/b_0).$$

Thus we need consider only one of these two cases and we will restrict our attention to (19).

First we note the homogeneous property, evident from (19), that

$$L(\lambda a_0, \lambda b_0) = \lambda L(a_0, b_0)$$

for any positive λ, a_0, b_0. Thus it suffices to consider the case where, say, $b_0 = 1$ and $a_0 = 1 + x$, with $x > -1$. It follows by induction that, for any $n \geqslant 1$,

(21a) $$a_n = 2^{-n}\prod_{r=1}^{n}(2^{2r-1} + x)\Big/\prod_{r=1}^{n-1}(2^{2r} + x),$$

(21b) $$b_n = 2^n\prod_{r=1}^{n}\left[(2^{2r-1} + x)/(2^{2r} + x)\right].$$

In analyzing the limit of this sequence we find it convenient to define

$$F(x) = L(1 + x, 1) = \lim_{n\to\infty} b_n,$$

so that

(22) $$F(x) = \prod_{r=1}^{\infty}\left[(1 + 2x/4^r)/(1 + x/4^r)\right].$$

It follows immediately from (22) that

(23) $$(1 + \tfrac{1}{4}x)F(x) = (1 + \tfrac{1}{2}x)F(\tfrac{1}{4}x).$$

Now we write

(24) $$F(x) = 1 + c_1x + c_2x^2 + \cdots.$$

On substituting (24) into (23) and comparing coefficients of x^m, we obtain

$$c_m + \tfrac{1}{4}c_{m-1} = c_m/4^m + 2c_{m-1}/4^m$$

for $m \geqslant 1$, with $c_0 = 1$. Hence we obtain

(25) $$c_m = (-1)^{m-1}\frac{(4^{m-1} - 2)\cdots(4 - 2)}{(4^m - 1)\cdots(4 - 1)},$$

so that

(26) $$F(x) = 1 + \frac{1}{3}x - \frac{2}{45}x^2 + \frac{4}{405}x^3 - \cdots$$

and an inspection of (25) shows that the series (26) is convergent for $|x| < 4$. Since we are concerned only with $x > -1$, the series (26) is valid for $-1 < x < 4$.

To obtain an expression for $F(x)$ valid for $x \geqslant 4$, we could apply (23) repeatedly and write

$$F(x) = \prod_{r=1}^{n}\left[(1 + 2x/4^r)/(1 + x/4^r)\right]\left(1 + \tfrac{1}{3}(x/4^n) - \frac{2}{45}(x/4^n)^2 + \cdots\right),$$

where the latter series is convergent for $|x| < 4^{n+1}$.

We now explore an alternative representation for $F(x)$ for large x. We define

(27) $$\psi(x) = \log F(x) = \sum_{r=1}^{\infty}\left(\log\left(1 + \frac{2x}{4^r}\right) - \log\left(1 + \frac{x}{4^r}\right)\right)$$

and write $x = 4^t$ where $m \leqslant t < m + 1$ and m is a positive integer. We express

$$\psi(x) = S_1(x) + S_2(x),$$

where $S_1(x)$ is the sum of the first m terms on the right of (27). Thus

$$S_2(x) = \sum_{r=m+1}^{\infty}\left(\log\left(1 + \frac{2}{4^{r-t}}\right) - \log\left(1 + \frac{1}{4^{r-t}}\right)\right)$$

and, on using the monotonicity of $\log(1 + x)$ and the inequality

$$\log(1 + x) < x$$

for $x > 0$, we obtain

$$0 < S_2(x) < \sum_{r=m+1}^{\infty} \log\left(1 + \frac{2}{4^{r-t}}\right) < \frac{8}{3},$$

so that $S_2(x) = O(1)$ for large x. For $S_1(x)$ we write

$$\begin{aligned} S_1(x) &= \sum_{r=1}^{m} \left(\log\left(1 + \frac{2}{4^{r-t}}\right) - \log\left(1 + \frac{1}{4^{r-t}}\right)\right) \\ &= \sum_{r=1}^{m} \left(\log \frac{2}{4^{r-t}}\left(1 + \frac{1}{2} \cdot \frac{1}{4^{t-r}}\right) - \log \frac{1}{4^{r-t}}\left(1 + \frac{1}{4^{t-r}}\right)\right) \\ &= m \log 2 + \sum_{r=1}^{m} \left(\log\left(1 + \frac{1}{2} \cdot \frac{1}{4^{t-r}}\right) - \log\left(1 + \frac{1}{4^{t-r}}\right)\right). \end{aligned}$$

It follows that $S_1(x) = m \log 2 + O(1)$ and thus

(28) $$\psi(x) = \tfrac{1}{2} \log x + O(1).$$

We may similarly verify that

$$\psi(x) - \psi(2/x) = m \log 2 + \psi(u) - \psi(2/u),$$

where $u = 4^{t-m} = x/4^m$. This shows that

(29) $$\psi(x) - \psi(2/x) - \tfrac{1}{2} \log x$$

is unaltered when x is replaced by $x/4^m$. It turns out that the expression (29) provides the key to a full understanding of the function ψ and thus of the limit of the arithmetic-harmonic mean process. However, it is convenient to 'centralize' the function (29) so that it is zero when $x = \sqrt{2}$. We therefore now study the function

(30) $$\delta(x) = \psi(x) - \psi(2/x) - \tfrac{1}{2} \log x + \tfrac{1}{4} \log 2$$

and verify some of its properties.

6. The Function δ.

LEMMA 1. *For all $x > 0$, $\delta(1/x) = \delta(x)$.*

Proof. From (27) we have

$$\begin{aligned} \psi(1/x) - \psi(2x) &= \sum_{r=1}^{\infty} \left(\log\left(1 + \frac{2}{4^r x}\right) - \log\left(1 + \frac{1}{4^r x}\right)\right) \\ &\quad - \sum_{r=1}^{\infty} \left(\log\left(1 + \frac{4x}{4^r}\right) - \log\left(1 + \frac{2x}{4^r}\right)\right) \\ &= \sum_{r=1}^{\infty} \left(\log\left(1 + \frac{2}{4^r x}\right) - \log\left(1 + \frac{4}{4^r x}\right)\right) + \log\left(1 + \frac{1}{x}\right) \\ &\quad - \sum_{r=1}^{\infty} \left(\log\left(1 + \frac{x}{4^r}\right) - \log\left(1 + \frac{2x}{4^r}\right)\right) - \log(1 + x) \\ &= -\psi(2/x) + \psi(x) - \log x \end{aligned}$$

and Lemma 1 follows.

LEMMA 2. *For all* $x > 0$, $\delta(2/x) = -\delta(x)$.

Proof. This follows immediately from (30).

LEMMA 3. *For all* $x > 0$, $\delta(2x) = -\delta(x)$.

Proof. Applying Lemma 2 and then Lemma 1 we obtain

$$\delta(2x) = -\delta(1/x) = -\delta(x).$$

An immediate consequence of this last lemma is that δ is unaltered when x is replaced by $4x$. We note in passing that this confirms our earlier observation, derived from a somewhat tedious manipulation of the infinite series for $\psi(x)$, that (29) is unaltered when x is replaced by $x/4^m$.

Because of the symmetries of δ revealed by the above lemmas, we need sketch the graph of δ only over the interval, say, $[1, \sqrt{2}]$ to see how δ behaves for all $x > 0$. By direct calculation, $\delta(x)$ apparently decreases monotonically to zero over the interval $[1, \sqrt{2}]$ from a maximum value of $\delta(1) \simeq 2.62 \cdot 10^{-6}$. Thus, for all $x > 0$, using the above lemmas and the computational evidence over $[1, \sqrt{2}]$, $\delta(x)$ oscillates between the values $\pm\delta(1)$. These calculations further suggest that, for all $x > 0$,

$$\delta(x) \simeq \delta(1)\cos\left(\frac{\pi \log x}{\log 2}\right). \tag{31}$$

In order to test these conjectures, we use (30) to express

$$\begin{aligned}\delta(x) = &\sum_{r=1}^{\infty}\left(\log\left(1 + \frac{2x}{4^r}\right) - \log\left(1 + \frac{x}{4^r}\right)\right)\\ &- \sum_{r=1}^{\infty}\left(\log\left(1 + \frac{1}{4^{r-1}x}\right) - \log\left(1 + \frac{2}{4^r x}\right)\right) - \frac{1}{2}\log x + \frac{1}{4}\log 2\\ = &\sum_{r=1}^{\infty}\left(\log\left(1 + \frac{2x}{4^r}\right) - \log\left(1 + \frac{x}{4^r}\right)\right)\\ &+ \sum_{r=1}^{\infty}\left(\log\left(1 + \frac{2}{4^r x}\right) - \log\left(1 + \frac{1}{4^r x}\right)\right) + \frac{1}{2}\log x - \log(1 + x) + \frac{1}{4}\log 2.\end{aligned}$$

We now replace each logarithm above by its Maclaurin series and rearrange the order of the summations to give

$$\delta(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot \frac{1}{2^n + 1}\left(x^n + \frac{1}{x^n}\right) + \frac{1}{2}\log x - \log(1 + x) + \frac{1}{4}\log 2, \tag{32}$$

where this latter representation for $\delta(x)$ is valid for $\frac{1}{2} \leqslant x \leqslant 2$. (There are no difficulties in justifying the rearrangement of the double series.) We note that, happily, the range of validity of (32) occupies precisely one cycle of the oscillatory function δ.

Encouraged by the approximation (31) we put $x = e^{-t}$ in (32) and construct the Fourier series for $\delta(e^{-t})$ on $[-\log 2, \log 2]$ of the form

$$\tfrac{1}{2}a_0 + \sum_{r=1}^{\infty}\left(a_r\cos(r\pi t/\log 2) + b_r\sin(r\pi t/\log 2)\right).$$

Since $\delta(e^{-t})$ is an even function of t, as is shown by Lemma 1 and readily confirmed by the representation (32), we see that each $b_r = 0$ and

$$a_r = \frac{2}{\log 2}\int_0^{\log 2}\delta(e^{-t})\cos(r\pi t/\log 2)\,dt. \tag{33}$$

Further, let us express the above integral as a sum of two integrals

$$\int_0^{\log 2} = \int_0^{\frac{1}{2}\log 2} + \int_{\frac{1}{2}\log 2}^{\log 2}$$

and make the substitution $t = \log 2 - \tau$ in the latter integral. Then, on using Lemma 3, we deduce that $a_r = 0$ if r is even.

To pursue (33) for r odd, we need to evaluate several integrals. First we obtain

$$\int_0^{\log 2} e^{nt}\cos(r\pi t/\log 2)\,dt = -\frac{1}{n}(2^n + 1)\Big/\left[1 + \left(\frac{r\pi}{n\log 2}\right)^2\right], \tag{34}$$

for r odd, on integrating by parts twice. Second we derive

$$\int_0^{\log 2} t\cos(r\pi t/\log 2)\,dt = -2\left(\frac{\log 2}{r\pi}\right)^2,$$

for r odd. We also need to evaluate

$$\int_0^{\log 2}\log(1 + e^{-t})\cos(r\pi t/\log 2)\,dt$$

which we do by expressing $\log(1 + e^{-t})$ in powers of e^{-t} and using (34) for $n = -1, -2, \ldots.$

Thus we derive from (32) and (33) the Fourier coefficients

$$a_r = \frac{2}{\log 2}\left[\left(\frac{\log 2}{r\pi}\right)^2 - 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2 + (r\pi/\log 2)^2}\right] \tag{35}$$

for r odd and $a_r = 0$ for r even. The latter series may be summed by using a standard contour integration technique. We have

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2 + a^2} = \frac{1}{2a^2} + \frac{\pi}{2a}\operatorname{csch}\pi a.$$

(See, for example, Whittaker and Watson [7, Example 5 of p. 136].) Thus (35) simplifies greatly to give

$$a_r = \frac{2}{r}\operatorname{csch}\left(\frac{\pi^2 r}{\log 2}\right). \tag{36}$$

It is easily verified that this Fourier series converges to δ for all $x > 0$, and we may write

$$\delta(x) = 2\sum_{r=1}^{\infty}\frac{1}{2r-1}\operatorname{csch}\left(\frac{\pi^2(2r-1)}{\log 2}\right)\cos\left[\frac{(2r-1)\pi\log x}{\log 2}\right]. \tag{37}$$

We note that the coefficients a_r, given by (36), tend to zero very rapidly indeed. The first few values are approximately

$$a_1 = 2.62\cdot 10^{-6},\quad a_3 = 3.74\cdot 10^{-19},\quad a_5 = 9.64\cdot 10^{-32}.$$

This shows that the approximation to $\delta(x)$ conjectured in (31) is extremely good, the maximum error being of order 10^{-19}.

7. The Limit for Large x. Having investigated the function δ, we return to (30) and write

$$\psi(x) = \tfrac{1}{2}\log x - \tfrac{1}{4}\log 2 + \delta(x) + \psi(2/x), \tag{38}$$

so that

$$F(x) = 2^{-1/4}x^{1/2}e^{\delta(x)}F(2/x).$$

If $x > \frac{1}{2}$, we may use (26) to express $F(2/x)$ as a power series in $1/x$ and thus obtain

$$F(x) = 2^{-1/4}x^{1/2}e^{\delta(x)}\left(1 + \frac{2}{3x} - \frac{8}{45x^2} + \frac{32}{405x^3} - \cdots\right), \tag{39}$$

valid for $x > \frac{1}{2}$, where $\delta(x)$ is given by (37).

Having now attained our goal of obtaining an expression for $F(x)$ for large x, we remark on the subtle role played by the function δ. There is one very simple relation involving F which we did not use in the foregoing analysis. This is

$$F(x)\cdot F(2x) = 1 + x, \tag{40}$$

which follows immediately from (22).

Before discerning the involvement of the function δ, we falsely conjectured from (40) that, for large x, $F(x)$ had the form of (39) with the factor $\exp(\delta(x))$ missing. It is amusing to see that this conjecture is consistent with (40), due to the fact (Lemma 3) that

$$e^{\delta(x)}\cdot e^{\delta(2x)} = 1.$$

Finally we draw a comparison between the arithmetic-harmonic mean process (19) and the superficially similar process

$$a_{n+1} = \tfrac{1}{2}(a_n + b_n), \tag{41a}$$

$$1/b_{n+1} = \tfrac{1}{2}(1/a_n + 1/b_n). \tag{41b}$$

It is well known and readily verified that $a_n b_n$ is invariant and that (41) is the Newton square root process

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right),$$

where $a = a_0 b_0$ and (a_n) converges quadratically to $\sqrt{a}$. (See Carlson [1].) Thus the processes (19) and (41) both involve the square root function in their respective limits.

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