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Some Inequalities for Elementary Mean Values

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Abstract. Upper and lower bounds for the difference between the arithmetic and harmonic means of n positive numbers are obtained in terms of n and the largest and smallest of the numbers. Also, results of S. H. Tung [2], are used to obtain upper and lower bounds for the elementary mean values M_p of Hardy, Littlewood, and Pólya.

1. In 1975, S. H. Tung proved the following theorem [2]:

Let $0 < b = x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n = B$. Let A and G be the arithmetic and geometric means, respectively, of $x_1, \ldots, x_n$. Then

$$n^{-1}(B^{1/2} - b^{1/2})^2 \leqslant A - G \leqslant g(b, B),$$

where $g(b, B) = cb + (1 - c)B - b^c B^{1-c}$, and

$$c = \frac{\log[(b/B - b)\log B/b]}{\log B/b}.$$

We will derive somewhat similar bounds for the difference between the arithmetic and the harmonic means of n positive numbers.

2. In [1, Chapter 2] Hardy, Littlewood, and Pólya discussed the elementary mean values, which are defined as follows:

Let $x_1, x_2, \ldots, x_n$ be positive numbers, and let p be a real number. Then $M_p(x_1, \ldots, x_n)$ is defined as $[n^{-1}\sum_{k=1}^n x_k^p]^{1/p}$, if $p \neq 0$; $M_0(x_1, \ldots, x_n)$ is defined as $(\prod_{k=1}^n x_k)^{1/n}$. We denote M_1, the arithmetic mean, by A; M_0, the geometric mean, by G; and M_{-1}, the harmonic mean, by H. Since $M_p(kx_1, \ldots, kx_n) = kM_p(x_1, \ldots, x_n)$ for all p and for all $k > 0$, we may, without loss of generality, assume $x_1 = 1$.

THEOREM 1. *Let* $1 = x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n = B$. *Then*

$$\frac{(B-1)^2}{n(B+1)} \leqslant A(1, \ldots, B) - H(1, \ldots, B) \leqslant (B^{1/2} - 1)^2.$$

Proof. For each k, $2 \leqslant k \leqslant n$, let

$$A_k = A(x_1, x_2, \ldots, x_{k-1}, x_n) \quad \text{and} \quad H_k = H(x_1, x_2, \ldots, x_{k-1}, x_n).$$

Fix $x_1, x_2, \ldots, x_{n-2}, x_n$, and let $x_{n-1} = x$ vary in $[1, B]$. Let

$$D(x) = A_n - H_n = \frac{(n-1)A_{n-1} + x}{n} - \frac{nxH_{n-1}}{(n-1)x + H_{n-1}}.$$

Computation of $D'(x)$ shows that $x = H_{n-1}$ is its only positive zero, and standard methods of analysis show that a minimum for $D(x)$ is attained at $x = H_{n-1}$.

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Therefore,

$$A_n - H_n \geqslant D(H_{n-1}) = n^{-1}(n-1)(A_{n-1} - H_{n-1}).$$

This process may be repeated, giving

$$A_n - H_n \geqslant \frac{n-1}{n}(A_{n-1} - H_{n-1}) \geqslant \frac{n-2}{n}(A_{n-2} - H_{n-2})$$

$$\geqslant \cdots \geqslant \frac{2}{n}(A_2 - H_2) = \frac{(B-1)^2}{n(B+1)}.$$

The maximum of $D(x)$ must occur at an endpoint, 1 or B, as each of the variables $x_2, x_3, \ldots, x_{n-1}$ in turn varies from 1 to B. So

$$A_n - H_n \leqslant \frac{nB - (B-1)k}{n} - \frac{nB}{(B-1)k + n} = F(k),$$

for some k, $0 \leqslant k \leqslant n$. The maximum of $F(x)$ on $[0, n]$ will, then, be an upper bound for $A_n - H_n$. Again, computation of $F'(x)$ and standard methods of analysis show that a maximum is attained for $x = n(B^{1/2} - 1)/(B - 1)$. Hence

$$A_n - H_n \leqslant F\{n(B^{1/2} - 1)/(B - 1)\} = (B^{1/2} - 1)^2.$$

This completes the proof of Theorem 1.

Upper and lower bounds for $G_n - H_n$ may be obtained using Theorem 1 and Tung's Theorem, since

$$G_n - H_n = (A_n - H_n) - (A_n - G_n).$$

3. Tung's Theorem may be used to obtain upper and lower bounds for the elementary mean values M_p, by using the relation

$$M_p(x_1, \ldots, x_n) = \{A(x_1^p, \ldots, x_n^p)\}^{1/p}.$$

(See [1].)

THEOREM 2. *Let* $1 = x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n = B$, *and let* $p > 0$. *Then*

$$\left[n^{-1}(B^{p/2} - 1)^2 + G^p\right]^{1/p} \leqslant M_p(1, \ldots, B) \leqslant \left[g(1, B^p) + G^p\right]^{1/p}$$

where $G = G(1, \ldots, B)$, *and* g *is the function defined in Tung's Theorem.*

THEOREM 3. *Let* $1 = x_1 \leqslant x_2 \leqslant \cdots x_n = B$, *and let* $p < 0$. *Then*

$$\left[g(B^p, 1) + G^p\right]^{1/p} \leqslant M_p(1, \ldots, B) \leqslant \left[n^{-1}(1 - B^{p/2})^2 + G^p\right]^{1/p}$$

where $G = G(1, \ldots, B)$ *and* g *is the function of Tung's Theorem.*

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1. G. H. HARDY, J. E. LITTLEWOOD & G. PÓLYA, *Inequalities*, Cambridge Univ. Press, Cambridge, 1952.

2. S. H. TUNG, "On lower and upper bounds of the difference between the arithmetic and the geometric mean," *Math. Comp.*, v. 29, 1975, pp. 834–836