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Odd Triperfect Numbers

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Abstract. We prove that an odd triperfect number has at least ten distinct prime factors.

1. A positive number N is called a triperfect number if $\sigma(N) = 3N$, where $\sigma(N)$ is the sum of the positive divisors of N. Six even triperfect numbers are known:

$$2^{14} \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151,$$
$$2^{13} \cdot 3 \cdot 11 \cdot 43 \cdot 127,$$
$$2^{9} \cdot 3 \cdot 11 \cdot 31,$$
$$2^{8} \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73,$$
$$2^{5} \cdot 3 \cdot 7,$$
$$2^{3} \cdot 3 \cdot 5.$$

However, the existence of an odd triperfect (OT) number is an open question. McDaniel [1] and Cohen [2] proved that an OT number has at least nine distinct prime factors, and Beck and Najar [3] showed that it exceeds 10^{50}.

In this paper using the technique of [4], we prove

THEOREM. *If N is OT, N has at least ten distinct prime factors.*

2. Throughout this paper we let

$$N = \prod_{i=1}^{r} p_i^{a_i},$$

where p_i's are odd primes, $p_1 < \cdots < p_r$ and a_i's are positive integers.

The following lemmas are easy to prove:

LEMMA 1. *If N is OT,*

$$a_i\text{'s are even for } 1 \leqslant i \leqslant r. \tag{1}$$

LEMMA 2. *If N is OT and q is a prime factor of $\sigma(p_i^{a_i})$ for some i, then $q = 3$ or $q = p_j$ for some j, $1 \leqslant j \leqslant r$.*

LEMMA 3. *If N is OT and $r = 9$, $p_8 < 80$.*

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As in [4] we define

$$S(N) = \sigma(N)/N,$$
$$a(p) = \operatorname{Min}\{a \mid a \text{ is even and } p^{a+1} > 10^{11}\},$$
$$b_i = \operatorname{Min}\{a_i, a(p_i)\},$$
$$M = \prod_{i=1}^{r} p_i^{b_i}.$$

LEMMA 4. *If N is OT, then*

$$\log 3 - r \cdot 10^{-11} < \log S(M) \leqslant \log 3. \tag{2}$$

Proof. Since $M \mid N$, $S(M) \leqslant S(N) = 3$ and so $\log S(M) \leqslant \log 3$. In [4] we proved that if $a > a(p)$, then

$$0 < \log S(p^a) - \log S(p^{a(p)}) < 10^{-11}.$$

Hence

$$0 < \log S(N) - \log S(M) < r \cdot 10^{-11},$$

and we have

$$\log 3 - r \cdot 10^{-11} = \log S(N) - r \cdot 10^{-11} < \log S(M).$$

Q.E.D.

COROLLARY. *If N is OT*, $L = M/p_r^{b_r}$ *and if* $p_r > 3500$, *then*

$$\log 3 - r \cdot 10^{-11} - \log S(3499^2) < \log S(L) < \log 3. \tag{3}$$

Proof of Theorem. We used a computer (PDP 11 at the University of Toledo) to find

$$M = \prod_{i=1}^{9} p_i^{a_i}$$

satisfying (1), $a_i \leqslant a(p_i)$ for $1 \leqslant i \leqslant 9$, (2) with $r = 9$, and $p_9 < 3500$. There were 71 such M's; however, all of them had a factor $p_i^{a_i}$ such that $a_i < a(p_i)$, $\sigma(p_i^{a_i})$ had a prime factor $q > 3$, and $q \neq p_j$, $1 \leqslant j \leqslant 9$.

Next we tried to find

$$L = \prod_{i=1}^{8} p_i^{a_i}$$

satisfying (1), $a_i \leqslant a(p_i)$ for $1 \leqslant i \leqslant 8$, and (3) with $r = 9$. There were 12689 such L's; however, 12473 of them had a factor $p_i^{a_i}$ such that $a_i < a(p_i)$, $\sigma(p_i^{a_i})$ had a prime factor $q > 3$, $q \neq p_j$ for $1 \leqslant j \leqslant 8$, and

$$\log S(L) + \log S(q^2) > \log 3.$$

The remaining 216 of them had the following properties: there exist two consecutive primes u and v such that $3500 < u < v$,

$$\log S(L) + \log S(u^2) > \log 3, \quad \text{and}$$
$$\log S(L) + \log v/(v-1) < \log 3 - 9 \cdot 10^{-11}.$$

These three cases show that if N is an odd integer with $r = 9$, then N cannot be OT. Q.E.D.

The computer time was over five hours.

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1. W. McDaniel, "On odd multiply perfect numbers," *Boll. Un. Mat. Ital.*, No. 2, 1970, pp. 185–190.

2. G. L. Cohen, "On odd perfect numbers II, Multiperfect numbers and quasiperfect numbers," *J. Austral. Math. Soc.*, v. 29, 1980, pp. 369–384.

3. W. E. Beck & R. M. Najar, "A lower bound for odd triperfects," *Math. Comp.*, v. 38, 1982, pp. 249–251.

4. M. Kishore, "Odd integers N with five distinct prime factors for which $2 - 10^{-12} < \sigma(N)/N < 2 + 10^{-12}$," *Math. Comp.*, v. 32, 1978, pp. 303–309.