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On Zeros of Mellin Transforms of $SL_2(\mathbf{Z})$ Cusp Forms

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Dedicated to Derrick H. and Emma Lehmer

Abstract. We compute zeros of Mellin transforms of modular cusp forms for $SL_2(\mathbf{Z})$. Such Mellin transforms are eigenforms of Hecke operators. We recall that, for all weights k and all dimensions of cusp forms, the Mellin transforms of cusp forms have infinitely many zeros of the form $k/2 + t\sqrt{-1}$, i.e., infinitely many zeros on the critical line.

A new basis theorem for the space of cusp forms is given which, together with the Selberg trace formula, renders practicable the explicit computations of the algebraic Fourier coefficients of cusp eigenforms required for the computations of the zeros.

The first forty of these Mellin transforms corresponding to cusp eigenforms of weight $k \leqslant 50$ and dimension $\leqslant 4$ are computed for the sections of the critical strips, $\sigma + t\sqrt{-1}$, $k - 1 < 2\sigma < k + 1$, $-40 \leqslant t \leqslant 40$. The first few zeros lie on the respective critical lines $k/2 + t\sqrt{-1}$ and are simple. A measure argument, depending upon the Riemann hypothesis for finite fields, is given which shows that Hasse-Weil L-functions (including the above) lie among Dirichlet series which do satisfy Riemann hypotheses (but which need not have functional equations nor analytic continuations).

1. Introduction. Zeros are computed explicitly for an important family of examples associated with Hecke operators for the modular group $SL_2(\mathbf{Z})$. The modular cusp forms have specially multiplicative Fourier coefficients which we compute from the Selberg trace formula and from a new practical basis theorem for the cusp forms. Forty distinct L-functions were computed for k an even integer, $12 \leqslant k \leqslant 50$, dimensions 0 to 4. In each of these forty and thirteen other cases, the rectangle $\sigma + t\sqrt{-1}$, $k - 1 \leqslant 2\sigma \leqslant k + 1$, $-40 \leqslant t \leqslant 40$, was examined for all zeros. Rankin [31] proved along with a functional equation that there are no zeros in the respective half planes $2\sigma \leqslant k - 1$, $2\sigma \geqslant k + 1$, so that it is sufficient to restrict attention to the critical strip $k - 1 < 2\sigma < k + 1$. All zeros in the rectangles we have investigated, k fixed, lie on the line $2\sigma = k$ and are simple.

We show that there exists an infinitude of zeros on the critical line $\mathrm{Re}(s) = k/2$ of Mellin transforms of cusp forms of weight k for $SL_2(\mathbf{Z})$. We conclude that this is also true for Hasse-Weil L-functions of Kuga varieties.

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2. $SL_2(\mathbf{Z})$ L-Functions. Rankin's book [32] contains background on modular forms, as does that of Serre [37] and Lang [19]. We establish enough notation to attach a Dirichlet series to a modular form. The discrete matrix group $SL_2(\mathbf{Z})$ acts on the complex upper half-plane via fractional linear transformations. A modular form of weight an even integer k is a function on the upper half-plane with invariance properties with respect to the $SL_2(\mathbf{Z})$ action. A cusp form is a modular form which will vanish at the cusp and have Fourier coefficients $a(n)$. Let $f(u) = \sum_{n \geq 1} a(n) e^{-2\pi n u}$ and satisfy the functional equation $f(1/u) = (-1)^{k/2} u^k f(u)$. The dimension of the space of cusp forms C_k of weight $k > 2$ for $SL_2(\mathbf{Z})$ is

$$\dim C_k = \frac{(-1)^{k/2}}{4} + \frac{k-1}{12} - \frac{1}{2} - \frac{2\sin(k-1)\pi/3}{3\sqrt{3}}.$$

The dimensions and weights relevant to our computational work appear in Table I. Hecke [11] introduced a commutative algebra of operators Hermitian with respect to the Peterson inner product on cusp forms. Thus any $SL_2(\mathbf{Z})$ cusp form can be written as a linear combination of eigenforms of these Hecke operators, and the Fourier coefficients of a cusp form can be had from those of the cusp eigenfunctions. That a cusp eigenform has specially multiplicative Fourier coefficients implies that its Mellin transform has an Euler product. Take the Mellin transform of the cusp form to get the Dirichlet series

$$L_k(s) = \int_0^\infty t^s f(t)\, d\log t = \Gamma(s) \sum_{n \geq 1} \frac{a(n)}{(2\pi n)^s}$$
$$= \frac{\Gamma(s)}{(2\pi)^s} \prod_p 1/H_k(p^{-s}, f, p).$$

TABLE I

$\dim C_k$	associated weights k
0	2, 4, 6, 8, 10, 14,
1	12, 16, 18, 20, 22, 26,
2	24, 28, 30, 32, 34, 38,
3	36, 40, 42, 44, 46, 50,
4	48.

The Hecke polynomial $H_k(T, f, p) = 1 - a(p)T + p^{k-1}T^2$ divides the $k-1$st Betti polynomial of a Kuga variety; cf., Ihara [14], Kuga and Shimura [17], Deligne [5]. A consequence of the Riemann hypotheses for finite fields is the inequality $|a(n)| < d(n) n^{(k-1)/2}$, $d(n) =$ number of divisors of n; this is the Ramanujan-Petersson conjecture [30]. The Dirichlet series $L_k(s)$ was studied extensively in Rankin [31], Goldstein [9] has proven Merten-type conditions for $L_k(s)$, Moreno [24], [25] has proven von Mangoldt formulas for $L_k(s)$ in a more general setting, see e.g., Langlands [20]. From the functional equation for the cusp form f we have the following; cf., Barrucand [2],

$$L_k(s) = L_k(s, f) = \sum_{n \geq 1} a(n) \left\{ \frac{\Gamma(s, 2\pi n)}{(2\pi n)^s} + (-1)^{k/2} \frac{\Gamma(k-s, 2\pi n)}{(2\pi n)^{k-s}} \right\},$$

where

$$\Gamma(s, M) = \int_M^\infty u^s e^{-u}\, d\log u$$

is the usual incomplete gamma function.

Symmetry up to sign under the map $s \to k - s$ of the right-hand side of the above equality gives the functional equation

$$L_k(k - s) = (-1)^{k/2} L_k(s).$$

In particular, if k is not divisible by four, $s = k/2$ is a root of $L_k(s)$. Also note that $L_k(k/2 + t\sqrt{-1})$ is either totally real or purely imaginary depending upon the parity of $k/2$.

The incomplete gamma function $\Gamma(s, M)$ can be evaluated for complex s by continued fractions, Abramowitz, Stegun, Davis [1], Terras [43], [44] or more efficiently by a Laguerre expansion, Henrici [12, pp. 628–630].

To compute $L_k(s)$ one must estimate the tail, restricting our attention to the strip $k - 2 < 2\,\mathrm{Re}(s) < k + 2$. Keeping the above notations, we have

$$\left| \sum_{n > N} a(n) \int_1^\infty \left(u^s + (-1)^{k/2} u^{k-s}\right) e^{-2\pi n u}\, d\log u \right| < \frac{4}{\pi} N^{(k+1)/2} e^{-2\pi N},$$

for $|\mathrm{Re}(s) - k/2| \leqslant 1$, $N > (k + 1)/2\pi$. In the proof of this inequality the above has been used as well as the fact, Olver [27, p. 70], that if $M > 2\,\mathrm{Re}(s)$, $|\Gamma(s, M)| \leqslant 2M^{\mathrm{Re}(s)-1}e^{-M}$. Note that this series, geometric in $e^{2\pi} > 535$, is fairly rapidly convergent. The difficulty in computing these Dirichlet series in this way is that a good deal of cancellation takes place among the terms and so it is important to carry precision adequate for Im( ). Recall that the first term is essentially $e^{-2\pi}$, but that (by the Riemann-Lebesgue lemma) the function becomes small as $\mathrm{Im}(s)$ increases.

Explicit computation of the Fourier coefficients of the cusp eigenfunctions is required to compute $L_k(s)$. In general, these coefficients are algebraic numbers, Rankin-Rushforth [33]; however, the traces of the Hecke operators themselves are integers. It will be possible, using the basis theorem below, to calculate the algebraic numbers from the traces. Formulas for the traces have been known for some time, Selberg [35], Eichler [8], Duflo-Labesse [7], Lang [19], Zagier [52]. As can be seen from the Selberg trace formula, the precision size for the computation of $a(n)$ is about $n^{k/2}$, almost as efficient as the Ramanujan-Petersson inequality allows. We calculated the first thousand of these traces for k from 12 to 26. Many require multiple precision to compute, but those required for the computation of the L-functions are small enough to fit in double precision. For $k = 12$ our integers agree with the $\tau(n)$ of Ramanujan [30], Watson [49] and Lehmer [21]. For $k > 14$ these coefficients have not been computed extensively. Ours agree with the three offered by Lehmer [22] for $k = 24$.

If for each p there are $\dim C_k$ distinct relatively prime Hecke polynomials, then the product of all of them will divide the k − 1st Betti polynomial. The associated Dirichlet series $L_k^{(\delta)}(s)$, $1 \leqslant \delta \leqslant \dim C_k$, are parametrized by roots of a degree $\dim C_k$ polynomial equation in one variable with integer coefficients. Setting

$$L_k(s) = \prod_{1 \leqslant \delta \leqslant \dim C_k} L_k^{(\delta)}(s),$$

yields the functional equation

$$L_k(s) = (-1)^{k \dim C_k/2} L_k(k-s)$$

and the roots of this correspond to the union of roots of the individual $L_k^{(\delta)}(s)$, $1 \leqslant \delta \leqslant \dim C_k$. It is of interest to know if any of these roots of the union are duplicated, i.e., if the roots of $L_k(s)$ are simple or not. In the computations done for each dimension so far there are no duplications, $\mathrm{Im}(s) \neq 0$, suggesting that the roots of the "big" Hecke factor $L_k(s)$ are simple and lie on the critical line.

We pause to point out that another example of a zeta function defined by Hecke operators, Hecke [11], is $\zeta(s)\zeta(s-k+1)$, the Mellin transform of an Eisenstein series, a modular form which is not a cusp form. $\zeta(s)$ is the classical zeta function and the lines $\mathrm{Re}(s) = \frac{1}{2}$ and $\mathrm{Re}(s) = k - \frac{1}{2}$ have been extensively investigated, Brent [4].

3. A Basis Theorem for Cusp Forms and Computation of the Algebraic Numbers. We emphasize that it is not enough for our purposes to give characteristic polynomials for the Hecke operators, $T_k(n)$, or even the eigenvalues of $T_k(n)$ in some order, possibly different for each n; cf, Wada [46]. We require explicit and computable bases for the space of cusp forms C_k. When $\dim C_k = 0$, $k = 2, 4, 6, 8, 10, 14$, all the coefficients excepting the first are zero and $L_k(s) \equiv 1$ (so that the Riemann hypothesis in this case is trivially true). When $\dim C_k = 1$, then the trace is the coefficient. In general, for $\dim C_k \geqslant 1$, we define cusp forms

$$\Delta_k(u) = \sum_{n \geqslant 1} a_k(n) e^{-2\pi n u}$$

by setting

$$a_k(n) = \frac{\operatorname{trace} T_k(n)}{\dim C_k},$$

so that $a_k(1) = 1$. The Δ_k are cusp forms of weight k and they are sums of cusp eigenforms. For $\dim C_k = 1$ they are eigenforms, for example

$$\Delta_{12}(u) = \Delta(u) = e^{-2\pi u} \prod_{m \geqslant 1} (1 - e^{-2\pi m u})^{24} = \sum_{n \geqslant 1} \tau(n) e^{-2\pi n u}$$

the 24th power of the Dedekind η-function, Ramanujan [30], Gunning [10].

For $1 \leqslant \delta \leqslant \dim C_k$ let each cusp eigenform of the commuting family of Hecke operators, $T_k(n)\colon C_k \to C_k$, be denoted by

$$F_k^{(\delta)}(u) = \sum_{n \geqslant 1} \lambda_k^{(\delta)}(n) e^{-2\pi n u},$$

where the $\lambda_k^{(\delta)}(n)$ are the Fourier coefficients we seek for the L-function

$$L_k(s, \delta) = \Gamma(s) \sum_{n \geqslant 1} \frac{\lambda_k^{(\delta)}(n)}{(2\pi n)^s}.$$

Then by arguments similar to those for the proof of the basis Theorem 6.1.2 of Rankin [31, p. 198] we have

$$F_k^{(\delta)} = \Delta_k + \sum_{P \in \mathfrak{P}} x_{P,\delta} \prod_{l \in P} \Delta_l,$$

where $\mathcal{P}$ is a finite class of $\dim C_k - 1$ partitions of k into exactly $2, 3, \ldots, \dim C_k$ parts. The $x_{p,\delta}$ are algebraic numbers which can be determined from the special multiplicativity properties of the coefficients of $F_k^{(\delta)}$.

The advantage of this particular basis theorem is for numerical computation with exact integers. The magnitude of integers involved in computing the integers, trace $T_k(n)$, from the Selberg trace formula is $O(n^{k/2})$, whereas the integer magnitude required to compute just the coefficient of the basis functions for Rankin's, Hecke's or Ramunujan's versions could be as high as $O(n^{k+k^2/24})$, clearly impractical even for the range in which we have done the computation $12 \leqslant k \leqslant 50$.

The particular partitions for $\dim C_k > 1$ employed in the present work are listed in Table II.

TABLE II

Particular choices of partitions used in the present computation for applying the basis theorem of cusp forms. For example, $F_{42}^{(\delta)} = \Delta_{42} + x\Delta_{20}\Delta_{22} + y\Delta_{12}^2\Delta_{18}$. *When* $\dim C_k = 1$, $F_k = \Delta_k$.

$\dim C_k$	Partitions
2	$24 = 12 + 12$
2	$28 = 12 + 16$
2	$30 = 12 + 18$
2	$32 = 16 + 16$
2	$34 = 16 + 18$
2	$38 = 18 + 20$
3	$36 = 18 + 18 = 12 + 12 + 12$
3	$40 = 20 + 20 = 12 + 12 + 16$
3	$42 = 20 + 22 = 12 + 12 + 18$
3	$44 = 22 + 22 = 12 + 12 + 20$
3	$46 = 22 + 24 = 12 + 16 + 18$
3	$50 = 24 + 26 = 16 + 16 + 18$
4	$48 = 24 + 24 = 16 + 16 + 16 = 12 + 12 + 12 + 12$
...	
$\dim C_k$	$n = k_{11} = k_{21} + k_{22} = k_{31} + k_{32} + k_{33} = \cdots = k_{j1} + \cdots + k_{jj}$, $1 \leqslant j \leqslant \dim C_k$, k_{ij} are positive integers, none equal to weights l such that $\dim C_l = 0$, i.e., $k_{ij} \geqslant 12$, $k_{ij} \neq 14$.

The problem of finding the algebraic numbers $x_{p,\delta}$ is touched on for $k = 24$ in Hecke [11], sketched briefly for $\dim C_k = 2$ and $k = 36$ in Lehmer [22] and, with a complicating basis, dealt with for $\dim C_k = 2$ in Rankin [32]. It turns out that the particular bases chosen here lead to some coincidental simplifications in the size of the $x_{p,\delta}$ and the size of the coefficients of the polynomial equations they satisfy. For convenience set $t = e^{-2\pi u}$,

$$\Delta_k(u) = \sum_{n \geqslant 1} a_k(n) t^n,$$

$$\Delta_{k_1}(u)\Delta_{k_2}(u) = \sum_{n \geqslant 1} b_k(n) t^n, \qquad k = k_1 + k_2,$$

$$\Delta_{k_1}(u)\Delta_{k_2}(n)\Delta_{k_3}(u) = \sum_{n \geqslant 1} c_k(n) t^n, \qquad k = k_1 + k_2 + k_3,$$

$$\Delta_{k_1}(u)\Delta_{k_2}(u)\Delta_{k_3}(u)\Delta_{k_4}(u) = \sum_{n \geqslant 1} d_k(n) t^n, \qquad k = k_1 + k_2 + k_3 + k_4.$$

Then for the partitions in the above table, trace $T_k(2) = 3a_k(2) = c_k(8)/c_k(4)$ for $\dim C_k = 3$ and trace $T_k(2) = b_k(4)$ for $\dim C_k = 2$.

Consider a cusp eigenform F_k with coefficients $\lambda_k(n)$. From the special multiplicative property for eigenforms of Hecke operators and any prime p we have the relations

$$\lambda_k(p^2) = \lambda_k(p)^2 - p^{k-1},$$
$$\lambda_k(p^3) = \lambda_k(p)^3 - 2p^{k-1}\lambda_k(p),$$
$$\lambda_k(p^4) = \lambda_k(p)^4 - 3p^{k-1}\lambda_k(p)^2 + p^{2k-2}.$$

There are three cases for the present dimensions.

Case 1, $\dim C_k = 2$. Let $\lambda_k(n) = a_k(n) + xb_k(n)$. Then $a_k(4) + b_k(4) = (x + a_k(2))^2 - 2^{k-1}$. Since $2a_k(2) = b_k(4)$ as noted above, the coefficient of x in this quadratic equation is zero, and $x^2 = 2^{k-1} + a_k(4) - a_k^2(a)$, $\dim C_k = 2$.Thus,

$$F_{24}^{(\delta)} = \Delta_{24} + (-1)^{\delta}\Delta_{12} \cdot 2^2 \cdot 3\sqrt{144169}\,,$$
$$F_{28}^{(\delta)} = \Delta_{28} + (-1)^{\delta}\Delta_{12}\Delta_{16} \cdot 2^2 \cdot 3^3\sqrt{131 \cdot 139}\,,$$
$$F_{30}^{(\delta)} = \Delta_{30} + (-1)^{\delta}\Delta_{12}\Delta_{18} \cdot 2^5 \cdot 3\sqrt{51349}\,,$$
$$F_{32}^{(\delta)} = \Delta_{32} + (-1)^{\delta}\Delta_{16}^2 \cdot 2^2 \cdot 3\sqrt{67 \cdot 273067}\,,$$
$$F_{34}^{(\delta)} = \Delta_{34} + (-1)^{\delta}\Delta_{16}\Delta_{18} \cdot 2^3 \cdot 3^2\sqrt{479 \cdot 4919}\,,$$
$$F_{38}^{(\delta)} = \Delta_{38} + (-1)^{\delta}\Delta_{18}\Delta_{20} \cdot 2^4 \cdot 3\sqrt{181 \cdot 349 \cdot 1009}\,.$$

We see that for $\dim C_k = 2$ the Fourier coefficients of cusp eigenforms are quadratic integers. These discriminants agree, $k = 24$ with Hecke [11] and Rankin [32], all other $k \neq 32$ in this list with Lehmer [22] where for $k = 32$, 18295849 is given (evidently a misprint) instead of $18295489 = 67 \cdot 273067$. The algebraic number coefficients can be easily read off from the tables of coefficients of Δ_k and $\Delta_{k_1}\Delta_{k_2}$.

Case 2, $\dim C_k = 3$. Let $\lambda_k(n) = a_k(n) + xb_k(n) + yc_k(n)$, $n \geqslant 1$. Again, making use of the algebraic relations among $\lambda_k(2)$, $\lambda_k(4)$, $\lambda_k(8)$, eliminating y and the fact that $3a_k(2) = c_k(8)/c_k(4)$ (which makes the coefficient of x^2 zero) and omitting the details, we see that x must satisfy the cubic equation

$$\begin{aligned} x^3 &+ \left(3a_k(2)b_k(4) - b_k(8) - 3(a_k(2))^2 - 2^k\right)x \\ &\qquad + 3a_k(2)a_k(4) - a_k(8) - 2(a_k(2))^3 + 2^{k-1}a_k(2) \\ &= x^3 + A_1x + A_0 = 0. \end{aligned}$$

Since the Hecke operators are Hermitian this equation has three real roots and negative discriminant. There is a cusp eigenform corresponding to each root x and corresponding value of y such that

$$yc_k(4) = x^2 + (2a_k(2) - b_k(4))x + \left((a_k(2))^2 - 2^{k-1} - a_k(4)\right).$$

The prime factors of cubic discriminant for $k = 36$ agree with the prime factors found in the discriminant of Lehmer [22] for his more complicated cubic. The coefficients and discriminant for $\dim C_k = 3$ are in Table III.

TABLE III

The cubic equation $x^3 + A_1x + A_0 = 0$ for $\dim C_k = 3$ cusp eigenforms of Hecke operators for $SL_2(\mathbf{Z})$. The associated coefficients x, y for a given dimension can be distinguished by their signs.

k	A_1	A_0	discriminant $= A_1^3/27 + A_0^2/4$
36	$-2^6 \cdot 3 \cdot 719 \cdot 475991$	$-2^{10} \cdot 13 \cdot 17 \cdot 367 \cdot$ $\cdot 4133 \cdot 12893$	$-2^{28} \cdot 3^7 \cdot 5^2 \cdot 7^2 \cdot 23 \cdot 1259 \cdot$ $\cdot 269461929553$
40	$-2^6 \cdot 3^4 \cdot 137 \cdot 1281971$	$+2^{10} \cdot 3^7 \cdot 11 \cdot 17 \cdot$ $\cdot 126781843$	$-2^{24} \cdot 3^9 \cdot 5^2 \cdot 7^2 \cdot 13^2 \cdot 73 \cdot 59077 \cdot$ $\cdot 92419245301$
42	$-2^8 \cdot 3 \cdot 37^3 \cdot 164893$	$+2^{13} \cdot 11 \cdot 127 \cdot$ $\cdot 18791834161$	$-2^{32} \cdot 3^7 \cdot 5^2 \cdot 7^2 \cdot$ $\cdot 1465869841 \cdot 578879197969$
44	$-2^6 \cdot 3 \cdot$ $\cdot 90059756926$	$-2^{10} \cdot 13 \cdot 191 \cdot$ $\cdot 300441338170$	$-2^{26} \cdot 3^7 \cdot 5^2 \cdot 7^2 \cdot 17^2 \cdot 37 \cdot$ $\cdot 92013596772457847677$
46	$-2^{12} \cdot 3^3 \cdot 8273 \cdot$ $\cdot 53987$	$+2^{19} \cdot 3^3 \cdot 13 \cdot$ $\cdot 137341 \cdot$ $\cdot 2947853$	$-2^{37} \cdot 3^{11} \cdot 5^2 \cdot 7^2 \cdot 227 \cdot$ $\cdot 454287770269681529$
50	$-2^8 \cdot 3 \cdot 19 \cdot$ $\cdot 52245512611$	$-2^{13} \cdot 11 \cdot 13 \cdot 31 \cdot$ $\cdot 26107 \cdot$ $\cdot 8443679383$	$-2^{30} \cdot 3^9 \cdot 5^4 \cdot 7^4 \cdot$ $\cdot 1228462869413174261940$1

Case 3, $\dim C_k = 4$. We will only compute the $k = 48$ case. In general

$$\lambda_k(n) = a_k(n) + xb_k(n) + yc_k(n) + zd_k(n),$$

where for $k = 48$ we have chosen coefficients a, b, c, d according to the appropriate equations above and the partitions of Table II corresponding to $\dim C_k = 4$, Δ_{48}, Δ_{24}^2, Δ_{16}^3, Δ_{12}^4, respectively. Including the algebraic relation for $\lambda_k(16)$ and the fact that, for $k = 48$, and probably for all of $\dim C_k = 4$,

$$\operatorname{trace} T_k(2) = \frac{c_k(16) - c_k(4)d_k(16)}{c_k(8) - c_k(4)d_k(8)}$$

(making the coefficient of x^3 zero), we see that each x satisfying the quartic equation below yields a value for y and z.

$$\begin{aligned}
x^4 &+ \left(4a_k(2)d_k(8) - d_k(16) - 6(a_k(2))^2 - 3 \cdot 2^{k-1}\right)x^2 \\
&\quad + \left(2^k a_k(2) - 8(a_k(2))^3 + 8(a_k(2))^2 d_k(8)\right. \\
&\qquad - 2a_k(2)d_k(16) + b_k(4)d_k(16) \\
&\qquad \left. - b_k(16) + 4a_k(2)b_k(8) - 4a_k(2)b_k(4)d_k(8)\right)x \\
&\quad + 5 \cdot 2^{k-1}(a_k(2))^2 - 3(a_k(2))^4 + 4(a_k(2))^3 d_k(8) \\
&\quad - 2^{k+1}a_k(2)d_k(8) - (a_k(2))^2 d_k(16) + 2^{k-1}d_k(16) - a_k(16) + a_k(4)d_k(16) \\
&\quad + 4a_k(2)a_k(8) - 4a_k(2)a_k(4)d_k(8) + 2^{2k-2} \\
&= x^4 + B_2x^2 + B_1x + B_0 = 0,
\end{aligned}$$

$$\begin{aligned}
y(c_k(8) &- c_k(4)d_k(8)) \\
&= x^3 + (3a_k(2) - d_k(8))x^2 \\
&\quad + \left(b_k(4)d_k(8) - b_k(8) + 3(a_k(2))^2 - 2a_k(2)d_k(8) - 2^k\right)x \\
&\quad + (a_k(2))^3 + a_k(4)d_k(8) - a_k(8) - 2^k a_k(2) - (a_k(2))^2 d_k(8) + 2^{k-1}d_k(8), \\
z &= x^2 + (2a_k(2) - b_k(4))x - yc_k(4) + (a_k(2))^2 - a_k(4) - 2^{k-1}.
\end{aligned}$$

For $k = 48$, we have summarized these relations explicitly in Table IV.

TABLE IV

Relations among the x, y, z algebraic number coefficients for $k = 48$, $\dim C_{48} = 4$. These four can be distinguished by their signs.

$$
\begin{aligned}
x^4 &- 479694638227032x^2 \\
&- 627209402058296734417848000x \\
&+ 34879678574083307331 16234450 = 0.
\end{aligned}
$$

$$
\begin{aligned}
&7115931648y \\
&\quad = 1441290x^2 - 453537008666100x \\
&\quad\quad - 3072271779027069512 40.
\end{aligned}
$$

$$
\begin{aligned}
z = {} & x^2 + 2261700x - 648y \\
& - 239847319112868.
\end{aligned}
$$

4. Computations and Tables of Coefficients and Zeros. The approximate locations of three zeros of L_{12} ($k = 12$) were found by Spira [40]. We checked his tables and found them to agree with ours, to the extent of his precision, although his method of computation was somewhat different. Other kinds of L-functions have been dealt with in Purdy, Terras and Williams [29] and Lagarias and Odlyzko [18].

For each L_k, k such that the dimensions of the spaces of cusp forms of weight for $SL_2(\mathbf{Z})$ be less than four and $k = 48$ for $\dim C_k = 4$, the winding number was computed for the boundary of the strip $|\mathrm{Re}(s) - k/2| \leqslant 1/2$, $|\mathrm{Im}(s)| \leqslant 40$ by taking small steps in real and/or imaginary parts. The functional equation of holomorphic $L_k(s)$ reduces this computation to essentially a fourth of the boundary. All zeros found were simple. A few zeros were computed for $\mathrm{Im}(s) > 40$. Since the functions $L_k(k/2 + t\sqrt{-1})$ are either real or purely imaginary all the zeros on this line $k/2 + t\sqrt{-1}$ were found by standard real techniques, a combination of Newton and secant methods.

The positional values of the positive zeros of eigenforms appear in Figure 1. The signs following the weights distinguish among the eigenforms of the same dimension and are the signs of the coefficients of the x's, x, y's or x, y, z's. For each k the position of t for a zero $k/2 + t\sqrt{-1}$ is given. For $\dim C_k < 4$ there are no other roots in the critical rectangle $k/2 \pm \varepsilon + t\sqrt{-1}$, $0 < \varepsilon \leqslant \frac{1}{2}$, $-40 \leqslant t \leqslant 40$, for $\dim C_k = 4$, $-30 \leqslant t \leqslant 30$. In the UMT deposit zeros have been listed with an accuracy beginning with twenty decimal places; in many cases zeros beyond $t = 40$ have been found.

In addition to zeros of the $L_k(s)$, zeros, number and simplicity were computed for Mellin transforms of the Δ_k, sums of L_k's. Of course, the gamma factors of the L_k, k fixed, are the same so the functional equations persist for Mellin transforms of cusp forms even though there may not be an Euler product.

Figure 2 contains positional values of t, $|t| \leqslant 30$ similar to those in Figure 1. The $\dim C_k = 1$ part of Figure 1, where $\Delta_k = L_k$, reappears in Figure 2 for comparison. Again, the zeros are simple and linearly arranged.

A comment on available precision is that as long as the coefficients of the series are integers $< 10^{28}$ precision is not limited by the coefficient accuracy, but for $\dim C_k > 1$ the coefficients are real numbers but not integers. A measure of their precision comes from the specially multiplicative properties of the coefficients. The precision of the zeros given in the UMT file is a reflection of these and similar roundoff considerations.

Preliminary factorizations of up to the first twenty Fourier coefficients arising for the dimensions considered here appear in the UMT file. For example, the nth coefficient of $\Delta_{k_1}\Delta_{k_2}$ is listed after the equal sign following (k_1, k_2, n). All prime factors less than or equal to 8009879 have been sieved out and are printed to the appropriate power. Any remaining integer not followed by a period is prime. If an integer is followed by a period, its factorization may be investigated by the techniques of [26] and [47], [48]. We thank Hugh C. Williams of the University of Manitoba for identifying factors and/or primality of most of these larger integers.

TABLE V

Zeros of $L_{12}(s) = \Gamma(s) \cdot \sum_{n \geqslant 1} \tau(n)/(2\pi n)^s$,

the Dirichlet series of Ramanujan-Rankin.

t values of roots $6 \pm t\sqrt{-1}$, $0 < t < 40$, truncated to twenty or the number of significant decimal digits available for the function value. Those below the line with $t > 40$ have not been counted nor proven simple. There are no other roots in the critical rectangle $6 \pm \varepsilon + t\sqrt{-1}$, $0 < \varepsilon \leqslant \frac{1}{2}$, $-40 \leqslant t \leqslant 40$.

	t
1	9.222379399 2110252224
2	13.9075498613 921344064
3	17.4427769782 34473313
4	19.6565131419 5496099
5	22.3361036372 09867
6	25.2746365481 12379
7	26.8043911583 5055
8	28.8316826241 89
9	31.1782094982 8
10	32.7748753809
11	35.196995715
12	36.74146106
13	37.7539078
14	40.219064
15	41.73076
16	43.5975
17	45.118
18	48.88

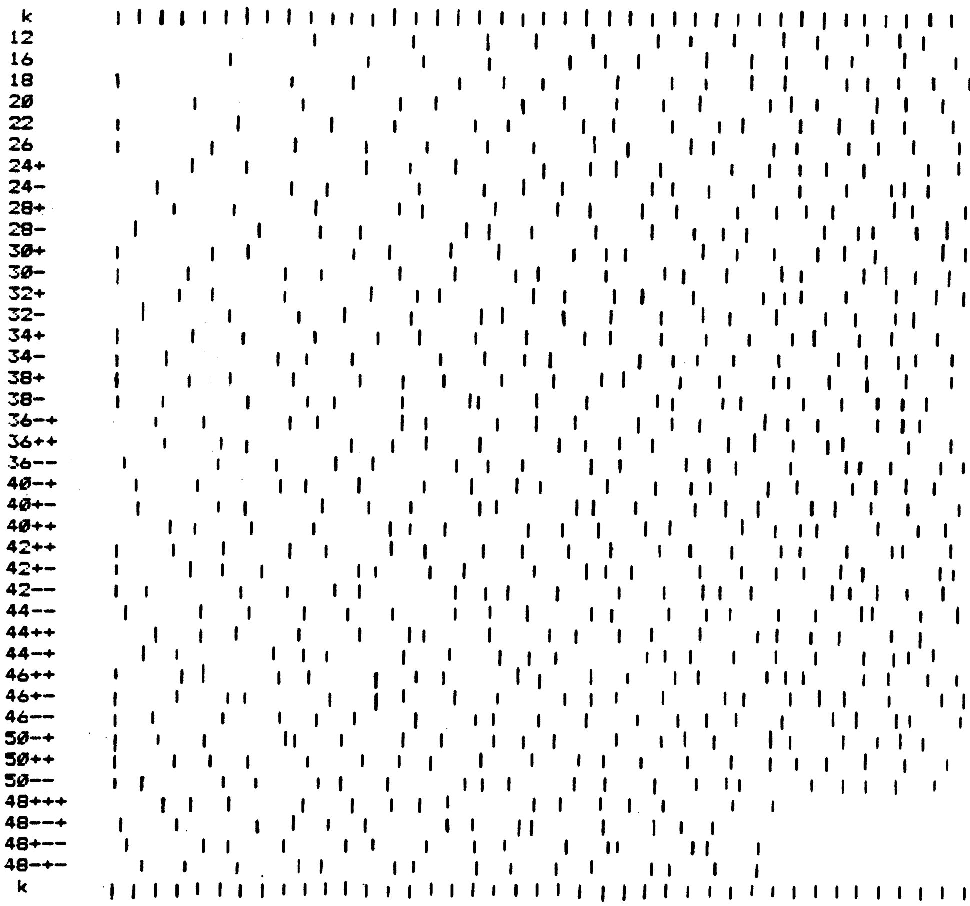

FIGURE 1

Zeros of the first forty Mellin transformed cusp eigenforms of dimensions 1, 2, 3, 4 located in the associated critical strips. The strips are $k - 1 < 2\,\mathrm{Re}(s) < k + 1$, where the k's are given in Table I, $|\mathrm{Im}(s)| \leqslant 40$ for dimensions 1, 2, 3 and $|\mathrm{Im}(s)| \leqslant 30$ for dimension 4. Fixing $\mathrm{Re}(s) = k/2$, $\mathrm{Im}(s) = t$, the scale markers top and bottom are spaced one unit apart beginning at $t = 0$ on the left with t increasing to the right. The lines of zero locations are identified by integers k followed by the distinguishing signs of the algebraic number coefficients of the basis cusp forms as decribed in the text. All zeros in the respective strips are simple and are arranged on the line $\mathrm{Re}(s) = k/2$.

5. Infinitely Many Zeros of $L_k(s, f), f \in C_k$, Lie on the Critical Line $k/2 + t\sqrt{-1}$. We have for $f \in C_k$, f a cusp form, that

$$\begin{aligned}
\Xi_k(t) = \Xi_k(t, f) &= L_k\left(k/2 + t\sqrt{-1}, f\right) \\
&= \int_1^{\infty}\left(u^{k/2+t\sqrt{-1}} + (-1)^{k/2}u^{k/2-t\sqrt{-1}}f(u)\right)d\log u \\
&= \begin{cases} 2\displaystyle\int_0^{\infty} e^{ku/2}f(e^u)\cos tu\,du, & \text{if } k/2 \text{ even}, \\ 2\sqrt{-1}\displaystyle\int_0^{\infty} e^{ku/2}f(e^u)\sin tu\,du, & \text{if } k/2 \text{ odd}. \end{cases}
\end{aligned}$$

By the functional equation of L_k,

$$\Xi_k(-t) = (-1)^{k/2}\Xi_k(t),$$

so that $\Xi_k(t)$ is an even or odd function of t according as $k/2$ is even or odd.

Let us write $F_k(u) = e^{ku/2}f(e^u)$. A direct translation of the facts that f is a cusp form (even though not necessarily a cusp eigenform) and has a functional equation is that F_k satisfies the functional equation

$$F_k(-u) = (-1)^{k/2}F_k(u).$$

Thus, $\Xi_k(t)$ is a Fourier transform of an even or odd function according as $k/2$ is even or odd.

By the Riemann-Lebesgue lemma, or directly by integration by parts, $\Theta_k(t) \to 0$ as $t \to \infty$. Note that $F_k(u)$ is an exponentially rapidly decreasing function of real t so that $\Theta_k(t)$ is an entire function.

There is a criterion of Hardy-Polya, Polya [28], which (after slight modification to include the odd case) is applicable to the present context of the Fourier transform of $F_k(u)$: the cosine (sine) transform of an expontially rapidly decreasing even (odd) function. Essentially, the criterion states that if the Fourier transform $\Xi_k(t)$ has only a finite number of real zeros, then there is some integer N, such that for $n > N$, $|F_k^{(n)}(i\omega)|$ is a monotone increasing function of ω, $0 < \omega < W$, where iW is the singular point of $F_k(u)$ which is next to the origin. In the present case, $W = \pi/2$ and from the functional equation for f, $|F_k^{(n)}(i\omega)| \to 0$ as $\omega \to \pi/2$, for all $n > 0$. The last relation follows from the way in which the cusp form $F(\varepsilon) \to 0$ as $\varepsilon \to 0$, so that the proof sketched here requires that f be a cusp form, cf. Hafner [13].

THEOREM. *Let C_k be the vector space of cusp forms of weight k for $SL_2(\mathbf{Z})$. For each $f \in C_k$ with real Fourier coefficients let $L_k(s, f)$ be the Mellin transform of f with functional equation*

$$L_k(k - s, f) = (-1)^{k/2}L_k(s, f).$$

Then infinitely many of the countable number of zeros of $L_k(s, f)$ have real part $k/2$, i.e., are of the form $k/2 + t\sqrt{-1}$, t real.

6. A Measure Theoretic Argument About Hasse-Weil L-Functions. We observe that the L-functions we introduced previously, the L-series attached to cusp forms on $SL_2(\mathbf{Z})$, are factors of the Hasse-Weil L-functions of Kuga varieties. Our computations provide examples of calculating some zeros of nontrivial Hasse-Weil L-functions. Our theorem on infinitely many zeros of cusp forms is an assertion about nontrivial Hasse-Weil L-functions.

Let X denote a projective smooth scheme. Let $X/F(p^n)$, $n \geqslant 1$, cardinality $N_n(p)$, denote the version of X over the finite field $F(p^n)$. There exist finite integers $\dim X$, $0 \leqslant b_i$, $0 \leqslant i \leqslant \dim X$, algebraic integers a_{ij}, $1 \leqslant j \leqslant b_i$, $|a_{ij}|^2 = p^i$ (Riemann hypothesis over finite fields), polynomials with coefficients in $\mathbf{Z}$,

$$P_i(T) = \prod_{1 \leqslant j \leqslant b_i} (1 - a_{ij}T), \qquad 0 \leqslant i \leqslant \dim X,$$

such that, $P_i(0) = 1$,

$$\exp \sum_{n \geqslant 1} N_n(p)T^n/n = \prod_{0 \leqslant j \leqslant \dim X} P_{2j-1}(T)/P_{2j}(T).$$

The ith Hasse-Weil L-function is defined to be

$$L_i(s;\, X) = \prod_p 1/P_i(p^{-s}),$$

where the product is over all but finitely many primes p. For definitions see Weil [50], Deligne [5], Katz [15], Serre [38]. The generalized Riemann hypothesis is the conjecture that all nontrivial zeros lie on the line $\mathrm{Re}(s) = (i+1)/2$. For the example of $X = \{\text{point}\}$, $P_0(T) = 1 - T$, $L_0(s;\, pt) = \zeta(s)$, the classical Riemann zeta function, Rosser et al. [34] and subsequently Brent [4] exclude counterexamples from a strip, e.g., $0 < \mathrm{Re}(s) < 1$, $|\mathrm{Im}(s)| < C$ containing 150 million zeros, $C = 32{,}585{,}736.4$. The examples of cusp forms arise from X = a Kuga variety. Then $H_k(T, f, p)$, the Hecke polynomial introduced above, divides the Betti polynomial $P_{k-1}(T)$. Our computations described above exclude counterexamples from various strips $k - 1 < 2\,\mathrm{Re}(s) < k + 1$, $\mathrm{Im}(s) < 40$, $k = i + 1$.

Parametrize the unit interval $[0, 1]$ in the essentially one-to-one way by binary expansions $t \to \Sigma(1 + t_{p_n})/2^{n+1}$, $t_{p_n} = \pm 1$, $n \geqslant 1$ for the montone sequence of prime numbers $\{p_n\} = \{p\}$. Then measure the product space $\{\pm 1\}^{\infty}$ of sequences $\{t_p\}$ with the Lebesgue measure from $[0, 1]$.

Let z_n, n in some countable index set, be fixed complex numbers. Then almost all of the series $\Sigma t_n z_n$ are convergent if $\Sigma|z_n|^2$ is convergent. Cf., Kolmogorov [16], Levy [23], Wintner [51], Billingsley [3].

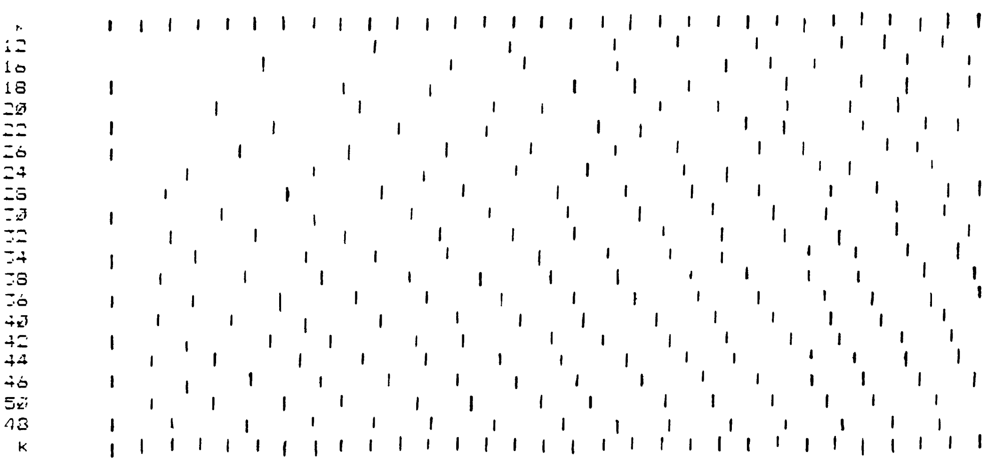

FIGURE 2

Zeros of the first nineteen Mellin transformed cusp forms Δ_k with Fourier coefficients the traces of the Hecke operators T_k, k all possible weights for the dimensions 1, 2, 3, 4. All zeros in the strips $k - 1 < 2\,\mathrm{Re}(s) < k + 1$, $|\mathrm{Im}(s)| \leqslant 30$ are simple and lie on the line $\mathrm{Re}(s) = k/2$. For $t = \mathrm{Im}(s)$, the scale markers top and bottom are positioned at $t = 0, 1, 2, \ldots, 30$, increasing to the right. Each line of zero locations is identified by the weight k of the associated cusp form. These numerical facts together with the theorem in the text on infinitude of zeros on the critical line suggest that a Riemann type hypothesis may be true for *any* cusp form of $SL_2(\mathbf{Z})$. It may be this hypothesis is false for cusp forms of discrete groups for which the Ramunujan-Petersson conjecture fails.

The convergence of the reciprocal of the L-series $L_t(s; X)$ is dependent on the convergence of the product and the sum, $t_p \equiv +1$,

$$\prod_p (1 - t_p a_{ij}(p) p^{-s}), \qquad \sum_p t_p a_{ij}(p) p^{-s}.$$

In general, define the pseudo L-function $L_{i,t}(s, X)$ for arbitrary $t \in [0, 1]$. For finite fields, $|a_{ij}(p)|^2 = p^i$, so the sum converges for $i/2 - \operatorname{Re}(s) < -1$, or $1 + i/2 < \operatorname{Re}(s)$ as previously noted. But for almost all $t \in [0, 1]$, the sums converge as long as $i - 2\operatorname{Re}(s) < -1$, or $\operatorname{Re}(s) > (1 + i)/2$. Thus for almost all $t \in [0, 1]$ the pseudo L-functions, $L_{i,t}(s, X)$, have reciprocals $1/L_{i,t}$ which converge in the half-plane $\operatorname{Re}(s) > (1 + i)/2$; so most $L_{i,t}$ have no zeros there and can be said to satisfy a generalized Riemann hypothesis. They may not have functional equations nor analytic continuations although they are Euler products. Of course, the set of (nonpseudo) geometric L-functions is countable and has measure zero in the measure used here.

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