MATHEMATICS OF COMPUTATION
VOLUME 42, NUMBER 166
APRIL 1984, PAGES 417-426

A Collocation-H^{-1}-Galerkin Method for Some Elliptic Equations

By Mitsuhiro Nakao

Abstract. A collocation-H^{-1}-Galerkin method is defined for some elliptic boundary value problems on a rectangle. The method uses tensor products of discontinuous piecewise polynomial spaces and collocation based on Jacobi points with weight function $x^2(1-x)^2$. Optimal order of L^2 rates of convergence is established for the approximation solution. A numerical example which confirms these results is presented.

1. Introduction. In this paper we define and analyze a collocation-H^{-1}-Galerkin method, for some elliptic equations on a rectangular domain in two-dimensional Euclidean space. The method uses tensor products of discontinuous piecewise polynomial spaces as the trial functions family.

The H^{-1}-Galerkin method was introduced by Rachford and Wheeler [8] for the numerical solution of the two-point boundary value problem. Later Douglas, Dupont, Rachford and Wheeler [6] applied this scheme to the elliptic boundary value problem on a unit square, and derived optimal L^2 and L^∞ error estimates. The collocation-Galerkin method, which is a mixed scheme of a collocation method and an L^2-Galerkin method, was first introduced by Diaz and in [2] he obtained optimal error estimates for the two-point boundary value problem. Also, in [3] Diaz applied the scheme to the Poisson equation and established optimal order L^2 estimates. For elliptic equations, Wheeler [9] proposed a collocation-L^2-Galerkin method with interior penalties and derived optimal L^2 estimates. That method uses discontinuous trial function spaces, but not tensor products, similar to that of the present paper. On the collocation-H^{-1}-Galerkin method, Dunn and Wheeler [5] obtained some optimal estimation results for the two-point boundary value problem. Diaz [4] extended those results to one space dimensional parabolic problems. Using similar ideas, Archer and Diaz described in [1] a discontinuous collocation-Galerkin scheme for a first order hyperbolic initial boundary value problem. On the other hand, Percell and Wheeler [7] reported about a collocation method using C^1 approximation spaces for elliptic equations.

In the following sections, we define a collocation-H^{-1}-Galerkin method for the boundary value problem: $Lu \equiv \Delta u + qu = f$ on a rectangular domain with $u = 0$ on the boundary, and establish optimal order L^2 error estimates. Finally, we present

Received July 31, 1981; revised February 18, 1983.
1980 *Mathematics Subject Classification*. Primary 65N30, 65N35.

some results of computational experiments comparing this method with the collocation-L^2-Galerkin method in [3]. In the error analysis we reduce the problem to a one-dimensional case but do not intend to use the existing estimation results. That is, this paper is essentially self-contained.

2. The Problem and Notations. Let E be a bounded open set in an n-dimensional Euclidean space ($n = 1$ or 2). For any integer $s \geqslant 0$, we denote the usual Sobolev space of order s by $H^s(E)$, i.e., $H^s(E)$ is the completion of $C^\infty(\bar{E})$ under the norm

$$\|u\|_{H^s(E)} = \left(\sum_{|\alpha| \leqslant s} \|D^\alpha u\|^2_{L^2(E)} \right)^{1/2},$$

where α are multi-integers and $\|u\|^2_{L^2(E)} = \int_E |u|^2\,dx$.

Consider the following elliptic boundary value problem on a rectangular domain

$$\text{(1)} \qquad Lu \equiv \Delta u + qu = f, \quad \text{in } R, \qquad u = 0, \quad \text{on } \partial R,$$

where $R = (0,1) \times (0,1)$ and $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Assume that $q \in H^2(R)$ and given $f \in L^2(R)$, (1) has a unique solution.

Let δ: $0 = x_0 < x_1 < \cdots < x_n = 1$, be a quasi-uniform partition of $I = (0,1)$. Also let $I_i = (x_{i-1}, x_i)$, $h_i = x_i - x_{i-1}$, $h = \max_{1 \leqslant i \leqslant N} h_i$ and $y_k = x_k$, $0 \leqslant k \leqslant N$.

For a positive integer r ($\geqslant 2$) and $E \subset I$, let $P_r(E)$ denote the set of polynomials of degree at most r on E. Let

$$\mathfrak{M}^r_{-1}(\delta) = \{ v : v|_{I_i} \in P_r(I_i), 1 \leqslant i \leqslant N \},$$
$$\mathfrak{M}^r_k(\delta) = \mathfrak{M}^r_{-1} \cap C^k(I) \cap H^1_0(I), \quad \text{for } k \geqslant 0,$$
$$Z^r_k(\delta) = \left\{ v : v \in \mathfrak{M}^r_k, \frac{d^j v}{dx^j}(x_i) = 0, 0 \leqslant i \leqslant N, 0 \leqslant j \leqslant k \right\},$$
$$\mathfrak{M}(\delta) = \mathfrak{M}^r_{-1}(\delta) \otimes \mathfrak{M}^r_{-1}(\delta),$$
$$\mathfrak{N}(\delta) = \mathfrak{M}^{r+2}_1(\delta) \otimes \mathfrak{M}^{r+2}_1(\delta).$$

Usually, we shall suppress the dependency on the partition in these notations. The following partition into direct sum is immediately obtained by the fact that $\mathfrak{M}^{r+2}_1 = Z^{r+2}_1 \oplus \mathfrak{M}^3_1$

$$\mathfrak{N} = \mathfrak{N}_1 \oplus \mathfrak{N}_2 \oplus \mathfrak{N}_3 \oplus \mathfrak{N}_4,$$

where $\mathfrak{N}_1 = Z^{r+2}_1 \otimes Z^{r+2}_1$, $\mathfrak{N}_2 = Z^{r+2}_1 \otimes \mathfrak{M}^3_1$, $\mathfrak{N}_3 = \mathfrak{M}^3_1 \otimes Z^{r+2}_1$ and $\mathfrak{N}_4 = \mathfrak{M}^3_1 \otimes \mathfrak{M}^3_1$.

3. A Collocation-H^{-1}-Galerkin Method. Let σ_j ($1 \leqslant j \leqslant r-1$) be zeros of the Jacobi polynomial on I with weight function $x^2(1-x)^2$. We adopt as collocation points the $(r-1)^2$ points (x_{ij}, y_{kl}), $1 \leqslant j, l \leqslant r-1$ on each subrectangle $I_i \times I_k$ which are the following affine transformations of σ_j:

$$x_{ij} = x_{i-1} + h_i \sigma_j, \qquad y_{kl} = y_{k-1} + h_k \sigma_l, \qquad 1 \leqslant j, l \leqslant r-1.$$

Now we define a collocation-H^{-1}-Galerkin approximation to (1) by $U \in \mathfrak{M}$ satisfying:

$$\text{(2-i)} \qquad \Delta U(x_{ij}, y_{kl}) + q(x_{ij}, y_{kl}) U(x_{ij}, y_{kl}) = f(x_{ij}, y_{kl}),$$
$$1 \leqslant i, k \leqslant N, 1 \leqslant j, l \leqslant r-1,$$

(2-ii) $$\int_I \frac{\partial^2}{\partial x^2} U(x_{ij}, \eta) v(\eta)\, d\eta + \int_I U(x_{ij}, \eta) v''(\eta)\, d\eta + \int_I q(x_{ij}, \eta) U(x_{ij}, \eta) v(\eta)\, d\eta = \int_I f(x_{ij}, \eta) v(\eta)\, d\eta,$$
$$1 \leqslant i \leqslant N,\ 1 \leqslant j \leqslant r-1,\ v \in \mathfrak{M}_1^3,$$

(2-iii) $$\int_I \frac{\partial^2}{\partial y^2} U(\xi, y_{kl}) v(\xi)\, d\xi + \int_I U(\xi, y_{kl}) v''(\xi)\, d\xi + \int_I q(\xi, y_{kl}) U(\xi, y_{kl}) v(\xi)\, d\xi = \int_I f(\xi, y_{kl}) v(\xi)\, d\xi,$$
$$1 \leqslant k \leqslant N,\ 1 \leqslant l \leqslant r-1,\ v \in \mathfrak{M}_1^3,$$

(2-iv) $$\iint_R U(\xi, \eta)(\Delta v(\xi, \eta) + q(\xi, \eta) v(\xi, \eta))\, d\xi d\eta = \iint_R f(\xi, \eta) v(\xi, \eta)\, d\xi d\eta, \qquad v \in \mathfrak{N}_4.$$

In order to make the error estimates easy, we represent (2) in a semidiscrete variational equation. First, as in [4], we define for a function ϕ defined on I and $v \in Z_1^{r+2}$

$$\langle \phi, v \rangle_i = \sum_{j=1}^{r-1} h_i \omega_j \frac{\phi(x_{ij}) v(x_{ij})}{\sigma_j^2 (1-\sigma_j)^2} \quad \text{and} \quad \langle \phi, v \rangle = \sum_{i=1}^{N} \langle \phi, v \rangle_i,$$

where ω_j are postiive constants determined by

$$\int_I x^2 (1-x)^2 p(x)\, dx = \sum_{j=1}^{r-1} \omega_j p(\sigma_j), \qquad p \in P_{2r-3}(I).$$

Note that if $\phi \cdot v \in P_{2r+1}(I_i)$, then

(3) $$\langle \phi, v \rangle_i = \int_{I_i} \phi(x) v(x)\, dx.$$

Furthermore, for $\phi \in \mathfrak{M}_1^{r+2}$

(4) $$\langle \phi'', \phi_1 \rangle + (\phi'', \phi_2) \leqslant -(\phi', \phi'),$$

where $\phi = \phi_1 + \phi_2$ such that $\phi_1 \in Z_1^{r+2}$ and $\phi_2 \in \mathfrak{M}_1^3$.

Next we define for the function ψ defined on R and $w \in \mathfrak{N}_1$,

(5) $$\langle\langle \psi, w \rangle\rangle = \sum_{i,k=1}^{N} \left\{ \sum_{j,l=1}^{r-1} h_i h_k \omega_j \omega_l \frac{\psi(x_{ij}, y_{kl}) w(x_{ij}, y_{kl})}{\sigma_j^2 (1-\sigma_j)^2 \sigma_l^2 (1-\sigma_l)^2} \right\}.$$

Now, using the following unique partition for $v \in \mathfrak{N}$,

(6) $$v = v_1 + v_2 + v_3 + v_4, \qquad v_m \in \mathfrak{N}_m,$$

a semidiscrete bilinear form $\mathfrak{L}(\cdot, \cdot)$ is defined as follows:

(7) $$\mathfrak{L}(\psi, v) = \langle\langle L\psi, v_1 \rangle\rangle + \int_I \langle L_x \psi, v_2 \rangle_x\, dy + \int_I \langle \psi, L_x v_3 \rangle_y\, dx + \int_I \langle \psi, L_y v_2 \rangle_x\, dy + \int_I \langle L_y \psi, v_3 \rangle_y\, dx + (\psi, L v_4),$$

where $\langle \cdot , \cdot \rangle_x$, $\langle \cdot , \cdot \rangle_y$ mean discrete bilinear forms defined earlier with respect to x and y, respectively, and $(\cdot, \cdot)$ implies the L^2 inner product in R. Furthermore, let

$$L_x\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}q\psi \quad \text{and} \quad L_y\psi = \frac{\partial^2\psi}{\partial y^2} + \frac{1}{2}q\psi.$$

In particular, when $L = \Delta$, we denote $\mathfrak{L}(\cdot, \cdot)$ by $D(\cdot, \cdot)$. Another bilinear form $\mathfrak{F}(\psi, v)$ is defined by

$$\mathfrak{F}(\psi, v) = \langle\langle \psi, v_1 \rangle\rangle + \int_I \langle \psi, v_2 \rangle_x \, dy + \int_I \langle \psi, v_3 \rangle_y \, dx + (\psi, v_4).$$

Then it can be easily seen that (2) is equivalent to

$$\mathfrak{L}(U, v) = \mathfrak{F}(f, v), \qquad v \in \mathfrak{N}. \tag{8}$$

Since $\mathfrak{M}$ and $\mathfrak{N}$ are of the same dimension, existence of U satisfying (8) is equivalent to uniqueness, but the uniqueness will immediately follow from the theorems in the next section as it is required that h is sufficiently small except for $L = \Delta$.

We now provide some well-known inequalities for later use. When $\phi \in H^2(I_l)$, we have ([4] and [5])

$$\|\phi_x\|_{L^\infty(I_l)} \leqslant C\left(h_l^{1/2}\|\phi_{xx}\|_{L^2(I_l)} + h_l^{-3/2}\|\phi\|_{L^2(I_l)}\right), \tag{9}$$

and

$$\|\phi\|_{L^\infty(I_l)} \leqslant C\left(h_l^{1/2}\|\phi_x\|_{L^2(I_l)} + h_l^{-1/2}\|\phi\|_{L^2(I_l)}\right). \tag{10}$$

Also, let $v \in P_s(I_l) \otimes P_t(I_k)$ and $\rho = I_l \times I_k$. Then, by the quasi-uniformity assumption on the partition δ, we have the following inverse properties [7]:

$$\|v\|_{L^\infty(\rho)} \leqslant Ch^{-1}\|v\|_{L^2(\rho)}, \tag{11}$$

and

$$\|\nabla v\|_{L^2(\rho)} \leqslant Ch^{-1}\|v\|_{L^2(\rho)}. \tag{12}$$

Here and throughout this paper, we use C to denote a generic constant not necessarily the same in any two places. Also v_m $(1 \leqslant m \leqslant 4)$ denotes the mth component of v in (6) unless otherwise stated.

4. Error Estimates.

4.1. For $L = \Delta$.

Let P: $H^3(I) \to \mathfrak{M}_{-1}^r$ be a projection determined by the realtions

$$\langle g'' - (Pg)'', w_1 \rangle = 0, \qquad w_1 \in Z_1^{r+2},$$
$$(g - Pg, w_2'') = 0, \qquad w_2 \in \mathfrak{M}_1^3.$$

Since the definition of P is local, the following lemma is easily obtained from elementary approximation theory.

LEMMA 1. *For any s such that $3 \leqslant s \leqslant r + 1$, if $g \in H^s(I_l)$, then there exists a constant $C > 0$, independent of h, such that*

$$\|g - Pg\|_{H^m(I_l)} \leqslant Ch^{s-m}\|g\|_{H^s(I_l)}, \qquad 0 \leqslant m \leqslant s.$$

Using this lemma for $s = 3$ and $m = 0$, by the triangle inequality we obtain

COROLLARY 1. *For any $g \in H^3(I_l)$, we have*

$$\|Pg\|_{L^2(I_l)} \leqslant C\left(\|g\|_{L^2(I_l)} + h^3\|g\|_{H^3(I_l)}\right).$$

For $u \in H^6(R)$, let $P_x u$ be the above projection of u in the x direction for each fixed y. Similarly $P_y u$ is defined. From the definition, we have $P_y P_x u = P_x P_y u \in \mathfrak{M}$.
Now we have the following lemma.

LEMMA 2. *Let $3 \leqslant s \leqslant r + 1$ and $u \in H^{s+3}(\rho)$ for $\rho = I_l \times I_k$. Then there exists a constant $C > 0$, independent of h, such that*

$$\|u - P_y P_x u\|_{H^m(\rho)} \leqslant Ch^{s-m}\left(\|u\|_{H^s(\rho)} + h^3\|u\|_{H^{s+3}(\rho)}\right), \qquad 0 \leqslant m \leqslant s.$$

Proof. First, by Lemma 1, Corollary 1 and the property of the operator P, we have

$$\begin{aligned}
\|u - P_y P_x u\|_{L^2(\rho)} &\leqslant \|u - P_x u\|_{L^2(\rho)} + \|P_x u - P_y P_x u\|_{L^2(\rho)} \\
&\leqslant Ch^s\|u\|_{H^s(\rho)} + Ch^s\left(\sum_{\alpha=0}^{s}\left\|\frac{\partial^\alpha}{\partial y^\alpha}P_x u\right\|^2_{L^2(\rho)}\right)^{1/2} \\
&= Ch^s\left\{\|u\|_{H^s(\rho)} + \left(\sum_{\alpha=0}^{s}\left\|P_x\frac{\partial^\alpha u}{\partial y^\alpha}\right\|^2_{L^2(\rho)}\right)^{1/2}\right\} \\
&\leqslant Ch^s\left\{\|u\|_{H^s(\rho)} + \sum_{\alpha=0}^{s}\left(\left\|\frac{\partial^\alpha u}{\partial y^\alpha}\right\|^2_{L^2(\rho)} + h^6\sum_{\beta=0}^{3}\left\|\frac{\partial^{\alpha+\beta} u}{\partial x^\beta \partial y^\alpha}\right\|^2_{L^2(\rho)}\right)^{1/2}\right\} \\
&\leqslant Ch^s\left(\|u\|_{H^s(\rho)} + h^3\|u\|_{H^{s+3}(\rho)}\right).
\end{aligned}$$

This implies that the lemma is valid for $m = 0$. Next, for $m \geqslant 1$, choosing $Q \in P_r(I_l) \otimes P_r(I_k)$ appropriately to approximate u (e.g., the L^2-projection of u), (12) and the above inequality yield

$$\begin{aligned}
\|u - P_y P_x\|_{H^m(\rho)} &\leqslant \|u - Q\|_{H^m(\rho)} + \|Q - P_y P_x u\|_{H^m(\rho)} \\
&\leqslant Ch^{s-m}\|u\|_{H^s(\rho)} + Ch^{-m}\|Q - P_y P_x u\|_{L^2(\rho)} \\
&\leqslant C\left(h^{s-m}\|u\|_{H^s(\rho)} + h^{-m}\|u - Q\|_{L^2(\rho)} + h^{-m}\|u - P_y P_x u\|_{L^2(\rho)}\right) \\
&\leqslant Ch^{s-m}\left(\|u\|_{H^s(\rho)} + h^3\|u\|_{H^{s+3}(\rho)}\right),
\end{aligned}$$

which proves the lemma.

We now prove the following theorem which provides optimal order L^2 error estimates for $u - U$ when $L = \Delta$.

THEOREM 1. *Let u and U be solutions to* (1) *and* (2), *respectively, for $L = \Delta$. If $u \in H^{r+4}(R)$, then there exists a constant $C > 0$, independent of h, such that*

$$\|u - U\|_{L^2(R)} \leqslant Ch^{r+1}\left(\|u\|_{H^{r+1}(R)} + h^3\|u\|_{H^{r+4}(R)}\right).$$

Proof. Let $W = P_y P_x u$, $\xi = u - W$ and $\eta = U - W$. Then from (1) and (8),

$$D(\eta, v) = D(\xi, v), \qquad v \in \mathfrak{M}. \tag{13}$$

Let $v \in \mathfrak{M}$ satisfy $v_{xxyy} = -\eta$. Using (3) and (4) one can verify that

$$(14)\qquad D(\eta, v) \geqslant \|v_{xxy}\|^2_{L^2(R)} + \|v_{yyx}\|^2_{L^2(R)} = \|\nabla v_{xy}\|^2_{L^2(R)}.$$

Next we estimate the right-hand side of (13). By the definition,

$$\begin{aligned} D(\xi, v) &= \langle\langle \xi_{xx}, v_1\rangle\rangle + \int_I \langle \xi_{xx}, v_2\rangle_x\, dy + \int_I \langle \xi, v_{3xx}\rangle_y\, dx + (\xi, v_{4xx}) \\ &\quad + \langle\langle \xi_{yy}, v_1\rangle\rangle + \int_I \langle \xi, v_{2yy}\rangle_x\, dy + \int_I \langle \xi_{yy}, v_3\rangle_y\, dx + (\xi, v_{4yy}) \\ &= \nu_x + \nu_y, \end{aligned}$$

where ν_x and ν_y are the sums of first and last four terms, respectively.

We now estimate each term in ν_x separately. Let $\mathfrak{R} = \{\rho\colon \rho = I_i \times I_k,\ 1 \leqslant i, k \leqslant N\}$.

First, for any $Q \in \mathfrak{M}$, we have

$$\begin{aligned} (15)\qquad \langle\langle \xi_{xx}, v_1\rangle\rangle &\leqslant Ch^2 \sum_{\rho\in\mathfrak{R}} \|\xi_{xx}\|_{L^\infty(\rho)} \|v_1\|_{L^\infty(\rho)} \\ &\leqslant Ch^2 \sum_{\rho\in\mathfrak{R}} \left(\left\| \frac{\partial^2}{\partial x^2}(u - Q) \right\|_{L^\infty(\rho)} + \left\| \frac{\partial^2}{\partial x^2}(Q - W) \right\|_{L^\infty(\rho)} \right) \|v_1\|_{L^\infty(\rho)}. \end{aligned}$$

If we select Q appropriately to approximate u (e.g., local L^2-projection of u into $\mathfrak{M}$), then, by simple calculations using (9) and (10), we obtain

$$\left\| \frac{\partial^2}{\partial x^2}(u - Q) \right\|_{L^\infty(\rho)} \leqslant Ch^{r-2} \|u\|_{H^{r+1}(\rho)}.$$

Hence, from (15), (11), (12) and Lemma 2,

$$\begin{aligned} (16)\qquad \langle\langle \xi_{xx}, v_1\rangle\rangle &\leqslant Ch^2 \sum_{\rho\in\mathfrak{R}} \left(h^{r-2}\|u\|_{H^{r+1}(\rho)} + h^{-3}\|Q - W\|_{L^2(\rho)} \right) h^2 \|v_{1xxy}\|_{L^2(\rho)} \\ &\leqslant Ch^4 \sum_{\rho\in\mathfrak{R}} \left(h^{r-2}\|u\|_{H^{r+1}(\rho)} + h^{-3}\|u - W\|_{L^2(\rho)} \right) \|v_{1xxy}\|_{L^2(\rho)} \\ &\leqslant Ch^{r+2} \left(\|u\|_{H^{r+1}(R)} + h^3 \|u\|_{H^{r+4}(R)} \right) \|v_{1xxy}\|_{L^2(R)}. \end{aligned}$$

Next, by the definition of P and (3), for any map $w\colon I_{(x)} \to \mathfrak{M}^r_{-1(y)}$

$$\begin{aligned} (17)\qquad \int_I \langle \xi_{xx}, v_2\rangle_x\, dy &= \int_I \left\langle \frac{\partial^2}{\partial x^2}(u - P_x u), v_2 \right\rangle_x dy \\ &\quad + \int_I \left\langle \frac{\partial^2}{\partial x^2}(P_x u - P_y P_x u), v_2 \right\rangle_x dy \\ &= \int\int_R \frac{\partial^2}{\partial x^2}(P_x u - P_y P_x u) v_2\, dx dy \\ &= \int_I \left(\int_I \left(\frac{\partial^2}{\partial x^2} P_x u - P_y \frac{\partial^2}{\partial x^2} P_x u \right) (v_2 - w_{yy})\, dy \right) dx. \end{aligned}$$

Notice that we can choose w satisfying

$$\|(v_2 - w_{yy})(x, \cdot)\|_{L^2(I)} \leqslant Ch \|v_2(x, \cdot)\|_{H^1(I)}.$$

Thus, by (17), (12) for its one-dimensional version and Corollary 1, we obtain

$$(18)\quad \int_I \langle \xi_{xx}, v_2 \rangle_x \, dy \leqslant C \int_I h^{r+2} \left\| \frac{\partial^2}{\partial x^2} P_x u(x, \cdot) \right\|_{H^{r+1}(I)} \|v_{2y}(x, \cdot)\|_{L^2(I)} \, dx$$

$$\leqslant Ch^{r+2} \sum_{i=1}^{N} \left(\sum_{\alpha=0}^{r+1} h^2 \left\| \frac{\partial^2}{\partial x^2} P_x \frac{\partial^\alpha u}{\partial y^\alpha} \right\|_{L^2(I_i \times I)} \right) \|v_{2xxy}\|_{L^2(I_i \times I)}$$

$$\leqslant Ch^{r+2} \sum_{i=1}^{N} \left(\sum_{\alpha=0}^{r+1} \left\| P_x \frac{\partial^\alpha u}{\partial y^\alpha} \right\|_{L^2(I_i \times I)} \right) \|v_{2xxy}\|_{L^2(I_i \times I)}$$

$$\leqslant Ch^{r+2} \sum_{i=1}^{N} \left\{ \sum_{\alpha=0}^{r+1} \left(\left\| \frac{\partial^\alpha u}{\partial y^\alpha} \right\|_{L^2(I_i \times I)} + \sum_{\beta=0}^{3} h^3 \left\| \frac{\partial^{\alpha+\beta} u}{\partial x^\beta \partial y^\alpha} \right\|_{L^2(I_i \times I)} \right) \right\} \|v_{2xxy}\|_{L^2(I_i \times I)}$$

$$\leqslant Ch^{r+2} \sum_{i=1}^{N} \left(\|u\|_{H^{r+1}(I_i \times I)} + h^3 \|u\|_{H^{r+4}(I_i \times I)} \right) \|v_{2xxy}\|_{L^2(I_i \times I)}$$

$$\leqslant Ch^{r+2} \left(\|u\|_{H^{r+1}(R)} + h^3 \|u\|_{H^{r+4}(R)} \right) \|v_{2xxy}\|_{L^2(R)}.$$

On the third term, we see that

$$\int_I \langle \xi, v_{3xx} \rangle_y \, dx = \int_I \langle u - P_y u, v_{3xx} \rangle_y \, dx + \int_I \langle P_y u - P_x P_y u, v_{3xx} \rangle_y \, dx$$

$$= \int_I \left\{ \langle u - P_y u, v_{3xx} \rangle_y - \left(u - P_y u, v_{3xx} \right)_y \right\} dx$$

$$+ \int_I \left(u - P_y u, v_{3xx} \right)_y dx.$$

But, it is shown in [4, Lemma 4.1] that

$$\left| \langle u - P_y u, v_{3xx} \rangle_y - \left(u - P_y u, v_{3xx} \right)_y \right| \leqslant Ch^2 \sum_{k=1}^{N} \|u - P_y u\|_{H^1(I_k)} \|v_{3xxy}\|_{L^2(I_k)}.$$

Also, using a similar estimation to that of the preceding term

$$\left| \left(u - P_y u, v_{3xx} \right)_y \right| \leqslant Ch \|u - P_y u\|_{L^2(I)} \|v_{3xxy}\|_{L^2(I)}.$$

Thus, by Lemma 1 we have

$$(19)\qquad \left| \int_I \langle \xi, v_{3xx} \rangle_y \, dx \right| \leqslant Ch^{r+2} \|u\|_{H^{r+1}(R)} \|v_{3xxy}\|_{L^2(R)}.$$

The last term is estimated by an argument similar to a part of the previous terms and Lemma 1. That is,

$$(20)\qquad |(\xi, v_{4xx})| = \left| \left(u - P_y u, v_{4xx} \right) \right| \leqslant Ch^{r+2} \|u\|_{H^{r+1}(R)} \|v_{4xxy}\|_{L^2(R)}.$$

Now, note that

$$(21)\qquad \|v_{mxxy}\|_{L^2(R)} \leqslant C \|v_{xxy}\|_{L^2(R)}, \qquad 1 \leqslant m \leqslant 4.$$

Therefore, from (16), (18), (19), (20) and (21), we have

$$(22)\qquad |\nu_x| \leqslant Ch^{r+2} \left(\|u\|_{H^{r+1}(R)} + h^3 \|u\|_{H^{r+4}(R)} \right) \| \nabla v_{xy} \|_{L^2(R)}.$$

Since v_y is estimated in a similar manner, by (13) and (14)

$$\|\nabla v_{xy}\|_{L^2(R)} \leqslant Ch^{r+2}\left(\|u\|_{H^{r+1}(R)} + h^3\|u\|_{H^{r+4}(R)}\right).$$

Thus by (12)

$$\|U - W\|_{L^2(R)} \leqslant Ch^{r+1}\left(\|u\|_{H^{r+1}(R)} + h^3\|u\|_{H^{r+4}(R)}\right). \tag{23}$$

The proof of the theorem now follows from (23), Lemma 2 and the triangle inequality.

It is easily seen that, from the above process of the proof and Lemma 2, Theorem 1 is extended to the following form.

COROLLARY 2. *Let u and U be solutions to* (1) *and* (2), *respectively, for $L = \Delta$. If $u \in H^{r+4}(\rho)$ for $\rho \in \mathfrak{R}$, there exists a constant $C > 0$, independent of h, such that for $0 \leqslant m \leqslant r$,*

$$\left(\sum_{\rho\in\mathfrak{R}} \|u - U\|^2_{H^m(\rho)}\right)^{1/2} \leqslant Ch^{r+1-m}\left(\sum_{\rho\in\mathfrak{R}} \|u\|^2_{H^{r+1}(\rho)} + h^6\|u\|_{H^{r+4}(\rho)}\right)^{1/2}.$$

4.2. For $L = \Delta + q$.

We define a projection $Y \in \mathfrak{M}$ by

$$D(u - Y, v) = 0, \qquad v \in \mathfrak{M}. \tag{24}$$

Now, let $\eta = U - Y$ and $\zeta = u - Y$. Then, from (8),

$$\mathfrak{L}(\eta, v) - (\eta, qv_4) = \mathfrak{L}(\zeta, v) - (\eta, qv_4), \qquad v \in \mathfrak{M}. \tag{25}$$

One can easily verify that if we take $v \in \mathfrak{M}$ satisfying $v_{xxyy} = -\eta$, then

$$|\mathfrak{L}(\eta, v) - (\eta, qv_4) - D(\eta, v)| \leqslant Ch^2\|\nabla v_{xy}\|^2_{L^2(R)}.$$

Hence, if h is sufficiently small, by (14), there exists a constant $C > 0$, independent of h, such that

$$C\|\nabla v_{xy}\|^2_{L^2(R)} \leqslant \mathfrak{L}(\eta, v) - (\eta, qv_4). \tag{26}$$

This relation will play an essential role in the proof of the following theorem.

THEOREM 2. *Let u and U be solutions to* (1) *and* (2), *respectively. If $u \in H^{r+4}(R)$, then, for h sufficiently small, there exists a constant $C > 0$, independent of h, such that*

$$\|u - U\|_{L^2(R)} \leqslant Ch^{r+1}\left(\|u\|_{H^{r+1}(R)} + h^3\|u\|_{H^{r+4}(R)}\right).$$

Proof. From (24), we have

$$\mathfrak{L}(\zeta, v) - (\eta, qv_4) = \langle\langle q\zeta, v_1\rangle\rangle + \int_I \langle q\zeta, v_2\rangle_x \, dy + \int_I \langle q\zeta, v_3\rangle_y \, dx + (u - U, qv_4). \tag{27}$$

First, by Sobolev's lemma, Corollary 2 and (11)

$$\begin{aligned} |\langle\langle q\zeta, v_1\rangle\rangle| &\leqslant Ch^2 \sum_{\rho\in\mathfrak{R}} \|\zeta\|_{L^\infty(\rho)}\|v_1\|_{L^\infty(\rho)} \\ &\leqslant Ch^3 \sum_{\rho\in\mathfrak{R}} \|\zeta\|_{H^2(\rho)}\|\nabla v_{1xy}\|_{L^2(\rho)} \\ &\leqslant Ch^{r+2}\left(\|u\|_{H^{r+1}(R)} + h^3\|u\|_{H^{r+4}(R)}\right)\|\nabla v_{xy}\|_{L^2(R)}. \end{aligned}$$

Similarly, we have

$$\left|\int_I \langle q\zeta, v_2\rangle_x \, dy\right| \leqslant C\int_I \sum_{l=1}^{N} h\|\zeta(\cdot, y)\|_{L^\infty(I_l)}\|v_2(\cdot, y)\|_{L^\infty(I_l)}\, dy$$
$$\leqslant C\int_I \sum_{l=1}^{N} h^2\|\zeta(\cdot, y)\|_{H^1(I_l)}\|v_{2xx}(\cdot, y)\|_{L^2(I_l)}\, dy$$
$$\leqslant Ch^{r+2}\left(\|u\|_{H^{r+1}(R)} + h^3\|u\|_{H^{r+4}(R)}\right)\|v_{2xxy}\|_{L^2(R)}.$$

The third term is estimated in the same manner.

In order to estimate the last term, let $\xi = u - U$, and we consider the following boundary value problem.

$$L\Phi = qv_4 \quad \text{in } R,$$
$$\Phi = 0 \quad \text{on } \partial R.$$

Choosing $\hat{\Phi} \in \mathfrak{M}_4$ appropriately to approximate Φ (e.g. piecewise Hermite interpolant), by (2-iv) and elliptic regularity, we have

$$|(\xi, qv_4)| = |(\xi, L(\Phi - \hat{\Phi}))| \leqslant \sum_{\rho\in\mathfrak{R}} |(\xi, L(\Phi - \hat{\Phi}))_\rho|$$
$$\leqslant C\sum_{\rho\in\mathfrak{R}} \|\xi\|_{L^2(\rho)}h^2\|\Phi\|_{H^4(\rho)} \leqslant Ch^2\|\xi\|_{L^2(R)}\|qv_4\|_{H^2(R)}$$
$$\leqslant Ch^2\|\xi\|_{L^2(R)}\|\nabla v_{4xy}\|_{L^2(R)}.$$

Thus, by (25), (26), (27) and (12)

$$\|\eta\|_{L^2(R)} \leqslant C\left\{h^{r+1}\left(\|u\|_{H^{r+1}(R)} + h^3\|u\|_{H^{r+4}(R)}\right) + h\|u - U\|_{L^2(R)}\right\}.$$

Therefore, if h is sufficiently small, by Theorem 1 and the triangle inequality we obtain the desired result.

Remarks. 1. While Theorem 2 is valid for sufficiently small h, Theorem 1 is so for an arbitrary mesh size.

2. For $r \geqslant 5$, it is possible that the norms on the right-hand side of all inequalities in the theorems can be weakened up to optimal size.

5. A Numerical Example. We made a numerical experiment with our method for the following problem.

$$\Delta u = -2\pi^2 \sin\pi x \sin\pi y, \qquad (x, y) \in R, \tag{28}$$
$$u = 0, \qquad (x, y) \in \partial R.$$

The exact solution of the above is $u(x, y) = \sin\pi x \sin\pi y$. We used piecewise quadratic polynomials and uniform partitions. Table I shows the results of this experiment. Each item in Table I reads as follows:

N: Number of partitions of I ($h = 1/N$).
E_{mesh}: Maximum of the errors at all interior mesh points.
E_{midd}: Maximum of the errors at the middle points of all subrectangles.
E_{oth}: Maximum of the errors at the points of the form $(x_i - 0.75h, y_j - 0.75h)$, $1 \leqslant i, j \leqslant N$.

Here, we adopted the maximum value of four different limits as the interior mesh point error for the discontinuity of the approximate solution. These results will be sufficient to confirm the cubic rates of convergence.

TABLE I

Errors for collocation-H^{-1}-Galerkin method

N	E_{mesh}	E_{midd}	E_{oth}
3	0.1211E − 1	0.1146E − 2	0.4716E − 2
5	0.2327E − 2	0.1575E − 3	0.9639E − 3
7	0.8373E − 3	0.4181E − 4	0.3501E − 3

Now, in order to compare our method, we also computed numerical solutions by the collocation-L^2-Galerkin method in [3] to (28) using the same degree ($r = 2$). The results are illustrated in Table II.

TABLE II

Errors for collocation-L^2-Galerkin method

N	E_{mesh}	E_{midd}	E_{oth}
3	0.3286E − 3	0.5131E − 2	0.9266E − 2
5	0.8165E − 4	0.6895E − 3	0.1979E − 2
7	0.2521E − 4	0.1816E − 3	0.7154E − 3

From these two tables it is seen that our method is inferior in the mesh points error. However, it seems to be a rather natural result because our trial functions were permitted discontinuity across the elements. On the other hand, it is clear that our method is superior for the error at the middle points. Similar phenomena were also observed at several other interior points in each subrectangle (see E_{oth} in these tables).

Department of Applied Mathematics
Kyushu Institute of Technology
Tobata, Kitakyushu 804, Japan

1. D. ARCHER & J. C. DIAZ, "A collocation-Galerkin method for a first order hyperbolic equation with space and time-dependent coefficient," *Math. Comp.*, v. 38, 1982, pp. 37–53.

2. J. C. DIAZ, "A collocation-Galerkin method for the two point boundary value problem using continuous piecewise polynomial spaces," *SIAM J. Numer. Anal.*, v. 14, 1977, pp. 844–858.

3. J. C. DIAZ, "A collocation-Galerkin method for Poisson's equation on rectangular regions," *Math. Comp.*, v. 33, 1979, pp. 77–84.

4. J. C. DIAZ, "Collocation-H^{-1}-Galerkin method for parabolic problems with time dependent coefficients," *SIAM J. Numer. Anal.*, v. 16, 1979, pp. 911–922.

5. R. J. DUNN, JR. & M. F. WHEELER, "Some collocation-Galerkin methods for two point boundary value problems," *SIAM J. Numer. Anal.*, v. 13, 1976, pp. 720–733.

6. J. DOUGLAS, T. DUPONT, H. H. RACHFORD & M. F. WHEELER, "Local H^{-1} Galerkin and adjoint local H^{-1} Galerkin procedures for elliptic equations," *RAIRO Anal. Numer.*, v. 11, 1977, pp. 3–12.

7. P. PERCELL & M. F. WHEELER, "A C^1 finite element collocation method for elliptic equations," *SIAM J. Numer. Anal.*, v. 17, 1980, pp. 605–622.

8. H. H. RACHFORD & M. F. WHEELER, "An H^{-1}-Galerkin procedure for the two point boundary value problem," *Mathematical Aspects of Finite Elements in Partial Differential Equations* (C. de Boor, Ed.), Academic Press, New York, 1974, pp. 353–382.

9. M. F. WHEELER, "An elliptic collocation-finite element method with interior penalties," *SIAM J. Numer. Anal.*, v. 15, 1978, pp. 152–161.