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Numerical Analysis of the Exterior Boundary Value Problem for the Time-Harmonic Maxwell Equations by a Boundary Finite Element Method Part 2: The Discrete Problem

By A. Bendali

Abstract. With the help of curved and mixed finite elements, we introduce an approximate surface on which the discrete problem is defined and construct surface currents and charges which approximate the solution of the continuous problem studied in a previous part. We study the existence and uniqueness of the solution of the discrete problem and give estimates for the error between currents, charges, corresponding fields and their calculated approximations.

1. Surface Approximation and Finite Element Spaces.

1.1. *Introduction.* The numerical method to be introduced must be a realistic one. Therefore, one has to consider an approximate surface which can be effectively handled by the computer. In the numerical analysis of the boundary finite element methods (see e.g. [12]), the introduction of a surface approximation resembles numerical integration. So, in order to avoid inessential complications in the proofs and notation, we assume that there is no error coming from numerical quadrature. Nevertheless, the subsequently developed analysis, which takes into account both the approximations of the surface and of the tangent plane, deals with the case where numerical quadratures are used and leads to the same inferences and error estimates.

Considering an approximate surface and using the mixed finite elements introduced by Raviart and Thomas [17] for a domain in the plane, we define mixed finite element approximation of tangent fields to a surface of $\mathbf{R}^3$. A suitable mapping insures the link between the fields defined on the surface and corresponding fields defined on the approximate surface. This leads to a discrete problem which has an easily computable solution.

The numerical analysis of this problem needs a generalization of Thomas' results for the coercive case [19] to that of a "Fredholm alternative" discussed in the first part of this work. To keep the presentation short, we do not introduce an "abstract framework". However, as in the first part, we shall give the study in a form which can be easily adapted to other problems of the same type. In particular, our approach leads to a slight generalization (when one also considers numerical integration) of the Fix and Nicolaides results [5] with the improvement that there is no need to introduce the adjoint problem.

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Here, we retain the notation and the definitions introduced in the first part of this work. We refer to formulas, theorems, etc. of the former part by giving their numbers and specifying that they are given in Part 1. C indicates, as usual, various constants not necessarily the same in all instances.

1.2. *Geometrical Approximation.* Let $\{\omega_i, D_i, \Phi_i\}_{i=1}^{i=p}$ be a "triangulation" of the surface Γ introduced in Part 1; i.e. $\{\omega_i, \Phi_i\}_{i=1}^{i=p}$ is an atlas for Γ and D_i is a closed polygonal domain contained in ω_i such that

$$\Gamma = \bigcup_{i=1}^{i=p} \Phi_i(D_i), \tag{1.1}$$

$$\Phi_i(D_i) \cap \Phi_j(D_j) \text{ for } i \neq j, \text{ is either a common vertex, empty or a common "curvilinear edge".} \tag{1.2}$$

Such a triangulation always exists, since Γ is an oriented compact manifold (cf. [3]). Following Nedelec's ideas (cf. [12]), we introduce a triangulation $\mathscr{T}_*^h = \bigcup_{i=1}^{i=p} \mathscr{T}_i^h$ such that $\mathscr{T}_i^h$, for each $i \in \{1,\dots,p\}$, is a common regular triangulation of D_i, i.e. as used in finite element discretization (cf. e.g. [4]). We partition each D_i into a collection of triangles T satisfying

$$h_T = \text{diameter of } T \leqslant h, \tag{1.3}$$

$$\frac{h_T}{\rho_T} \leqslant c, \qquad c \text{ constant independent of } T \text{ and } h, \tag{1.4}$$

ρ_T is the diameter of the largest circle which can be inscribed in the triangle T. We also assume the following compatibility conditions between the different triangulations $\mathscr{T}_i^h$: if T_1 and T_2 are any triangles belonging respectively to $\mathscr{T}_i^h$ and $\mathscr{T}_j^h$, then if $\partial T_{l,s_l}$ denotes the s_lth edge of the triangle T_l, $\Phi_i(\partial T_{1,s_1}) \cap \Phi_j(\partial T_{2,s_2})$ is either a curvilinear edge, a vertex, or empty for any edge $\partial T_{1,s_1}$ of T_1 and any edge $\partial T_{2,s_2}$ of T_2. We denote this triangulation by $\mathscr{T}_*^h$ because it is just used to define a triangulation $\mathscr{T}^h$ which will be the basis of subsequent constructions and which is defined as follows. Let Φ_i^T, $i = 1,\dots,p$, be the linear interpolant of Φ_i on T lying in D_i. Then, at least for h small enough, the juxtaposition $\mathscr{T}^h$ of the triangles

$$K = \Phi_i^T(T), \qquad T \in \mathscr{T}_i^h, \; i = 1,\dots,p, \tag{1.5}$$

defines a polyhedral surface denoted by $\tilde{\Gamma}_h$. $\tilde{\Gamma}_h$ is a closed Lipschitzian surface in the sense of Nečas [11].

For the lowest-order geometrical approximation, we take $\tilde{\Gamma}_h$ as the approximate surface Γ_h. For closer approximations, we proceed as follows. Fix an integer $l > 1$. For all $K \in \mathscr{T}^h$, we introduce a C^0 finite element $\{K, \Sigma_K, \mathbf{P}_l\}$ of Lagrange type (cf. e.g. [4]). In the sequel, $\mathbf{P}_l$ will denote the space of complex coefficient polynomials of two variables of degree l or less. Σ_K is the corresponding set of degrees of freedom leading to a finite element of class C^0. Σ_K actually consists of the values taken by p in $\mathbf{P}_l$ on a finite set N_l of nodes contained in K. Let us denote by ψ the orthogonal projection defined in a neighborhood of Γ (cf. e.g. [12]). Using the fact once again that h can be taken sufficiently small, we can assume that ψ establishes a bijection between $\tilde{\Gamma}_h$ and Γ. Hence, we can consider the mapping F_K, defined for all $K \in \mathscr{T}^h$, as the interpolant of $\psi|_K$ through the finite element previously introduced. The approximate surface Γ_h is then defined by

$$\Gamma_h = \bigcup_{K \in \mathscr{T}^h} F_K(K). \tag{1.6}$$

It is clear, since the different interpolants meet at the edges of $\tilde{\Gamma}_h$, that Γ_h is a closed Lipschitzian surface in the sense of Nečas [11].

Notice that the previous construction keeps $\tilde{\Gamma}_h$ unchanged if $l = 1$. In the sequel, we shall introduce other geometrical approximations and, when needed, we shall recall the error estimates given in [12].

1.3. *Some Finite Element Spaces*. For each triangle K belonging to $\mathscr{T}^h$, we introduce an orthonormal frame $\{O_K, \vec{f}_1(K), \vec{f}_2(K)\}$ of the triangle plane. For an integer $m \geqslant 1$, we consider the following space $\mathbf{D}_m$ introduced by Raviart and Thomas (cf. e.g. [17] and [20]):

$$\begin{cases} \vec{p} \in \mathbf{D}_m \text{ if and only if there exists } q_0, q_1, q_2 \in \mathbf{P}_{m-1}(K) \\ \text{such that } q_0 \text{ homogeneous of degree } (m-1) \\ \vec{p}(\xi) = p^\alpha(\xi)\vec{f}_\alpha(K), \\ p^\alpha(\xi) = q_\alpha(\xi) + \xi^\alpha q_0(\xi); \qquad \alpha = 1,2; \\ \xi = (\xi^1, \xi^2) \quad \text{in the frame } \{O_K, \vec{f}_1(K), \vec{f}_2(K)\}. \end{cases} \tag{1.7}$$

With the above space, we associate the following set of degrees of freedom

$$\int_{\partial K_j} (\vec{p}, \vec{\nu}_j) w \, dl; \qquad j = 1,2,3 \; \forall w \in \mathbf{P}_{m-1}(\partial K_j), \tag{1.8}$$

where $\vec{\nu}_j$ denotes the unit normal to the edge ∂K_j outwardly directed to the triangle K, for $j = 1,2,3$,

$$\int_K p^\alpha w \, d\xi; \qquad \alpha = 1,2 \; \forall w \in \mathbf{P}_{m-2}(K) \tag{1.9}$$

(this latter condition is omitted if $m = 1$).

The dimension of $\mathbf{D}_m$ is $m(m+2)$. The space $\mathbf{D}_m$, the degrees of freedom (1.8) and (1.9), and the domain K constitute a (mixed) finite element; see [20] or [17].

The essential generalization will be to suitably define a corresponding finite element over Γ or Γ_h. In this section, we only define finite element spaces over Γ. The corresponding definitions over Γ_h will follow later and will be related to suitable approximations of the sesquilinear and linear forms which are used to formulate the problem. To this end, we recall some differential geometry notation (cf. [3]).

The parametrization of the curvilinear triangle

$$\tilde{K} = \psi \circ F_K(K) \tag{1.10}$$

of Γ leads to the definition of the basis of the tangent vectors to Γ

$$\vec{e}_\alpha = \frac{\partial}{\partial \xi^\alpha}(\psi \circ F_K) \tag{1.11}$$

which provides the expression of such vectors from their contravariant components. The Riemannian metric is defined through the tensor

$$g_{\alpha\beta} = \left(\vec{e}_\alpha, \vec{e}_\beta\right). \tag{1.12}$$

The determinant $g = \det(g_{\alpha\beta})$ of this tensor leads to the 2-form τ, having $\sqrt{g}$ as strict component and giving the surface measure in a neighborhood of $\tilde{K}$.

Now, let φ and $\vec{v}$ be respectively a C^∞ function and a C^∞ tangent vector to Γ. $\vec{v} = v^\alpha \vec{e}_\alpha$ in local coordinates near $\tilde{K}$. Stokes' formula yields

$$\int_{\tilde{K}} \operatorname{div}_\Gamma(\varphi\vec{v})\tau = \int_{\partial\tilde{K}} i(\varphi\vec{v})\tau, \tag{1.13}$$

where $i(\varphi\vec{v})\tau$ is the 1-form defined as the interior product of the form τ and the vector $\varphi\vec{v}$. The covariant components of $i(\varphi\vec{v})\tau$ are given by

$$\left(i(\varphi\vec{v})\tau\right)_\beta = \varphi v^\alpha \varepsilon^{12}_{\alpha\beta}\sqrt{g}, \tag{1.14}$$

where $\varepsilon^{\gamma\delta}_{\alpha\beta}$ is the Kronecker tensor (cf. [3]).

Returning to less general but more familar notation, the previous equality (1.13) may be written

$$\int_{\tilde{K}} \{\bar{\varphi} \operatorname{div}_\Gamma \vec{v} + (\vec{v}, \operatorname{grad}_\Gamma \varphi)\} d\gamma = \sum_{j=1}^{3} \int_{\partial K_j} \bar{\varphi}\left(R_K \vec{v}, \vec{\nu}_j\right) dl, \tag{1.15}$$

where $\operatorname{grad}_\Gamma \varphi$ is the surface gradient of φ and $d\gamma$ is the surface measure of Γ. $R_K\vec{v}$ is a vector in the triangle K plane given by

$$R_K \vec{v} = \sqrt{g}\, v^\alpha(\xi)\, \vec{f}_\alpha(K). \tag{1.16}$$

This review makes clear the following definition of the finite element approximation X_h of the space X introduced in the previous part:

$$\begin{cases} \vec{p} \in X_h \quad \text{if, and only if, } R_K \vec{p} \in \mathbf{D}_m \ \forall K \in \mathcal{T}^h, \\ \left(R_{K_1}\vec{p}, \vec{\nu}_{K_{1,s_1}}\right) + \left(R_{K_2}\vec{p}, \vec{\nu}_{K_{2,s_2}}\right) = 0, \\ \text{on each curvilinear edge } \partial\tilde{K} = \left(\psi \circ F_{K_j}\right)\left(\partial K_{j,s_j}\right), \qquad j = 1, 2. \end{cases} \tag{1.17}$$

We have written $\vec{\nu}_{K_{j,s_j}}$ for the unit normal to the s_jth edge $\partial K_{j,s_j}$ of K_j outwardly directed to K_j. The last equality in (1.17) is defined as that of two polynomials of the real variable t defining the usual parametrization of the edge $\partial K_{1,s_1} \equiv \partial K_{2,s_2}$.

Clearly, denoting by φ a C^∞ function defined on Γ and by $\vec{v}$ a tangent vector field to Γ which belongs to X_h, definitions (1.13) and (1.14) yield

$$\int_\Gamma \bar{\varphi} \operatorname{div}_\Gamma \vec{p}\, d\gamma + \int_\Gamma (\vec{p}, \operatorname{grad}_\Gamma \varphi) d\gamma = 0, \tag{1.18}$$

where $\operatorname{div}_\Gamma \vec{p}$ is a function lying in the finite-dimensional space

$$W_h = \left\{\lambda \in L^2(\Gamma); \lambda|_{\tilde{K}} = \frac{1}{\sqrt{g}} q(\xi); q(\xi) \in \mathbf{P}_{m-1}(K)\ \forall K \in \mathcal{T}^h\right\}. \tag{1.19}$$

If we choose $\varphi \equiv 1$ in (1.18), we see that $\operatorname{div}_\Gamma \vec{p}$ actually belongs to the subspace M_h of M (cf. Part 1 for the definition of M) defined by (cf. (1.19))

$$M_h = \left\{\lambda \in W_h;\ \sum_{K \in \mathcal{T}^h} \int_K q(\xi) d\xi = 0\right\}. \tag{1.20}$$

We also need to describe a subspace of X_h containing divergence-free elements. To this end, we introduce another finite element space:

$$S_h = \{ v \in C^0(\Gamma);\, v|_{\hat{K}} \in \mathbf{P}_m(K)\ \forall K \in \mathcal{T}^h \}. \tag{1.21}$$

We have written $v|_{\hat{K}}$ for the function $v \circ \psi$.

It is well known (cf. [13]) that this space is a subspace of $H^1(\Gamma)$ and that, for $v \in S_h$, the tangent vector field to Γ defined by

$$\overrightarrow{\operatorname{curl}}_\Gamma v|_{\hat{K}} = \frac{1}{\sqrt{g}} \varepsilon_{12}^{\alpha\beta} \frac{\partial v}{\partial \xi^\beta} \vec{e}_\alpha \tag{1.22}$$

belongs to X_h and satisfies

$$\operatorname{div}_\Gamma(\overrightarrow{\operatorname{curl}}_\Gamma v) = 0. \tag{1.23}$$

1.4. *Approximation Properties of Finite Element Spaces.* First, we recall Thomas' [19] results on the approximation of plane fields:

PROPOSITION 1.1 (THOMAS). *For all K of $\mathcal{T}^h$ and $s > \frac{1}{2}$, one can define an interpolation operator Π_h: $\{H^s(K)\}^2 \to H(\operatorname{div}, K)$, where*

$$H(\operatorname{div}, K) = \left\{ \vec{p} \in \{L^2(K)\}^2;\ \operatorname{div} \vec{p} \in L^2(K) \right\}. \tag{1.24}$$

For $\vec{p} \in \{H^s(K)\}^2$, $\Pi_h \vec{p}$ is given by

$$\Pi_h \vec{p} \in \mathbf{D}_m; \tag{1.25}$$

$$\int_{\partial K_j} (\Pi_h \vec{p}, \vec{\nu}_j) w\, dl = \int_{\partial K_j} (\vec{p}, \vec{\nu}_j) w\, dl \quad \forall w \in \mathbf{P}_{m-1}, j = 1,2,3; \tag{1.26}$$

$$\int_K (\Pi_h \vec{p})^\alpha w\, d\xi = \int_K p^\alpha w\, d\xi; \qquad \alpha = 1,2\ \forall w \in \mathbf{P}_{m-2}. \tag{1.27}$$

Moreover, we have the estimates: $\frac{1}{2} < s \leqslant m$,

$$\|\vec{p} - \Pi_h \vec{p}\|_{0,K} \leqslant C_s h^s \|\vec{p}\|_{s,K} \quad \forall \vec{p} \in \{H^s(K)\}^2, \tag{1.28}$$

where C_s is a constant independent of h and K.

Proof. The proof is given in Thomas [19] for s an integer. Since K has the strong m-extension property, the proof for real s, $1 \leqslant s \leqslant m$, follows from interpolation theory (cf. [10]) which gives intermediate estimates between those obtained in the cases where $s = 1$ and $s = m$ (cf. [1]). The proof for $\frac{1}{2} < s < 1$ has not yet been published. Therefore, we reproduce it below.

Clearly, trace theorems (cf. [10]) show that the interpolation operator Π_h is well defined. Then, we have to estimate

$$\|\vec{p} - \Pi_h \vec{p}\|_{0,K}^2 = \int_K |\vec{p} - \Pi_h \vec{p}|^2\, d\xi. \tag{1.29}$$

Consider the affine mapping defined by

$$\xi = B\hat{\xi} + b, \tag{1.30}$$

which maps the unit reference triangle $\hat{K}$ (cf. [4]) into the triangle K:

$$\hat{K} = \{ \xi \in \mathbf{R}^2;\ \xi_1, \xi_2 \geqslant 0;\ \xi_1 + \xi_2 \leqslant 1 \}. \tag{1.31}$$

Then, we define

$$J = \det B, \tag{1.32}$$

$$\hat{\vec{p}}(\xi) = JB^{-1}\vec{p}(\xi). \tag{1.33}$$

A linear variable change yields

$$\|\vec{p} - \Pi_h \vec{p}\|_{0,K}^2 = \frac{1}{J}\|B\|^2 \|\hat{\vec{p}} - \hat{\Pi}\hat{\vec{p}}\|_{0,\hat{K}}^2, \tag{1.34}$$

where $\hat{\Pi}$ is the interpolation operator over $\hat{K}$ built in the same way as Π_h.

The Bramble-Hilbert Lemma leads to:

$$\begin{cases} \text{there exists a constant } C_s \text{ depending only on } s \text{ and } m, \text{ such that} \\ \|\hat{\vec{p}} - \hat{\Pi}\hat{\vec{p}}\|_{0,\hat{K}} \leqslant C_s |\hat{\vec{p}}|_{s,\hat{K}}. \end{cases} \tag{1.35}$$

We have used the following notation

$$|\hat{\vec{p}}|_{s,\hat{K}}^2 = \int_{\hat{K}\times\hat{K}} \frac{|\hat{\vec{p}}(x) - \hat{\vec{p}}(y)|^2}{|x-y|^{2+2s}}\, dx\, dy. \tag{1.36}$$

Returning to the previous variables and using the well-known estimates given by regularity assumptions on the mesh (1.3) and (1.4), we obtain

$$\|\vec{p} - \Pi_h \vec{p}\|_{0,K}^2 \leqslant C_s h^{2s} |\vec{p}|_{s,K}^2, \tag{1.37}$$

where C_s is again a constant independent of $\vec{p}$, h and K. This completes the proof of the proposition. □

PROPOSITION 1.2. *For any $s > \frac{1}{2}$, one can define an interpolation operator Π_h from $TH^s(\Gamma)$ into X_h by*

$$R_K(\Pi_h \vec{p}) = \Pi_h R_K \vec{p} \quad \forall K \in \mathcal{T}^h. \tag{1.38}$$

(*We also denote by Π_h the interpolation operator from $TH^s(\Gamma)$ into X_j.*) *For $\frac{1}{2} < s \leqslant m$, the following estimate holds*

$$\|\vec{p} - \Pi_h \vec{p}\|_{0,\Gamma} \leqslant C_s h^s \|\vec{p}\|_{s,\Gamma} \quad \forall \vec{p} \in TH^s(\Gamma). \tag{1.39}$$

Proof. The first part of the proposition is clearly a direct consequence of the previous definitions. Since the partial derivatives of $\psi \circ F_K$ up to the order m are uniformly bounded on K by a constant not depending on K or h, the estimate (1.39) easily follows from (1.28). □

Remark 1.3. Green's formula and the definition of $\Pi_h \vec{p}$ give

$$\int_K \operatorname{div}(R_K \Pi_h \vec{p}) \overline{w}\, d\xi = \int_K \operatorname{div}(R_K \vec{p})\, \overline{w}\, d\xi, \tag{1.40}$$

for all $w \in \mathbf{P}_{m-1}(K)$, all $K \in \mathcal{T}^h$ and all $\vec{p} \in TH^1(\Gamma)$. This may also be written as

$$\int_{\tilde{K}} \overline{w} \operatorname{div}_\Gamma(\Pi_h \vec{p}) d\gamma = \int_{\tilde{K}} \overline{w} \operatorname{div}_\Gamma \vec{p}\, d\gamma \quad \forall w \in \mathbf{P}_{m-1}(K);\ \forall K \in \mathcal{T}^h. \tag{1.41}$$

In other words, defining the projection operator P_h from $L^2(\Gamma)$ onto W_h by

(1.42) $$\int_K \overline{w}(P_h\lambda)\sqrt{g}\,d\xi = \int_K \overline{w}\,\lambda\sqrt{g}\,d\xi \quad \forall w \in \mathbf{P}_{m-1}(K);\ \forall K \in \mathcal{T}^h,$$

we can write

(1.43) $$P_h \operatorname{div}_\Gamma \vec{p} = \operatorname{div}_\Gamma(\Pi_h \vec{p}) \quad \text{for all } \vec{p} \in TH^s(\Gamma);\ s \geqslant 1.$$

Moreover, we have

(1.44) $$P_h \lambda \in M_h \quad \text{for all } \lambda \in M \cap L^2(\Gamma).$$

The arguments developed by Nedelec [12] yield

(1.45) $$\begin{cases} \text{for } 0 \leqslant s, t \leqslant m, \text{ there exists a constant } C \text{ independent of } h: \\ \|\lambda - P_h \lambda\|_{-s,\Gamma} \leqslant Ch^{s+t}\|\lambda\|_{t,\Gamma} \quad \forall \lambda \in H^t(\Gamma). \end{cases}$$

Hence, it results from (1.43) that

(1.46) $$\begin{cases} \text{for } 0 \leqslant s, t \leqslant m, \text{ there exists a constant } C \text{ independent of } h: \\ \|\operatorname{div}_\Gamma \vec{p} - \operatorname{div}_\Gamma(\Pi_h \vec{p})\|_{-s,\Gamma} \leqslant Ch^{s+t}\|\vec{p}\|_{t+1,\Gamma} \quad \forall \vec{p} \in TH^{t+1}(\Gamma). \end{cases}$$

We shall also use the so-called inverse inequalities:

Let $0 \leqslant s \leqslant t \leqslant 1$, there exists a constant C such that

(1.47) $$\|\lambda\|_{-s,\Gamma} \leqslant \frac{c}{h^{t-s}}\|\lambda\|_{-t,\Gamma} \quad \forall \lambda \in W_h,$$

(1.48) $$\|\vec{p}\|_{-s,\Gamma} \leqslant \frac{c}{h^{t-s}}\|\vec{p}\|_{-t,\Gamma} \quad \forall \vec{p} \in X_h.$$

These latter estimates can be obtained in the same way as in [12], using the regularity property of the mesh (1.4). □

The operator C (cf. (2.42) in Part 1) is a bounded mapping from $H^t(\Gamma)$ into $H^{t+1}(\Gamma)$ for all real t. Hence, for $t \geqslant 0$ and $\vec{p} \in TH^t(\Gamma)$, we can define the affine subspace of X_h:

(1.49) $$V_h(\vec{p}) = \{\vec{q} \in X_h;\ b(\nu, \vec{q}) = b(\nu, \vec{p});\ \forall \nu \in M_h\}$$

(cf. Part 1 for the definition of the sesquilinear form b. We recall that $b(\nu, \vec{p}) = \langle C\nu, div_\Gamma \vec{p}\rangle$).

The subsequent error estimates need a bound of the quantity $\inf_{\vec{q} \in V_h(\vec{p})}\|\vec{p} - \vec{q}\|_{-s,\Gamma}$ for some values of s.

First, we treat the case $s = 0$. To this end, we introduce the following problem:

(1.50) $$\begin{cases} \text{Find } \vec{p}_h \in X_h, \lambda_h \in M_h, \text{ such that} \\ (\vec{p}_h - \vec{p}, \vec{q}) + b(\lambda_h, \vec{q}) = 0 \quad \forall \vec{q} \in X_h,^* \\ b(\nu, \vec{p}_h - \vec{p}) = 0 \quad \forall \lambda \in M_h. \end{cases}$$

$\vec{p}_h$ is actually the orthogonal projection of $\vec{p}$ over the affine subspace $V_h(\vec{p})$; λ_h plays the role of the associated Lagrange multiplier.

THEOREM 1.4. *The problem* (1.50) *has one and only one solution* $(\vec{p}_h, \lambda_h)$. *Moreover, for* $0 \leqslant t \leqslant m$ *and* $0 < \sigma \leqslant \frac{1}{2}$, *we have*

(1.51) $$\|\vec{p} - \vec{p}_h\|_{0,\Gamma} \leqslant Ch^{t-1/2-\sigma}\|\vec{p}\|_{t,\Gamma},$$

where C is a constant independent of h and of $\vec{p} \in TH^t(\Gamma)$.

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* We have denoted by $(\vec{p}, \vec{q})$ the scalar product of $\vec{p}$ and $\vec{q}$ lying in the Hilbert space $TH^0(\Gamma)$.

Proof. It will be carried out in several steps.

Step 1: *inf-sup condition*. Fix λ in M_h. Classical results on elliptic problems and standard properties of the Laplace-Beltrami operators Δ_Γ on Γ lead to: there exists one and only one $w \in H^2(\Gamma) \cap M$ such that

$$\Delta_\Gamma w = \operatorname{div}_\Gamma(\operatorname{grad}_\Gamma w) = \lambda \quad \text{in } \mathscr{D}'(\Gamma), \tag{1.52}$$

and for σ in the interval $0 \leqslant \sigma \leqslant \frac{1}{2}$, w satisfies the estimate

$$\|w\|_{3/2+\sigma,\Gamma} \leqslant C\|\lambda\|_{-1/2+\sigma}, \tag{1.53}$$

where C is a constant only depending on σ and Γ; cf. [21].

We set $\vec{p} = \operatorname{grad}_\Gamma w$. The interpolant $\Pi_h \vec{p}$ of $\vec{p}$ is then well defined and, in view of (1.43), satisfies

$$\operatorname{div}_\Gamma(\Pi_h \vec{p}) = \lambda. \tag{1.54}$$

The coercivity of the operator C (cf. (2.43) in Part 1) yields

$$b(\lambda, \Pi_h \vec{p}) = \langle C\lambda, \lambda \rangle \geqslant \alpha\|\lambda\|^2_{-1/2,\Gamma}, \tag{1.55}$$

where $\alpha > 0$ only depends on Γ.

The estimate (1.28) and the inverse inequality (1.47) give

$$\sup_{\vec{q} \in X_h} \left\{ \frac{1}{\|\vec{q}\|_{0,\Gamma}} |b(\lambda, \vec{q})| \right\} \geqslant \alpha h^{1/2+\sigma}\|\lambda\|_{0,\Gamma}, \tag{1.56}$$

where again $\alpha > 0$ depends on $0 < \sigma \leqslant \frac{1}{2}$ and on the surface Γ.

Step 2: *the case* $t = 0$. Since X_h and M_h are finite-dimensional spaces, Brezzi's theorem (cf. [2]), by (1.56), insures the existence and the uniqueness of the solution of the problem (1.50). Moreover, we have the following estimate:

$$\alpha h^{1/2+\sigma}\|\lambda_h\|_{0,\Gamma} \leqslant \|\vec{p} - \vec{p}_h\|_{0,\Gamma}. \tag{1.57}$$

Therefore, since $\vec{p}_h \in V_h(\vec{p})$,

$$\|\vec{p}_h\|^2_{0,\Gamma} \leqslant \|\vec{p}\|_{0,\Gamma}\|\vec{p}_h\|_{0,\Gamma} + Ch^{-1/2-\sigma}\|\vec{p}\|_{0,\Gamma}\|\vec{p} - \vec{p}_h\|_{0,\Gamma}. \tag{1.58}$$

The estimate (1.51), for $t = 0$, is then an easy consequence of the inequality

$$|ab| \leqslant \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2 \quad \text{for all } a, b \in \mathbf{R} \text{ and } \varepsilon > 0. \tag{1.59}$$

Step 3: *the general case*. Let $\vec{p} \in TH^m(\Gamma)$. The solution $(\vec{p}_h, \lambda_h)$ of problem (1.50) satisfies

$$\begin{cases} (\vec{p}_h - \Pi_h \vec{p}, \vec{q}) + b(\lambda_h, \vec{q}) = (\vec{p} - \Pi_h \vec{p}, \vec{q}) & \forall \vec{q} \in X_h, \\ b(\nu, \vec{p}_h - \Pi_h \vec{p}) = b(\nu, \vec{p} - \Pi_h \vec{p}) & \forall \nu \in M_h. \end{cases} \tag{1.60}$$

Following the lines of the proof given in the previous step and using the estimates (1.39) and (1.46), we obtain (1.51) for $t = m$. The intermediate case results from the standard techniques of interpolation between two estimates (cf. [10]). □

In order to deal with the case $s < 0$, we prove the following lemma:

LEMMA 1.5. *For $\frac{1}{2} \leqslant t \leqslant m+1$ and $-\frac{1}{2} \leqslant s \leqslant m$, there exists a constant C, independent of h and of $\vec{p} \in TH^t(\Gamma)$, such that the solution $(\vec{p}_h, \lambda_h)$ of the problem (1.50) satisfies*

$$\|\operatorname{div}_\Gamma \vec{p}_h - \operatorname{div}_\Gamma \vec{p}\|_{-s-1,\Gamma} \leqslant Ch^{s+t}\|\vec{p}\|_{t,\Gamma}. \tag{1.61}$$

Proof. The sesquilinear form defined for λ and $\mu \in H^{-1/2}(\Gamma)$ by $\langle C\lambda, \mu\rangle$, C being the previously introduced operator, is a scalar product on $H^{-1/2}(\Gamma)$. Denote by π_h the orthogonal projection operator from $H^{-1/2}(\Gamma)$ onto W_h associated with this scalar product. The orthogonal projection with respect to this scalar product also leads to an operator σ_h acting from W_h onto M_h. Standard techniques which now are well-known (cf. [8], [12]) lead to the estimates

$$\begin{cases} \text{for } -\frac{1}{2} \leqslant s \leqslant m,\ \frac{1}{2} \leqslant t \leqslant m+1, \\ \|\lambda - \pi_h\lambda\|_{-s-1,\Gamma} \leqslant Ch^{s+t}\|\lambda\|_{t-1,\Gamma}, \end{cases} \tag{1.62}$$

where C is a constant independent of h and of $\lambda \in H^{t-1}(\Gamma)$.

Let h be fixed. Consider the function $\mu_0^h \in W_h$ defined in local coordinates by

$$\sqrt{g}\,\mu_0^h = 1 \quad \text{on each } K \in \mathcal{T}^h. \tag{1.63}$$

It follows that

$$\int_\Gamma \mu_0^h\, d\gamma = \sum_{K\in\mathcal{T}^h} \int_K d\xi \geqslant C_1 > 0, \tag{1.64}$$

and also

$$\int_\Gamma |\mu_0^h|^2\, d\gamma = \sum_{K\in\mathcal{T}^h} \int_K \frac{1}{\sqrt{g}} d\xi \leqslant C_2, \tag{1.65}$$

where C_1 and C_2 are constants independent of h.

The function μ_0^h is used to write

$$\sigma_h\pi_h\lambda = \pi_h\lambda + \frac{\langle \pi_h\lambda, 1\rangle}{(\mu_0^h, 1)}\left(\sigma_h\mu_0^h - \mu_0^h\right). \tag{1.66}$$

Using (1.64), we arrive at

$$\|\sigma_h\pi_h\lambda - \lambda\|_{-s-1,\Gamma} \leqslant \|\pi_h\lambda - \lambda\|_{-s-1,\Gamma} + C|\langle \pi_h\lambda, 1\rangle|\,\|\sigma_h\mu_0^h - \mu_0^h\|_{-s-1,\Gamma}. \tag{1.67}$$

Let $\lambda \in M$. It follows that

$$|\langle \pi_h\lambda, 1\rangle| = |\langle \pi_h\lambda - \lambda, 1\rangle| \leqslant C\|\pi_h\lambda - \lambda\|_{-m-1,\Gamma}. \tag{1.68}$$

Therefore, using (1.62), (1.65), (1.68), we easily obtain

$$\|\sigma_h\pi_h\lambda - \lambda\|_{-s-1,\Gamma} \leqslant Ch^{s+t}\|\lambda\|_{t-1,\Gamma}, \tag{1.69}$$

where C is a constant independent of h and of $\lambda \in H^{t-1}(\Gamma) \cap M$.

To complete the proof, we have only to remark that

$$\operatorname{div}_\Gamma \vec{p}_h = \sigma_h\pi_h \operatorname{div}_\Gamma \vec{p}. \qquad \square \tag{1.70}$$

Following the lines of the proof given in [12] for $s = -\frac{1}{2}$, one can establish that the orthogonal projection operator

$$s_h \colon L^2(\Gamma) \to S_h \tag{1.71}$$

satisfies the estimates

$$\begin{cases} \text{for } -(m+1) \leqslant s \leqslant 1;\ 0 \leqslant t \leqslant m+1 \text{ and } s \leqslant t, \\ \|s_h v - v\|_{s,\Gamma} \leqslant C h^{t-s} \|v\|_{t,\Gamma} \quad \forall v \in H^t(\Gamma). \end{cases} \tag{1.72}$$

Using the fact that a tangential derivative defines a bounded mapping from $H^s(\Gamma)$ into $H^{s-1}(\Gamma)$ for all real s, we immediately deduce

$$\begin{cases} \text{Let } s \text{ and } t \text{ satisfy the conditions of } (1.72); \\ \|\overrightarrow{\operatorname{curl}}_\Gamma s_h v - \overrightarrow{\operatorname{curl}}_\Gamma v\|_{s-1,\Gamma} \leqslant C h^{t-s} \|v\|_{t,\Gamma} \quad \forall v \in H^t(\Gamma). \end{cases} \tag{1.73}$$

We have now laid the groundwork for the following theorem:

THEOREM 1.6. *Let there be given s, t and σ satisfying $\frac{1}{2} \leqslant t \leqslant m$; $0 < s \leqslant m$ and $0 < \sigma \leqslant \frac{1}{2}$. Then, there exists a constant C independent of h and of $\vec{p} \in TH^t(\Gamma)$ such that, if $(\vec{p}_h, \lambda_h)$ denotes the solution of the problem* (1.50), *we have*

$$\|\vec{p} - \vec{p}_h\|_{-s,\Gamma} \leqslant C h^{s+t-1/2-\sigma} \|\vec{p}\|_{t,\Gamma}. \tag{1.74}$$

Proof. The proof is based on the following Hodge decomposition (cf. e.g. [21]). The mapping from $\{H^{s+1}(\Gamma) \cap M\}^2 \times \mathbf{C}^{2n}$ into $TH^s(\Gamma)$, which assigns to each $(u, v, \{\alpha_i\}_{i=1}^{i=2n})$ in the former space the tangent vector $\vec{p}$ to Γ, defined by

$$\vec{p} = \operatorname{grad}_\Gamma u + \overrightarrow{\operatorname{curl}}_\Gamma v + \sum_{i=1}^{2n} \alpha_i \vec{\theta}_i, \tag{1.75}$$

is an (algebraic and topological) isomorphism. The surface Γ is supposed to have n "holes". We denote by $\{\vec{\theta}_i\}_{i=1}^{i=2n}$ a basis of the harmonic 1-forms of Γ. Therefore, for $s \geqslant 0$, there exists a constant C not depending on $\vec{p} \in TH^t(\Gamma)$ such that

$$\begin{aligned} \|\vec{p} - \vec{p}_h\|_{-s,\Gamma} &\leqslant C \left\{ \sup_{u \in M \cap H^{s+1}(\Gamma)} \frac{|(\vec{p} - \vec{p}_h, \operatorname{grad}_\Gamma u)|}{\|u\|_{s+1,\Gamma}} \right. \\ &\quad \left. + \sup_{v \in M \cap H^{s+1}(\Gamma)} \frac{|(\vec{p} - \vec{p}_h, \overrightarrow{\operatorname{curl}}_\Gamma v)|}{\|v\|_{s+1,\Gamma}} + \sum_{i=1}^{2n} (\vec{p} - \vec{p}_h, \vec{\theta}^i) \right\}. \end{aligned} \tag{1.76}$$

Integration by parts yields

$$\begin{aligned} |(\vec{p} - \vec{p}_h, \operatorname{grad}_\Gamma u)| &= |\langle \operatorname{div}_\Gamma \vec{p} - \operatorname{div}_\Gamma \vec{p}_h, u \rangle| \\ &\leqslant \|\operatorname{div}_\Gamma \vec{p} - \operatorname{div}_\Gamma \vec{p}_h\|_{-s-1,\Gamma} \|u\|_{s+1,\Gamma}. \end{aligned} \tag{1.77}$$

$\operatorname{div}_\Gamma(\overrightarrow{\operatorname{curl}}_\Gamma s_h v) = \operatorname{div}_\Gamma \Pi_h \vec{\theta}^i = 0$ leads to

$$\begin{aligned} |(\vec{p} - \vec{p}_h, \overrightarrow{\operatorname{curl}}_\Gamma v)| &= |(\vec{p} - \vec{p}_h, \overrightarrow{\operatorname{curl}}_\Gamma v - \overrightarrow{\operatorname{curl}}_\Gamma s_h v)| \\ &\leqslant C h^s \|\vec{p} - \vec{p}_h\|_{0,\Gamma} \|v\|_{s+1,\Gamma}, \end{aligned} \tag{1.78}$$

$$|(\vec{p} - \vec{p}_h, \vec{\theta}^i)| = |(\vec{p} - \vec{p}_h, \vec{\theta}^i - \Pi_h \vec{\theta}^i)| \leqslant C h^m \|\vec{p} - \vec{p}_h\|_{0,\Gamma}. \tag{1.79}$$

The end of the proof is then achieved by (1.51) and (1.61). □

2. The Discrete Problem. In this section, we set up the discrete problem and show it to be well-posed (under the assumption that k^2 is not an eigenvalue of the interior problem). A realistic way to discretize the equations is to define the problem on the approximate surface. Moreover, in the present case, we are faced with sesquilinear forms (for instance, a and r) which not only involve the surface but also its tangent plane. The natural way of doing the approximation consists of replacing the tangent plane to the surface Γ by the tangent plane to the approximate surface Γ_h. This consists in replacing the tangent plane basis $\vec{e}_\alpha$ (cf. (1.11)) by $\partial F_K/\partial \xi^\alpha$ (cf. (1.5)). The following analysis shows that, in this case, there is a loss of one convergence order. In the case of the lowest-order method (i.e. $l = m = 1$), which is the method proposed by Rao et al. [16], there is even danger of loss of consistency. In order to overcome this defect, we propose the following treatment. For all $K \in \mathscr{T}^h$, a Lagrange finite element $\{K, \tilde{\Sigma}_K, \mathbf{P}_{l+1}\}$ of order $l+1$ is introduced. If $\tilde{\rho}_K$ denotes the interpolation operator associated with this finite element, the approximation of the basis $\vec{e}_\alpha$ is carried out as follows:

$$\vec{e}^{\,h}_\alpha = \frac{\partial}{\partial \xi^\alpha} \tilde{\rho}_K(\psi \circ F_K). \tag{2.1}$$

This basis can be computed using only the coordinates of the surface points

$$\psi \circ F_K(\xi) \quad \forall \xi \in \tilde{N}_K, \tag{2.2}$$

where $\tilde{N}_K$ is the set of nodes defining $\tilde{\Sigma}_K$.

For instance, if the surface is approximated by the juxtaposition of planar triangles, the vectors $\vec{e}^{\,h}_\alpha$ are calculated from the coordinates of the vertices and the orthogonal projections of the midpoint of the edges.

We give the study only in the case where approximation (2.1) is used. In the case where $l > 1$ and $\vec{e}^{\,h}_\alpha = \partial F_K/\partial \xi^\alpha$, the same analysis leads to similar results with a loss of one convergence order in the geometrical (consistency) error.

2.1. *Formulation of the Discrete Problem.* We now introduce approximate sesquilinear forms a_h, b_h, etc., of those a, b, etc., defining the continuous problem. This will be achieved by suitably relating finite element spaces defined in the previous section to finite element spaces defined on the approximate surface Γ_h. We recall that, in order to simplify the notation, we do not distinguish between an element defined on the surface Γ and the corresponding one defined on the approximate surface Γ_h through a suitable mapping.

Let $\vec{p} \in X_h$ be given in local coordinates by

$$\vec{p}\,|_{\tilde{K}} = \frac{1}{\sqrt{g}}\, p^\alpha \vec{e}_\alpha, \tag{2.3}$$

where $R_K \vec{p} = p^\alpha \vec{f}_\alpha(K)$ (cf. (1.16)) lies in $\mathbf{D}_m$. Then, the corresponding vector is defined on Γ_h by

$$\vec{p}\,|_{\tilde{K}_h} = \frac{1}{\sqrt{g_h}}\, p^\alpha \vec{e}^{\,h}_\alpha. \tag{2.4}$$

We have written

$$\tilde{K}_h = F_K(K) \tag{2.5}$$

and $\sqrt{g_h}$ for the strict component of the 2-form defining the surface measure associated with the parametrization (2.5) of $\tilde{K}_h$.

In the same way, if $\lambda \in W_h$ is given in local coordinates by

$$\lambda|_{\tilde{K}} = \frac{1}{\sqrt{g}} r; \qquad r \in \mathbf{P}_{m-1}(K), \tag{2.6}$$

the associated function is defined on Γ_h by

$$\lambda|_{\tilde{K}_h} = \frac{1}{\sqrt{g_h}} r. \tag{2.7}$$

It must be emphasized that the vector $\vec{p}$ defined in (2.4) is not a tangent vector to Γ_h. Its surface divergence on Γ_h is defined by

$$\operatorname{div}_{\Gamma_h} \vec{p}|_{\tilde{K}_h} = \frac{1}{\sqrt{g_h}} \frac{\partial p^\alpha}{\partial \xi^\alpha}. \tag{2.8}$$

We can then introduce approximate sesquilinear forms by

$$a_h(\vec{p}, \vec{q}) = \int_{\Gamma_h \times \Gamma_h} G_0(x, y)(\vec{p}(x), \vec{q}(y)) d\gamma_h(x) d\gamma_h(y); \tag{2.9}$$

$$b_h(\lambda, \vec{p}) = \int_{\Gamma_h \times \Gamma_h} G_0(x, y)\lambda(x) \overline{\operatorname{div}_{\Gamma_h} \vec{p}(y)}\, d\gamma_h(x) d\gamma_h(y), \tag{2.10}$$

for $\vec{p} \in X_h$ and $\lambda_h \in W_h$. We have written $d\gamma_h$ for the surface measure of Γ_h.

The other sesquilinear forms r, $\tilde{a} = a + r$, etc., are obtained in the same way.

Remark 2.1. The sesquilinear forms (2.9) and (2.10) actually have the following expressions (cf. (2.3),...,(2.8)):

$$a_h(\vec{p}, \vec{q}) = \sum_{K,T \in \mathcal{T}^h} \int_{K \times T} G_0(x_h, y_h) p^\alpha(\xi) \overline{q^\beta(\eta)}\, (\vec{e}_\alpha^{\,h}(\xi), \vec{e}_\alpha^{\,h}(\eta)) d\xi\, d\eta, \tag{2.11}$$

$$b_h(\lambda, \vec{p}) = \sum_{K,T \in \mathcal{T}^h} \int_{K \times T} G_0(x_h, y_y) r(\xi) \overline{\frac{\partial p^\alpha}{\partial \eta^\alpha}}(\eta) d\xi\, d\eta. \tag{2.12}$$

We have written

$$x_h = F_K(\xi); \qquad y_h = F_T(\eta), \tag{2.13}$$

$$\vec{q}|_{\tilde{T}} = \frac{1}{\sqrt{g}} q^\beta \vec{e}_\beta. \tag{2.14}$$

We remark that the above expressions only involve integrations of polynomials weighted by the kernel G_0 over planar triangles. □

We also denote by $\vec{c}$ a field, defined around Γ, having as tangential trace on Γ the previously introduced field $\vec{c}$. Given $\vec{q} \in X_h$, the following integral can be seen through the duality pairing as an element $\vec{c}_h$ approximating $\vec{c}$ in H'

$$\langle \vec{c}_h, \vec{q} \rangle = \int_{\Gamma_h} (\vec{c}, \vec{q}) d\gamma_h. \tag{2.15}$$

The discrete problem can now be stated:

$$\begin{cases} \text{find } (\vec{p}_h, \lambda_h) \in X_h \times M_h \text{ such that} \\ \tilde{a}_h(\vec{p}_h, \vec{q}) + \tilde{b}_h(\lambda_h, \vec{q}) = \langle \vec{c}_h, \vec{q} \rangle \quad \forall \vec{q} \in X_h; \\ \overline{b_h(\nu, \vec{p}_h)} + c_h(\lambda_h, \nu) = 0 \quad \forall \nu \in M_h. \end{cases} \tag{2.16}$$

Indeed, we shall see below that any solution $(\vec{p}_h, \lambda_h)$ of this problem satisfies

$$\operatorname{div}_\Gamma \vec{p}_h + k^2\lambda_h = 0 \quad \text{on } \Gamma. \tag{2.17}$$

Thus, the problem which is to be numerically solved is

$$\begin{cases} \text{find } \vec{p}_h \in X_h \text{ such that} \\ \displaystyle\int_{\Gamma_h\times\Gamma_h} G(x, y)\Big\{-\frac{1}{k^2}\operatorname{div}_{\Gamma_h}\vec{p}_h(x)\,\overline{\operatorname{div}_{\Gamma_h}\vec{q}(x)} \\ \qquad\qquad + (\vec{p}_h(x), \vec{q}(y))\Big\}d\gamma_h(x)d\gamma_h(y) \\ \quad = \displaystyle\int_{\Gamma_h}(\vec{c}, \vec{q})d\gamma_h \quad \forall\vec{q} \in X_h. \end{cases} \tag{2.18}$$

The actual form of the above problem which can be handled by the computer is the linear system

$$[Z][I] = [U], \tag{2.19}$$

where $[Z]$ is a complex symmetric (but not Hermitian) matrix. The symmetry of $[Z]$ indeed results from a general principle known in electromagnetism as the reciprocity principle (cf. [7] and [22] for more details).

THEOREM 2.1. *Consistency or geometrical error*: *The approximate sesquilinear forms* a_h, s_h, *etc., satisfy*

$$|a_h(\vec{p}, \vec{q}) - a(\vec{p}, \vec{q})| \leqslant Ch^l\|\vec{p}\|_H\|\vec{q}\|_H, \tag{2.20}$$

$$|s_h(\lambda, \vec{p}) - s(\lambda, \vec{p})| \leqslant Ch^l\|\lambda\|_M\|\vec{p}\|_X, \tag{2.21}$$

where C is a constant independent of h, $\vec{p}, \vec{q} \in X_h$ and $\lambda \in M_h$. (The other forms satisfy identical estimates.)

Proof. We only outline the essential features of the proof which is obtained by the arguments developed in [12].

Variable changes in both the integrals over Γ and Γ_h yield (cf. (2.11) and (2.12)):

$$\begin{aligned} &|a_h(\vec{p}, \vec{q}) - a(\vec{p}, \vec{q})| \\ &\quad = \Big| \sum_{K,T\in\mathscr{T}^h} \int_{K\times T} p^\alpha(\xi)\bar{q}^\beta(\eta)\big\{G_0(x_h, y_h)\big(\vec{e}^{\,h}_\alpha(\xi), \vec{e}^{\,h}_\beta(\eta)\big) \\ &\qquad\qquad - G_0(x, y)\big(\vec{e}_\alpha(\xi), \vec{e}_\beta(\eta)\big)\big\}d\xi\, d\eta \Big|, \end{aligned} \tag{2.22}$$

where $x = \psi\circ F_K(\xi)$, $y = \psi\circ F_T(\eta)$, and

$$\begin{aligned} &|s_h(\lambda, \vec{p}) - s(\lambda, \vec{p})| \\ &\quad = \Big| \sum_{K,T\in\mathscr{T}^h} \int_{K\times T} r(\xi)\,\overline{\frac{\partial p^\alpha}{\partial\eta^\alpha}}(\eta)(S(x_h, y_h) - S(x, y))d\xi\, d\eta \Big|, \end{aligned} \tag{2.23}$$

where $S(x, y) = (e^{ik|x-y|} - 1)/4\pi|x - y|$.

In [12], it is proved that

$$|G_0(x_h, y_h) - G_0(x, y)| \leqslant C \frac{h^{l+1}}{|x-y|}. \tag{2.24}$$

In the same way, one can prove (cf. [9])

$$|S(x_h, y_h) - S(x, y)| \leqslant Ch^{l+1}. \tag{2.25}$$

The usual finite element estimates of interpolation error give

$$|\vec{e}^{\,h}_\alpha - \vec{e}_\alpha| \leqslant Ch^{l+1}. \tag{2.26}$$

Combining all these estimates, we obtain

$$|a_h(\vec{p}, \vec{q}) - a(\vec{p}, \vec{q})| \leqslant Ch^{l+1}\|\vec{p}\|_{0,\Gamma}\|\vec{q}\|_{0,\Gamma}, \tag{2.27}$$

$$|s_h(\lambda, \vec{p}) - s(\lambda, \vec{p})| \leqslant Ch^{l+1}\|\lambda\|_{0,\Gamma}\|\vec{p}\|_{0,\Gamma}. \tag{2.28}$$

Then, the inverse inequalities (1.47) and (1.48) lead to (2.20) and (2.21). □

Remark 2.3. It must be emphasized that the sesquilinear form s bounded on $M \times H$ can be approximated only in $M \times X$. □

PROPOSITION 2.4. *There exist positive constants h^*, α^*, β^* such that, for $0 < h < h^*$,*

$$a_h(\vec{p}, \vec{p}) \geqslant \alpha^*\|\vec{p}\|^2_H \quad \forall \vec{p} \in X_h, \tag{2.29}$$

$$\sup_{\vec{q} \in X_h} \left\{ \frac{1}{\|\vec{q}\|_X} |b_h(\lambda, \vec{q})| \right\} \geqslant \beta^*\|\lambda\|_M \quad \forall \lambda \in M_h. \tag{2.30}$$

Proof. The coercivity (2.29) follows from that of the sesquilinear form a and the inequality of consistency (2.20).

To establish (2.30), we introduce an operator π_h acting from X into X_h and defined as follows. Take $\vec{q}$ to be fixed in X. Remark 3.3 in Part 1 enables us to construct $\vec{p} \in TH^{1/2}(\Gamma)$ such that

$$\operatorname{div}_\Gamma \vec{p} = \operatorname{div}_\Gamma \vec{q}. \tag{2.31}$$

Now, consider the solution $(\vec{p}_h, \lambda_h)$ of problem (1.50) associated with $\vec{p}$ (i.e. $\vec{p}_h$ is the orthogonal projection of $\vec{p}$ onto the affine subspace $V_h(\vec{p})$ of X_h (cf. (1.49))). We set $\pi_h\vec{q} = \vec{p}_h$. Thus $\pi_h\vec{q}$ satisfies

$$b(\nu, \pi_h\vec{q}) = b(\nu, \vec{q}) \quad \forall \nu \in M_h. \tag{2.32}$$

The estimate (1.74) and the boundedness of the mapping which assigns to each $\vec{q}$ the vector $\vec{p}$, yield

$$\|\pi_h\|_{\mathscr{L}(X, X_h)} \leqslant C, \quad C \text{ constant independent of } h. \tag{2.33}$$

Then, estimate (2.30) easily follows from the consistency properties of the discrete problem (cf. Theorem 2.2) and the "inf-sup" condition satisfied by the exact sesquilinear form b (cf. Lemma 3.2 in Part 1). □

The above proposition insures that Brezzi's conditions are satisfied by the sesquilinear forms a_h and b_h. This enables us to define the operator

$$T_h \in \mathscr{L}(X_h \times M_h, X_h \times M_h)$$

in order to approximate the previously introduced isomorphism

$$T \in \mathscr{L}(H \times M, H \times M)$$

(cf. (3.27) in Part 1).

To each $(\vec{u}, \lambda) \in X_h \times M_h$, we assign

$$T_h(\vec{u}, \lambda) = (\vec{w}_h, \mu_h), \tag{2.34}$$

where $(\vec{w}_h, \mu_h)$ is the unique solution, given by Brezzi's theorem (cf. [2]), of the following problem:

$$\begin{cases} \text{find } (\vec{w}_h, \mu_h) \in X_h \times M_h \text{ such that} \\ a_h(\vec{w}_h, \vec{v}) + b_h(\mu_h, \vec{v}) = \tilde{a}_h(\vec{u}, \vec{v}) + \tilde{b}_h(\lambda, \vec{v}) \quad \forall \vec{v} \in X_h, \\ \overline{b_h(\nu, \vec{w}_h)} = \overline{b_h(\nu, \vec{u})} + c_h(\lambda, \nu) \quad \forall \nu \in M_h. \end{cases} \tag{2.35}$$

It is clear that $(\vec{w}, \mu) = T(\vec{u}, \lambda)$ is the unique solution of

$$\begin{cases} \text{find } (\vec{w}, \mu) \in X \times M \text{ such that} \\ a(\vec{w}, \vec{v}) + b(\mu, \vec{v}) = \tilde{a}(\vec{v}, \vec{v}) + \tilde{b}(\lambda, \vec{v}) \quad \forall \vec{v} \in X, \\ \overline{b(\nu, \vec{w})} = \overline{b(\nu, \vec{u})} + c(\lambda, \nu) \quad \forall \nu \in M. \end{cases} \tag{2.36}$$

In order to use the regularity results established in Part 1, we introduce

$$\vec{p} = \vec{w} - \vec{u}; \qquad \zeta = \mu - \lambda, \tag{2.37}$$

$$\vec{p}_h = \vec{w}_h - \vec{u}; \qquad \zeta_h = \mu_h - \lambda. \tag{2.38}$$

Given that $\vec{u} \in X_h$ and $\lambda \in M_h$, $(\vec{p}, \zeta)$ and $(\vec{p}_h, \zeta_h)$ are the respective solutions of the problems:

$$\begin{cases} \text{find } (\vec{p}, \xi) \in X \times M \text{ such that} \\ a(\vec{p}, \vec{v}) + b(\zeta, \vec{v}) = r(\vec{u}, \vec{v}) + s(\lambda, \vec{v}) \quad \forall \vec{v} \in X, \\ \overline{b(\nu, \vec{p})} = c(\lambda, \nu) \quad \forall \nu \in M, \end{cases} \tag{2.39}$$

and

$$\begin{cases} \text{find } (\vec{p}_h, \zeta_h) \in X_h \times M_h \text{ such that} \\ a_h(\vec{p}_h, \vec{v}) + b_h(\zeta_h, \vec{v}) = r_h(\vec{u}, \vec{v}) + s_h(\lambda, \vec{v}) \quad \forall \vec{v} \in X_h, \\ \overline{b_h(\nu, \vec{p}_h)} = c_h(\lambda, \nu) \quad \forall \nu \in M_h. \end{cases} \tag{2.40}$$

Remark 2.5. The last equation of (2.40) can be written in the explicit form

$$\int_{\Gamma_h \times \Gamma_h} G_0(x, y)\bar{\nu}(y)\big(\operatorname{div}_{\Gamma_h} \vec{p}_h(x) - k^2\lambda(x)\big)d\gamma_h(y)d\gamma_h(x) = 0 \quad \forall \nu \in M_h. \tag{2.41}$$

Theorem 2.2 and the coercivity of the operator C (cf. (3.18) in the previous part of this work) lead to

$$\operatorname{div}_{\Gamma_h} \vec{p}_h = k^2\lambda \quad \text{on } \Gamma_h. \tag{2.42}$$

Thus, the definitions of finite elements on Γ_h and on Γ (cf. (2.1),…,(2.8)) give an equivalent form of the above equation

$$\operatorname{div}_\Gamma \vec{p}_h = k^2\lambda \quad \text{on } \Gamma. \tag{2.43}$$

This, finally, turns into (cf. (1.49))

(2.44) $$\vec{p}_h \in V_h(\vec{p}).$$

In the same way, the "discrete conservation law" (2.17) yielding the effective resolution of the discrete problem is established. We set

(2.45) $$V_h = \{\vec{q} \in X_h;\, b(\nu, \vec{q}) = 0\ \forall \nu \in M_h\}.$$

The coercivity of the operator C (cf. (2.43) in Part 1) yields

(2.46) V_h is a subspace of V. □

Let us denote

(2.47) $$\vec{e}_h = \vec{p}_h - \vec{p} = \vec{w}_h - \vec{w}.$$

(2.48) $$\varepsilon_h = \zeta_h - \zeta = \mu_h - \mu.$$

Following Thomas' argumentation [19], we get

(2.49) $$\|\vec{e}_h\|_H \leqslant C \operatorname*{Inf}_{\vec{q} \in V_h(\vec{p})} \{\|\vec{p} - \vec{q}\|_H + h^l(\|\vec{q}\|_H + \|\vec{u}\|_H)\},$$

(2.50) $$\|\varepsilon_h\|_M \leqslant C \operatorname*{Inf}_{\tau \in M_h} \{\|\zeta - \tau\|_M + \|\vec{e}_h\|_H + h^l(\|\vec{u}\|_H + \|\vec{p}_h\|_H + \|\lambda\|_M + \|\tau\|_M)\},$$

C being a constant independent of $(\vec{u}, \lambda) \in X_h \times M_h$ and of h.

We have used geometrical error estimates (cf. Theorem 2.2), Brezzi's conditions [2] and the following property satisfied by any $\vec{v}$ in V_h:

(2.51) $$\operatorname{div}_\Gamma \vec{v} = 0.$$

Observing that $(\vec{p}, \lambda)$ actually satisfies $\Lambda(\vec{p}, \zeta) = \Theta(\vec{u}, \lambda)$ and using the regularization properties of R and S together with the regularity Theorem 3.6 given in Part 1, we find that there exists a constant C, independent of $(\vec{u}, \lambda)$ and of h, such that

(2.52) $$\|\vec{p}\|_{1/2,\Gamma} + \|\zeta\|_{1/2,\Gamma} \leqslant C(\|\vec{u}\|_X + \|\lambda\|_M).$$

Let $0 < \sigma < \frac{1}{2}$ be a fixed real number. Taking $\vec{q}$ and $\vec{\tau}$ as the respective projections of $\vec{p}$ and ζ defined in (1.50) and in the proof of Lemma 1.5, we obtain

(2.53) $$\|\vec{e}_h\|_H + \|\varepsilon_h\|_M \leqslant Ch^{1/2-\sigma}(\|\vec{u}\|_H + \|\tau\|_M),$$

where C is a constant independent of h, $\vec{u}$ and λ (but depending on σ).

Another way of writing (2.53) is

(2.54) $$\|T(\vec{u}, \lambda) - T_h(\vec{u}, \lambda)\|_{H\times M} \leqslant Ch^{1/2-\sigma}\|(\vec{u}, \lambda)\|_{H\times M}.$$

Recalling the estimate (3.28) given in Part 1, we have thus proved the following proposition:

PROPOSITION 2.6. *There exists $h^* > 0$ and $\gamma^* > 0$ such that, for all $0 < h < h^*$, the following lower bound holds*

(2.55) $$\|T_h(\vec{u}, \lambda)\|_{H\times M} \geqslant \gamma^*\|(\vec{u}, \lambda)\|_{H\times M},$$

for all $(\vec{u}, \lambda) \in X_h \times M_h$. □

The well-posedness of the discrete problem can now be established.

THEOREM 2.7. *The discrete problem (2.16) has one and only one solution.*

Proof. Obviously, it will be sufficient to check that the discrete problem can have at most one solution. Hence, let $(\vec{p}_h, \lambda_h)$ satisfy (2.16) with $\vec{c}_h = 0$. Thus, it follows from the definition of the operator T_h that $T_h(\vec{p}_h, \lambda_h) = 0$. The estimate (2.55) then implies $(\vec{p}_h, \lambda_h) = 0$. □

3. Error Estimates. We come now to the main results of this work. In this section, we shall establish several estimates of the error due to numerical approximations of some quantities like surface currents and charges, far-field pattern, etc., which are of practical interest and which are computed from the solution of the discrete problem.

In the sequel, $(\vec{p}, \lambda)$ and $(\vec{p}_h, \lambda_h)$ will, respectively, denote the solutions of the continuous and of the discrete problem (cf. (2.50) in Part 1). In general, $\vec{p}_h$ does not belong to $V_h(\vec{p})$. So, to compare $\vec{p}_h$ and $\vec{p}$, we first adapt Giroire's results [9] on the scalar Helmholtz equation. This and a kind of stability property given by (2.55) lead to an estimate of the errors

$$\vec{e}_h = \vec{p} - \vec{p}_h, \tag{3.1}$$

$$\varepsilon_h = \lambda - \lambda_h. \tag{3.2}$$

Finally, error estimates of approximations of the solution of the boundary value problem in different zones are given.

3.1. *Some Adaptations of the Scalar Case Results.* Consider the following sesquilinear forms given by

$$d(\lambda, \nu) = \langle \tilde{C}\lambda, \nu\rangle \quad \forall \lambda, \nu \in H^{-1/2}(\Gamma), \tag{3.3}$$

where $\tilde{C}$ is the operator defined by the kernel G (cf. (0.4) and (2.32) in Part 1) and

$$d_h(\lambda, \nu) = \int_{\Gamma_h\times\Gamma_h} G(x, y)\lambda(y)\overline{\nu(x)}\, d\gamma_h(y) d\gamma_h(x) \quad \forall \lambda, \nu \in W_h. \tag{3.4}$$

Then, d and d_h are related to the sesquilinear forms $\tilde{b}$ and $\tilde{b}_h$ by

$$\tilde{b}(\nu, \vec{p}) = d(\nu, \operatorname{div}_\Gamma \vec{p}), \qquad \nu \in M, \vec{p} \in X, \tag{3.5}$$

$$\tilde{b}_h(\nu, \vec{p}) = d_h(\nu, \operatorname{div}_{\Gamma_h} \vec{p}), \qquad \nu \in M_h, \vec{p} \in X_h. \tag{3.6}$$

Observe that this last relation may be equivalently written as

$$\tilde{b}_h(\nu, \vec{p}) = \int_{\Gamma_h\times\Gamma_h} G(x, y)\nu(y)\, \overline{\operatorname{div}_{\Gamma_h} \vec{p}(x)}\, d\gamma_h(y) d\gamma_h(x). \tag{3.7}$$

The following theorem summarizes the properties of the sesquilinear forms d and d_h which will be needed for the subsequent error estimates.

THEOREM 3.1. *There exists $h^* > 0$ such that, for all $0 < h \leqslant h^*$, λ being fixed in M, the problem*

$$\begin{cases} \text{Find } \lambda_h \in M_h \text{ such that} \\ d_h(\lambda_h, \nu) = d(\lambda, \nu) \quad \forall \nu \in M_h, \end{cases} \tag{3.8}$$

is well-posed. Moreover, we have

$$\|\lambda - \lambda_h\|_M \leqslant C \operatorname*{Inf}_{\mu \in M_h} \left(\|\lambda - \mu\|_M + h^l \|\mu\|_M\right), \tag{3.9}$$

where C is a constant independent of $\lambda \in M$ and of h.

Proof. We shall only establish the coerciveness estimate

$$\exists \alpha > 0\colon \sup_{\nu \in M} \frac{1}{\|\nu\|_M} |d(\lambda, \nu)| \geqslant \alpha \|\lambda\|_M \quad \forall \lambda \in M. \tag{3.10}$$

The proof is completed by standard arguments as, for instance, the approximation of d by d_h (see Theorem 2.2 above) and the regularization property of the operator S (see (2.48) in Part 1).

Suppose that (3.10) does not hold. Then, there exist two sequences $\{\lambda_n\}$ of functions in M and $\{\alpha_n\}$ of real numbers such that

$$\|\lambda_n\|_M = 1, \tag{3.11}$$

$$\lim \alpha_n = 0, \tag{3.12}$$

$$|d(\lambda_n, \nu)| \leqslant \alpha_n \|\nu\|_M \quad \forall \nu \in M. \tag{3.13}$$

Possibly passing to subsequences also denoted by $\{\lambda_n\}$ and $\{\alpha_n\}$, we may assume that

$$\lim \lambda_n = \lambda \quad \text{weakly in } M. \tag{3.14}$$

It follows that λ satisfies

$$\langle \tilde{C}\lambda, \nu \rangle = 0 \quad \forall \nu \in M. \tag{3.15}$$

$\tilde{C}\lambda$, having the same null space as the function 1, is thus constant on Γ. Hence, there exists a constant $a \in \mathbf{C}$ such that $\lambda = a\varphi$. φ is the C^∞-function defined from the solution u of the problem (here, the inversion of the operator C involves the assumption: k^2 is not an eigenvalue of the interior problem, cf. Part 1 and [9]):

$$\begin{cases} \text{find } u \in H^1_{\text{loc}}(\mathbf{R}^3) \text{ such that} \\ \Delta u + k^2 u = 0 \quad \text{in } \Omega^i \cup \Omega^e, \\ u = 1 \quad \text{on } \Gamma, \\ \partial u/\partial r - iku = o(1/r), \end{cases} \tag{3.16}$$

by

$$\varphi = [\gamma_1 u]. \tag{3.17}$$

Green's formula leads to

$$\int_\Gamma \gamma_1 u|_{\text{int}}\, d\gamma = |u|^2_{1,\Omega^i} - k^2 |u|^2_{0,\Omega^e}, \tag{3.18}$$

$$-\int_\Gamma \gamma_1 u|_{\text{ext}}\, d\gamma = |u|^2_{1,\Omega^e_R} - k^2 |u|^2_{0,\Omega^e_R} - \int_{S_R} \frac{\partial u}{\partial r} \bar{u}\, dS_R, \tag{3.19}$$

where

$$\Omega^e_R = \{x \in \Omega^e; |x| < R\} \quad \text{and} \quad S_R = \{x \in \mathbf{R}^3; |x| = R\}.$$

The radiation condition thus yields

$$\operatorname{Im} \int_\Gamma \varphi\, d\gamma = -k \lim_{R \to +\infty} \int_{S_R} |u|^2\, dS_R. \tag{3.20}$$

Rellich's lemma (cf. [18]) then gives

$$\int_\Gamma \varphi \, d\gamma \neq 0. \tag{3.21}$$

It follows from (3.15) and the fact that $\lambda \in M$ that

$$\lambda = 0. \tag{3.22}$$

In Part 1, we have seen that $\tilde{C}$ is split into the sum $\tilde{C} = C + S$. Using the compactness of the operator S and the coerciveness of the operator C, we arrive at

$$\lim \lambda_n = 0 \quad \text{strongly in } M, \tag{3.23}$$

which contradicts (3.11) and completes the proof. □

3.2. *Error Bounds and Convergence Properties.* First, we establish a stability property of the method.

THEOREM 3.2. *There exists a constant C independent of h such that, $(\vec{l}_h, \chi_h)$ being fixed in $H' \times M'$, if $(\vec{u}_h, \mu_h)$ satisfies*

$$\begin{cases} \tilde{a}_h(\vec{u}_h, \vec{q}) + \tilde{b}_h(\mu_h, \vec{q}) = \langle \vec{l}_h, \vec{q} \rangle & \forall \vec{q} \in X_h, \\ \overline{b_h(\nu, \vec{u}_h)} + c_h(\mu_h, \nu) = \langle \chi_h, \nu \rangle & \forall \nu \in M_h, \end{cases} \tag{3.24}$$

then,

$$\|(\vec{u}_h, \mu_h)\|_{H \times M} \leqslant C \left\| (\vec{l}_h, \chi_h) \right\|_{H' \times M'}. \tag{3.25}$$

Proof. Coerciveness estimates, (2.29) and (2.30), and standard arguments give the theorem for $(\vec{w}_h, \delta_h)$ satisfying

$$\begin{cases} a_h(\vec{w}_h, \vec{v}) + b_h(\delta_h, \vec{v}) = \langle \vec{l}_h, \vec{v} \rangle & \forall \vec{v} \in X_h, \\ \overline{b_h(\nu, \vec{w}_h)} = \langle \chi_h, \nu \rangle & \forall \nu \in M_h \end{cases} \tag{3.26}$$

(cf. e.g. [2], [19]). To get the general estimate, it is sufficient to set $(\vec{w}_h, \delta_h) = T_h(\vec{u}_h, \mu_h)$ and to use (2.55). □

Let $\vec{q}$ be fixed in $V_h(\vec{p})$ and ζ be the solution of (3.8) (we recall that $(\vec{p}, \lambda)$ is the solution of the exact problem). We denote

$$\vec{u} = \vec{p} - \vec{q}; \qquad \mu = \lambda - \zeta, \tag{3.27}$$

$$\vec{u}_h = \vec{p}_h - \vec{q}; \qquad \mu_h = \lambda_h - \zeta. \tag{3.28}$$

It follows that $(\vec{u}_h, \mu_h)$ satisfies

$$\tilde{a}_h(\vec{u}_h, \vec{v}) + \tilde{b}_h(\mu_h, \vec{v}) = \langle \vec{c}_h - \vec{c}, \vec{v} \rangle + \tilde{a}(\vec{u}, \vec{v}) + (\tilde{a} - \tilde{a}_h)(\vec{q}, \vec{v}), \tag{3.29}$$

$$\overline{b_h(\nu, \vec{u}_h)} + c_h(\mu_h, \nu) = c(\mu, \nu) + (c - c_h)(\zeta, \nu) + \overline{(b - b_h)(\nu, \vec{q})}. \tag{3.30}$$

Hence, the stability property (3.25) yields

$$\|\vec{u}_h\|_H + \|\mu_h\|_M \leqslant C\left\{ \|\vec{u}\|_H + \|\mu\|_M + \|\vec{c} - \vec{c}_h\|_{H'} + h^l\left(\|\vec{q}\|_X + \|\zeta\|_M\right) \right\}. \tag{3.31}$$

Finally, the estimate (3.9) leads to

$$(3.32)\quad \|\vec{p}-\vec{p}_h\|_H+\|\lambda-\lambda_h\|_M \leqslant C \inf_{\vec{q}\in V_h(\vec{p}),\mu\in M_h}\left\{\|\vec{p}-\vec{q}\|_H+\|\lambda-\mu\|_M+\|\vec{c}-\vec{c}_h\|_{H'}+h^l(\|\vec{q}\|_X+\|\lambda\|_M)\right\}.$$

Suppose now that $\vec{c}$ is infinitely differentiable in a neighborhood of Γ. This assumption is not restrictive in practice because the problem is stated under the physical hypothesis that the currents and charges creating the incident wave are far enough from the obstacle; see e.g. [15], [22].

Then, regularity results given in Part 1 insure that $\vec{p}$ and λ are infinitely differentiable as well.

We choose $\delta > 0$ so that Γ_h is contained in

$$(3.33)\qquad \Gamma_\delta=\left\{x\in\mathbf{R}^3;\ d(x,\Gamma)\leqslant\delta\right\}$$

($d(x,\Gamma)$ is the distance of x to Γ). Then, using Taylor's formula and the inverse inequality (1.48), we obtain

$$(3.34)\qquad \|\vec{c}-\vec{c}_h\|_{H'}\leqslant Ch^{l+1/2}\|\vec{c}\|_{C^1(\Gamma_\delta)},$$

where $C^1(\Gamma_\delta)$ indicates the space of continuously differentiable functions endowed with its usual norm.

Finally, given $0<\sigma\leqslant\frac{1}{2}$, using (1.74) and (1.69), we thus prove

THEOREM 3.3. *Under the above general assumptions, we have*

$$(3.35)\qquad \|\vec{p}-\vec{p}_h\|_H+\|\lambda-\lambda_h\|_M\leqslant CE(h,\vec{p},\lambda,\vec{c}),$$

where

$$(3.36)\quad E(h,\vec{p},\lambda,\vec{c}) = (h^{m-\sigma}+h^l)\|\vec{p}\|_{m,\Gamma}+(h^{m+1/2}+h^l)\|\lambda\|_{m,\Gamma}+h^{l+1/2}\|\vec{c}\|_{C^1(\Gamma_\delta)}$$

and C is a constant independent of h, $\vec{p}$, λ, $\vec{c}$. □

Remark 3.4. As a particular case, we see that the lower-order method $m=l=1$, i.e. the method of Rao et al. but improved by a suitable treatment of the approximation of the tangent plane, converges with an error estimate in $h^{1-\sigma}$ for all $0<\sigma$. □

Relevant quantities in many applications can be computed once the discrete problem is solved. For instance, here, we consider the approximation of the solution of the boundary value problem (cf. (2.9),…,(2.11) in Part 1), and that of the following vector, giving rise to far-field calculations,

$$(3.37)\qquad \vec{f}(\vec{\omega})=\int_\Gamma \exp(-ik(\vec{r}(y),\vec{\omega}))\vec{p}(y)\,d\gamma(y),$$

where $\vec{\omega}$ is any angular direction, i.e. $|\vec{\omega}|=1$, and $\vec{r}(y)$ is the radius vector of the point y.

The solution $\vec{e}$ of the boundary problem may be approximated in the Fresnel zone (far enough from the obstacle but not in the Fraunhofer, or far-field, zone, cf. e.g. [14]) by:

$$\vec{e}_h(x) = -\text{grad}\, v_h(x) + \vec{a}_h(x), \tag{3.38}$$

$$v_h(x) = \int_{\Gamma_h} G(x, y)\lambda_h(y)d\gamma_h(y), \tag{3.39}$$

$$\vec{a}_h(x) = \int_{\Gamma_h} G(x, y)\vec{p}_h(y)d\gamma_h(y). \tag{3.40}$$

In the same way, we obtain the following approximation of the vector $\vec{f}(\vec{\omega})$:

$$\vec{f}_h(\omega) = \int_{\Gamma_h} \exp(-ik(\vec{r}(y), \vec{\omega}))\vec{p}_h(y)d\gamma_h(y). \tag{3.41}$$

Following the plan used in the study of the scalar case (cf. [9], [12]), we easily prove

THEOREM 3.4. *Keeping the above notations and hypotheses, for h small enough and $|x| \geqslant \delta$, we have*

$$|\vec{e}_h(x) - \vec{e}(x)| \leqslant Ce_3(x, \Gamma)E(h, \vec{p}, \lambda, \vec{c}), \tag{3.42}$$

$$|\vec{f}_h(\vec{\omega}) - \vec{f}(\vec{\omega})| \leqslant CE(h, \vec{p}, \lambda, \vec{c}), \tag{3.43}$$

where C is a constant independent of h, $\vec{p}$, λ, $\vec{c}$, $\vec{\omega}$, and $e_j(x, \Gamma)$ is the function (*cf.* [12]) *defined by*

$$e_j(x, \Gamma) = \sum_{i=0}^{j} \frac{1}{d^i(x, \Gamma)}. \tag{3.44}$$

Final Remark 3.5. We have obtained quasi-optimal error estimates for the approximation of the currrents and charges (i.e. $\vec{p}$ and λ). The loss of an order $h^{1/2+\sigma}$, $0 < \sigma \leqslant \frac{1}{2}$, in the estimates is due to the use of the inverse inequalities and seems to be difficult to improve. However, we think that a suitable Aubin-Nitsche trick may lead to better estimates for field computations in the Fresnel and Fraunhofer zones, respectively. □

Centre de Mathématiques Appliquées
Ecole Polytechnique
91128 Palaiseau Cedex, France

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