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Approximation Properties of Quadrature Methods for Volterra Integral Equations of the First Kind

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Abstract. We present a unifying analysis of quadrature methods for Volterra integral equations of the first kind that are zero-stable and have an asymptotic repetition factor. We show that such methods are essentially collocation-projection methods with underlying subspaces that have nice approximation properties, and which are stable as projection methods. This is used to derive asymptotically optimal error estimates under minimal smoothness conditions. The class of quadrature methods covered includes the cyclic linear multistep and the reducible quadrature methods, but not (really) Runge-Kutta methods.

1. Introduction. We study approximation properties of quadrature methods for the numerical solution of Volterra integral equations of the first kind

$$(1.1)\qquad \mathscr{U}f(x) = \int_0^x \Phi(x, y, f(y))\,dy = g(x), \qquad x \in [0,1],$$

i.e., methods of the form

$$(1.2)\qquad n^{-1}\sum_{j=0}^{i} w_{ij}\Phi\big(x_i, x_j, f_n(x_j)\big) = g(x_i), \qquad i = s, s+1,\ldots,n,$$

for some fixed integer $s > 0$, where $x_i = i/n$. The additionally needed $f_n(x_i)$, $i = 0,1,\ldots,s-1$ are furnished by some other starting quadrature method, say of the form

$$(1.3)\quad n^{-1}\sum_{j=0}^{t-1} w_{ij}^{(s)}\Phi\big(x_i^{(s)}, y_j^{(s)}, f_n\big(y_j^{(s)}\big)\big) = g\big(x_i^{(s)}\big), \qquad i = 0,1,\ldots,t-1,$$

with $x_i^{(s)} = \theta_i/n$, for fixed $\theta_i > 0$, and the numbers x_j, $j = 0,1,\ldots,s-1$ contained in the set $\{y_j^{(s)}: j = 0,1,\ldots,t-1\}$. Generally, $t - s$ is nonnegative. We renumber the sequence $x_0, x_1^{(s)},\ldots,x_{t-1}^{(s)}, x_s, x_{s+1},\ldots,x_n$ as $y_0, y_1,\ldots,y_{n+d-1}$. For a review of quadrature methods, among others, see Brunner [4].

The system (1.2-3) is denoted by

$$(1.4)\qquad U_n r_n f_n = \rho_n g,$$

where

$$(1.5)\qquad r_n f = \big(f(y_0), f(y_1),\ldots,f(y_{n+d-1})\big)^T$$

and similarly for ρ_n,

$$(1.6)\qquad \rho_n g = \big(g\big(x_0^{(s)}\big), g(y_1), g(y_2),\ldots,g(y_{n+d-1})\big)^T.$$

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For general Φ, U_n is a *nonlinear* operator acting on $\mathbf{R}^{n+d}$, and its invertibility needs to be investigated.

In (1.4-5), we may consider r_n as a mapping of $C[0,1]$ onto $\mathbf{R}^{n+d}$, but f_n, the solution of (1.4), assuming it exists, is not (yet) defined on all of [0, 1], only at the y_j. This is characteristic for *discretization methods*, and is one of the underlying themes of the work of Vainikko [21], [22] (see in particular Taylor [20]), and Stummel [16], [17], [18], among others. The theme of this paper is that there *are*, more or less, *natural* ways to *interpolate* $r_n f_n$, at least for the simple case $\Phi(x, y, z) \equiv z$, such that (1.4) is the system of equations for a *collocation-projection* method for (1.1). This is used to prove error estimates for f_n which are optimal with respect to order of convergence and smoothness requirements on f and Φ.

The approach to the interpolation of $r_n f_n$ is to imitate the analysis of collocation-projection methods for the basic linear version of (1.1), cf. [5], [6]. So let $\Phi(x, y, z) = z$ and consider

$$(1.7) \qquad \mathscr{W}f(x) = \int_0^x f(y)\,dy = g(x), \qquad x \in [0,1],$$

and let W_n be the matrix for (1.2-3). Apparently, we have $W_n r_n f_n = \rho_n \mathscr{W} f$, so

$$(1.8) \qquad r_n(f_n - f) = W_n^{-1}(\rho_n \mathscr{W} - W_n r_n) f.$$

Now the mimicry starts: we may subtract from f in the right-hand side of (1.8) anything that is annihilated by $\rho_n \mathscr{W} - W_n r_n$. To formalize this, let

$$(1.9) \qquad \mathscr{Z}_n = \{\psi \in C[0,1] : \rho_n \mathscr{W}\psi = W_n r_n \psi\},$$

and then from (1.8) (all norms are uniform norms)

$$(1.10) \qquad \|r_n(f_n - f)\| \leqslant \left(\|W_n^{-1}\rho_n \mathscr{W}\| + 1\right) \inf\{\|f - \psi\| : \psi \in \mathscr{Z}_n\}.$$

Now there are two questions. First, can the quantity $\|W_n^{-1}\rho_n \mathscr{W}\|$ be bounded uniformly in n? The answer is yes if the quadrature method is *zero-stable* (and consistent). Second, what are the approximation properties of $\mathscr{Z}_n$? $\mathscr{Z}_n$ is certainly a large subspace of $C[0, 1]$, since it has codimension $n + d$, but the question is whether it contains anything useful for *approximation purposes* (how small is the infimum in (1.10)). We settle this question by exhibiting natural interpolators (prolongations) p_n: $\mathbf{R}^{n+d} \to \mathscr{Z}_n$ such that $\|f - p_n r_n f\| \leqslant c\|f^{(p)}\| n^{-p}$, if $f \in C^p[0,1]$, and $\{p_n\}_n$ is a *regular* family of embeddings. In addition, it turns out that the system (1.4) in this case is equivalent to

$$(1.11) \qquad \begin{cases} \rho_n \mathscr{W} f_n = \rho_n g, \\ f_n \in \mathscr{S}_n = \operatorname{im} p_n, \end{cases}$$

i.e., (1.4) is a *collocation-projection* method, and $\mathscr{S}_n$ has nice approximation properties (see above). For arbitrary linear systems, i.e., $\Phi(x, y, z) = K(x, y)z$, the system (1.4) may be considered as a fully discretized version of the collocation-projection method

$$(1.12) \qquad \begin{cases} \rho_n \mathscr{U} f_n = \rho_n g, \\ f_n \in \mathscr{S}_n, \end{cases}$$

and hence shares the basic stability and convergence properties of (1.11). In the general *nonlinear* case, there is a similar identification.

In this paper, we work out the above approach, and apply it to some well-known quadrature methods, such as the cyclic linear multistep methods of Holyhead et al. [9] and Andrade and McKee [1], and the reducible quadrature methods of Wolkenfelt [23], [24], which contain the methods of Gladwin [7], [8] and Taylor [19] as special cases. An added benefit of the above approach is that we obtain optimal error estimates with respect to smoothness requirements as opposed to the results in the above papers.

We now state the main result of this paper. Assume that the method (1.4) for the case $\Phi(x, y, z) \equiv z$ is

(i) *zero-stable*, i.e.,

$$\sup_n n^{-1}\|W_n^{-1}\| < \infty; \tag{1.13}$$

(ii) *asymptotically r-cyclic and composite of order p*, i.e.,

$$\|\Delta_n^r(W_n r_n f - \rho_n \mathscr{W} f)\| \leqslant c\|f^{(p)}\| n^{-p-1}, \tag{1.14}$$

where

$$\Delta_n^r z = \begin{cases} z_i, & i = 1,2,\ldots,r-1, \\ z_i - z_{i-r}, & i = r,\ldots,n+d. \end{cases} \tag{1.15}$$

Here r is the *asymptotic repetition factor*. It is easily seen that (1.14) implies that $|w_{ij}| < \infty$, uniformly in i, j, n, and

$$\sum_{j=0}^{i-r} |w_{ij} - w_{i-r,j}| < \infty \quad \text{uniformly in } i, n, \tag{1.16}$$

and that the quadrature method (1.4) has a truncation error of order n^{-p}. The condition (1.16) is well-known, see, e.g., Taylor [20] for $r = 1$, as is, of course, (1.13).

THEOREM 1.1. *Assume that* (1.13-14) *holds. If* $\Phi(x, y, z) = K(x, y)z$, *and* $\partial K/\partial x$ *and* f *have continuous (partial) derivatives of order* q, *then for* n *large enough the solution* f_n *of* (1.14) *exists and satisfies*

$$\|r_n(f_n - f)\| \leqslant c\|K\|_{q+1,T}\|f\|_q n^{-q}, \tag{1.17}$$

where $q = 1, 2, \ldots, p$. (*See* (2.3) *for the meaning of the norms.*)

The usual smoothness requirements are $\partial K/\partial x$ and f are C^{p+1} to get a bound $O(n^{-p})$, cf. [1], [7], [8], [9], [13], [19], [23], [24]. There is a similar theorem for the nonlinear case.

The remainder of this paper consists in proving the above theorem. The assumption (1.14) is used to *construct* a piecewise polynomial subspace $\mathscr{S}_n$ of $\mathscr{X}_n$ such that (1.11) is equivalent to $W_n r_n f_n = \rho_n g$, and the approximation properties of $\mathscr{S}_n$ are established (Section 3). The zero-stability (1.13) is used to prove that the projector onto $\mathscr{S}_n$, implicitly defined by (1.11), is bounded uniformly in n, and that for the general linear case, (1.4) is the fully discretized version of (1.12). This interpretation is used to prove Theorem 1.1 (Section 4). In Section 5, the general nonlinear equation is treated. In Section 6, we show that the aforementioned quadrature methods of [1], [9], [23], [24] fit the mold, i.e., the assumptions (1.13-14) are satisfied. This then provides a unified theory of quadrature methods for first kind Volterra integral equations.

2. Notations and Preliminaries. All norms in this paper are supremum or maximum norms unless indicated otherwise, so if $f \in L^\infty(\Omega)$ with $\Omega \subset \mathbf{R}^m$,

$$\|f\|_\Omega = \operatorname{ess\,sup}\{|f(x)| : x \in \Omega\}. \tag{2.1}$$

In this paper, $\Omega = [0,1] \subset \mathbf{R}$ or $\Omega = T = \{(x, y) : 0 \leqslant y \leqslant x \leqslant 1 \text{ or } 0 \leqslant y \leqslant \varepsilon\}$, or $\Omega = T \times \mathbf{R}$. For $\Omega = [0,1]$, the norm is denoted simply by $\| \ \|$. If $b \in \mathbf{R}^n$, its norm is defined by

$$\|b\| = \max\{|b_i| : i = 1, 2, \ldots, n\}. \tag{2.2}$$

For $q \geqslant 0$, an integer, and $\Omega \subset \mathbf{R}^m$, $C^q(\Omega)$ denotes the space of q-times boundedly and continuously differentiable functions on Ω, and we define a norm on $C^q(\Omega)$ by

$$\|f\|_{q,\Omega} = \sum_{|\alpha| \leqslant q} \|D^\alpha f\|_\Omega, \tag{2.3}$$

where $D^\alpha = \partial^{\alpha_1}/\partial x_1^{\alpha_1} \cdots \partial^{\alpha_m}/\partial x_m^{\alpha_m}$ in the usual multi-index notation, and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_m$. So $f \in C^q(\Omega)$ implies $\|f\|_{q,\Omega} < \infty$. We also define

$$C_0^q[0,1] = \{f \in C^q[0,1] : f(0) = 0\} \tag{2.4}$$

with a norm

$$\|f\|_{0,q} = \|f'\|_{q-1}. \tag{2.5}$$

We have already defined $\mathscr{U}$ and $\mathscr{W}$ in the introduction. We let

$$\mathscr{V} f(x) = \int_0^x K(x, y) f(y)\, dy, \qquad x \in [0,1], \tag{2.6}$$

and we restrict $\mathscr{U}$ and $\mathscr{V}$ by requiring that $\Phi \in \mathscr{N}_q$, and $K \in \mathscr{K}_q$, where

$$\mathscr{K}_q = \{K \in C^{q+1}(T) : K(x, x) \neq 0 \text{ for all } x \in [0,1]\}, \tag{2.7}$$

$$\mathscr{N}_q = \{\Phi_z \in C^{q+1}(T \times \mathbf{R}) : \inf\{|\Phi_z(x, x, z)| : (x, z) \in [0,1] \times \mathbf{R}\} > 0\}. \tag{2.8}$$

We have the following mapping properties of $\mathscr{U}$, $\mathscr{V}$ and $\mathscr{W}$.

LEMMA 2.1. (a) *$\mathscr{W}$ is an isometric isomorphism of $C^q[0,1]$ onto $C_0^{q+1}[0,1]$.*
(b) *$\mathscr{V}$ is an isomorphism of $C^q[0,1]$ onto $C_0^{q+1}[0,1]$ provided $K \in \mathscr{K}_q$.*
(c) *$\mathscr{U}$ is a homeomorphism of $C^q[0,1]$ onto $C_0^{q+1}[0,1]$ provided $\Phi \in \mathscr{N}_q$.*

Proof. Parts (a) and (b) are essentially well known, see, e.g., Krasnosel'skii et al. [11, Chapter 1, Section 1.2]. Part (c) can be proved as follows. The *Fréchet derivative* of $\mathscr{U}$ at f is given by

$$[\mathscr{U}'(f)h](x) = \int_0^x \Phi_z(x, y, f(y)) h(y)\, dy, \qquad x \in [0,1],$$

and it follows from part (b) that $\mathscr{U}'(f)$: $C^q[0,1] \to C_0^{q+1}[0,1]$ is a homeomorphism, whose inverse is bounded, uniformly in $f \in C^q[0,1]$, if $\Phi \in \mathscr{N}_q$. It then follows from Hadamard's theorem, Berger [3, (5.1.5)], applied to $\mathscr{U}$, that $\mathscr{U}$ is a homeomorphism. □

The above can be extended to prove that $\mathscr{U}^{-1}$ is Lipschitz continuous, i.e.,

$$\|\mathscr{U}^{-1}f - \mathscr{U}^{-1}g\| \leqslant c\|f - g\|,$$

where c depends only on Φ. It may also be phrased as: $\mathscr{U}f - \mathscr{U}g = \mathscr{W}h$ implies that $\|f - g\| \leqslant c\|h\|$. In Section 5, we prove it for the operators U_n, assuming the zero-stability of the quadrature method.

Finally, some remarks on notation. The subintervals $[y_{i-1}, y_i]$ of $[0,1]$ are denoted by σ_i, $i = 1, 2, \dots, n+d$. It is convenient to index components of vectors in $\mathbf{R}^{n+d}$ as $x = (x_0, x_1, \dots, x_{n+d-1})^T$, and we will do so consistently. Finally, the symbol c denotes a universal constant, but each occurrence usually denotes a different constant.

3. The Approximating Subspaces $\mathcal{S}_n$. Throughout this section, we assume that the quadrature method (1.4) satisfies property (1.14). We prove the following theorem and lemma.

THEOREM 3.1. *There exist prolongations* $p_n: \mathbf{R}^{n+d} \to C[0,1]$, *such that*

$$\|x\| \leqslant \|p_n x\| \leqslant c\|x\| \quad \text{for all } x \in \mathbf{R}^{n+d}, \tag{3.1}$$

and $\mathcal{S}_n = \{ p_n x : x \in \mathbf{R}^{n+d} \}$ *is a subspace of* $\mathcal{X}_n$ *which satisfies*

$$\inf\{ \|f - \psi\| : \psi \in \mathcal{S}_n \} \leqslant c\|f^{(q)}\| n^{-q}, \tag{3.2}$$

for all $f \in C^q[0,1]$, $q = 0, 1, \dots, p$.

LEMMA 3.2. *For* $i = 0, 1, \dots, n+d-1$,

$$\inf\{ \|\mathcal{W} f - \mathcal{W}\psi\|_{\sigma_i} : \psi \in \mathcal{S}_n \} \leqslant cn^{-1}\|f\|. \tag{3.3}$$

It should be remarked that instead of (3.3), we would rather have

$$\inf\{ \|\mathcal{W} f - \mathcal{W}\psi\| : \psi \in \mathcal{S}_n, \|\psi\| \leqslant c\|f\| \} \leqslant cn^{-1}\|f\|, \tag{3.4}$$

but no proof could be found (without assuming zero-stability). However, (3.3) is useful as is.

The proof of Theorem 3.1 is constructive. To construct elements of $\mathcal{S}_n$, let $z \in \mathbf{R}^{n+d}$. We want to find a (piecewise polynomial) function $\psi = p_n z$ such that

$$\begin{cases} W_n r_n \psi = \rho_n \mathcal{W}\psi, \\ r_n \psi = z. \end{cases} \tag{3.5}$$

A solution of (3.5) is given by $\psi(x) = \Psi_i'(x)$, $x \in \sigma_i$, $i = 1, \dots, n+d$, where Ψ_i is the cubic Hermite interpolating polynomial satisfying $\Psi_i(x_{i-j}) = (W_n r_n z)_{i-j}$, $\Psi_i'(x_{i-j}) = z_{i-j}$ for $j = 0, 1$. This choice of ψ does not give very good approximation properties though: we would like $f - p_n r_n f$ to be small. The correct approach is through certain Birkhoff interpolation problems. In Subsection 3.1, we review the existence of solutions and their approximation properties. In Subsection 3.2, we show how this solves the problem (3.3) for the case $r = 1$. In Subsection 3.3, we describe the ad hoc modification necessary for the case $r \geqslant 2$. This then will complete the proof of Theorem 3.1 and Lemma 3.2.

3.1. *Birkhoff Interpolation Problems.* Consider the following Birkhoff interpolation problems, given $z \in \mathbf{R}^{n+d}$, with $p^* = \max(p-2, 1)$,

(a) for $1 \leqslant i \leqslant rp^*$,

$$\begin{cases} \Psi_i(y_{i+j}) = (W_n z)_{i+j}, & j = 0, 1, \\ \Psi_i'(y_{i+j}) = z_{i+j}, & j = 0, 1, \dots, p^*, \\ \Psi_i \in \mathcal{P}_{p^*}, \end{cases} \tag{3.6}$$

where r is the asymptotic repetition factor, cf. (1.14-16), and $\mathscr{P}_{p^*+2}$ is the set of polynomials of degree $\leqslant p^* + 2$;

(b) for $rp^* + 1 \leqslant i \leqslant n + d$,

$$\left\{\begin{array}{ll}\Psi_i(y_{i-jr}) = (W_n z)_{i-jr}, & j = 0, 1,\\ \Psi_i'(y_{i-jr}) = z_{i-jr}, & j = 0, 1, \ldots, p^*,\\ \Psi_i \in \mathscr{P}_{p^*+2};\end{array}\right. \tag{3.7}$$

(c) for $i = 0$,

$$\left\{\begin{array}{l}\Psi_0(y_0) = 0, \qquad \Psi_0(y_1) = (W_n z)_1,\\ \Psi_0(x_0^{(0)}) = (W_n z)_0,\\ \Psi_0'(y_j) = z_j, \qquad j = 0, 1, \ldots, p^*,\\ \Psi_0 \in \mathscr{P}_{p^*+3}.\end{array}\right. \tag{3.8}$$

The problem (b) is the "main" problem. Parts (a) and (c) are just there to make things fit neatly.

LEMMA 3.3. *The interpolation problems* (3.6-8) *have unique solutions.*

Outline of Proof. In the terminology of Lorentz et al. [12, Chapter 1], the *interpolation matrices E* for (3.6-8) are given by

$$E_{(3.6)} = \begin{bmatrix} 1 & 1\\ 1 & 1\\ 0 & 1\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \uparrow\\ \Big| \; p^* - 1\\ \downarrow \end{matrix}; \qquad E_{(3.7)} = \begin{bmatrix} 0 & 1\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ 0 & 1\\ 1 & 1\\ 1 & 1 \end{bmatrix} \begin{matrix} \uparrow\\ \Big| \; p^* - 1\\ \downarrow \\ \\ \\ \end{matrix};$$

$$E_{(3.8)} = \begin{bmatrix} 1 & 1\\ 1 & 0\\ 1 & 1\\ 0 & 1\\ \vdots & \vdots\\ 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \uparrow\\ \Big| \; p^* - 1\\ \downarrow \end{matrix}.$$

For all $r \geqslant 1$, the matrices E can be extended with columns of zeros to *Polya matrices without odd supported sequences*, hence they *are order regular*, and the Ψ_i exist and are unique, see Lorentz et al. [12, Theorem 1.3], or the original reference Atkinson and Sharma [2]. □

A consequence of the lemma is that the functions Ψ_i depend linearly on z, so we may write them as

$$\Psi_i(x) = \Psi_i(z; x) = n^{-1} \sum_{j=0}^{p^*} z_{i-jr} B_j(n(x - y_i)) + \sum_{j=0}^{1} (W_n z)_{i-jr} b_j(n(x - y_i)), \tag{3.9}$$

for $rp^* + 1 \leqslant i \leqslant n + d$, and for $1 \leqslant i \leqslant rp^*$,

$$\Psi_i(x) = \Psi_i(z; x) = n^{-1} \sum_{j=0}^{p^*} z_{i+j} \tilde{B}_j(n(x - y_i)) + \sum_{j=0}^{1} (W_n z)_{i+j} \tilde{b}_j(n(x - y_i)), \tag{3.10}$$

and similarly for Ψ_0. Here the B_j, $\tilde{B}_j$, b_j and $\tilde{b}_j$ are *independent* of n.

We now turn to approximation properties.

LEMMA 3.4. *If $f \in C^q[0, 1]$, with $0 \leqslant q \leqslant p$, then*

$$\|\mathscr{W}f - \Psi_i(r_n f; \cdot)\|_{\sigma_{i+j}} \leqslant cn^{-q}\|f^{(q)}\|, \tag{3.11}$$

$$\|f - \Psi_i'(r_n f; \cdot)\|_{\sigma_{i+j}} \leqslant cn^{-q}\|f^{(q)}\|, \tag{3.12}$$

for $j = 0, \pm 1$.

Proof. We only prove the lemma for $j = 0$ and $i \geqslant rp^* + 1$, since the other cases are similar. Let

$$\Phi_i(f; x) = n^{-1} \sum_{j=0}^{p^*} (r_n f)_{i-jr} B_j(n(x - y_i)) + \sum_{j=0}^{1} (\rho_n \mathscr{W} f)_{i-jr} b_j(n(x - y_i)). \tag{3.13}$$

Since $\Phi_i(f; \cdot) = \mathscr{W}f$ if $f \in \mathscr{P}_{p^*+1}$, we obtain that

$$\|\Phi_i(f; \cdot) - \mathscr{W}f\|_{\sigma_i} \leqslant cn^{-q-1}\|f^{(q)}\|. \tag{3.14}$$

Consequently,

$$\|\Psi_i(r_n f; \cdot) - \mathscr{W}f\|_{\sigma_i} \leqslant cn^{-q-1}\|f^{(q)}\| + R,$$

where

$$R = \left\| \sum_{j=0}^{1} \{(\rho_n \mathscr{W} f)_{i-jr} - (W_n r_n f)_{i-jr}\} b_j(n(\cdot - y_i)) \right\|_{\sigma_i}.$$

It follows from (1.14) and the independence of n of the b_j, that $R \leqslant c\|f^{(q)}\|n^{-q}$, whence (3.11) follows. Similar to (3.14), we get

$$\|\Phi_i'(f; \cdot) - f\|_{\sigma_i} \leqslant cn^{-q}\|f^{(q)}\|, \tag{3.15}$$

and so

$$\|\Psi_i'(r_n f; \cdot) - f\|_{\sigma_i} \leqslant cn^{-q}\|f^{(q)}\| + R',$$

where

$$R' = \left\| \sum_{j=0}^{1} n\{(\rho_n \mathscr{W} f)_{i-jr} - (W_n r_n f)_{i-jr}\} b_j'(n(\cdot - y_i)) \right\|_{\sigma_i}.$$

We may rewrite this as

$$R' = |n\Delta_n^r(\rho_n \mathscr{W} f - W_n r_n f)_i| \, \|b_0'(n(\cdot - y_i))\|_{\sigma_i}, \tag{3.16}$$

since $\Phi_i' - \Psi_i'$ is not affected if we subtract a constant vector (all of whose components are equal) from $\rho_n \mathscr{W} f - W_n r_n f$. Now assumption (1.14) gives us that $R' \leqslant cn^{-q}\|f^{(q)}\|$, and (3.12) follows. □

We need two results which are related to (3.11), but which are not quite implied by it.

LEMMA 3.5. *For* $q = 0, 1, \ldots, p$ *and* $f \in C^q[0,1]$,

$$\|\Psi_i(r_n f;\ \cdot) - \Psi_{i+1}(r_n f;\ \cdot)\|_{\sigma_{i+j}} \leqslant cn^{-q-1}\|f^{(q)}\|,$$

with $j = 0, \pm 1$.

Proof. Let Ψ_i denote $\Psi_i(r_n f;\ \cdot)$. Apparently,

$$\|\Psi_i - \Psi_{i+1}\|_{\sigma_{i+j}} \leqslant \|\Psi_i - \Phi_i - (\Psi_{i+1} - \Phi_{i+1})\|_{\sigma_{i+j}} + \|\Phi_i - \Phi_{i+1}\|_{\sigma_{i+j}}$$

where $\Phi_i = \Phi_i(f;\ \cdot)$ is defined by (3.13). Now

$$\|\Phi_i - \Phi_{i+1}\|_{\sigma_{i+j}} \leqslant \|\Phi_i - \mathscr{W} f\|_{\sigma_{i+j}} + \|\mathscr{W} f - \Phi_{i+1}\|_{\sigma_{i+j}} \leqslant c\|f^{(q)}\|n^{-q-1},$$

by (3.14). As for the first term, we get for $i > rp^*$,

$$\Theta_i(x) = \Psi_i(x) - \Phi_i(x) = \sum_{j=0}^{1} (W_n r_n f - \rho_n \mathscr{W} f)_{i-jr} b_j(n(x - y_i)),$$

so by the mean value theorem

$$\Theta_i(x) - \Theta_{i+1}(x) = \sum_{j=0}^{1} (W_n r_n f - \rho_n \mathscr{W} f)_{i-jr} n(y_i - y_{i+1}) b_j'(\eta_i),$$

and similar to (3.16),

$$\|\Theta_i - \Theta_{i+1}\|_{\sigma_{i+j}} \leqslant |\Delta_n^r(W_n r_n f - \rho_n \mathscr{W} f)_i|\ \|b_0'(n(\cdot - y_i))\|_{\sigma_{i+j}}$$

and the assumption (1.14) gives the desired estimate.

For $i \leqslant rp^*$, the estimate follows likewise. □

The estimate (3.11) can be improved for $q = 0$ in the following sense.

LEMMA 3.6. *For each* $f \in C[0,1]$, *there exists a* $z \in \mathbf{R}^{n+d}$ *such that*

$$\|\Psi_{i+j}(z;\ \cdot) - \mathscr{W} f\|_{\sigma_i} \leqslant c\|f\|n^{-1},$$

with $j = 0, -1$.

Proof. Let i_0 be such that for all $i > i_0$ and for all n,

$$n^{-1} \sum_{j=0}^{i-p^*-1} w_{ij} \geqslant \frac{1}{2} y_i. \tag{3.17}$$

For $0 \leqslant x \leqslant i_0/n$ we have $|\mathscr{W} f(x)| \leqslant c\|f\| i_0/n$, hence $\|\mathscr{W} f\|_{\sigma_{i+j}} \leqslant \tilde{c}\|f\|n^{-1}$, and by taking $z = 0$, we prove the lemma. For $i > i_0$, choose $z \in \mathbf{R}^{n+d}$ as follows:

$$z_j = \begin{cases} 0, & j \geqslant i - p^*, \\ \zeta, & 0 \leqslant j \leqslant i - p^* - 1, \end{cases}$$

where

$$\zeta = \left(n^{-1} \sum_{j=0}^{i-p^*-1} w_{ij}\right)^{-1} \mathscr{W} f(y_i).$$

Then $\|z\| = |\zeta| \leqslant 2\|f\|$, and from (3.9) (w.l.o.g. $i_0 \geqslant rp^*$), we obtain

$$\Psi_i(z; x) - \mathscr{W}f(x) = \mathscr{W}f(y_i)\{b_0(n(x - y_i)) + b_1(n(x - y_i))\} - \mathscr{W}f(x).$$

Since $b_0(x) + b_1(x) = 1$ for all x, the lemma follows for $j = 0$. For $j = -1$, the lemma follows likewise. □

3.2. *The Case* $r = 1$. We now prove Theorem 3.1 and Lemma 3.2 for the case $r = 1$. The prolongations p_n are defined by

$$(3.18) \qquad p_n z(x) = \begin{cases} \Psi'_{i-1}(z; x), & x \in \sigma_i, i = 1, 2, \ldots, rp^*, \\ \Psi'_i(z; x), & x \in \sigma_i, i = rp^* + 1, \ldots, n + d, \end{cases}$$

where $z \in \mathbf{R}^{n+d}$, and set

$$(3.19) \qquad \mathscr{S}_n = \{p_n z : z \in \mathbf{R}^{n+d}\}.$$

It is easily seen that $\rho_n \mathscr{W} p_n z = W_n z$, hence (3.5) holds, and $\mathscr{S}_n$ is indeed a subspace of $\mathscr{X}_n$. The regularity property (3.1) follows from (3.12) by taking f to be the piecewise linear interpolant on z, so $r_n f = z$, and $j = 0$, $q = 0$. The approximation property (3.2) follows also from (3.12), with $j = 0$, by piecing together the σ_i. Finally, Lemma 3.2 follows from Lemma 3.6 with $j = 0$.

3.3. *The Case* $r \geqslant 2$. It is clear that the definitions (3.18-19) will not work for $r \geqslant 2$, since the function $\Psi(x) = \Psi_i(x)$, $x \in \sigma_i$, $i = rp^* + 1, \ldots, n + d$, is not even continuous. However, we may remedy this by taking some average of Ψ_i and Ψ_{i-1} on σ_i, and differentiate this, to obtain

$$(3.20) \quad p_n z(x) = \begin{cases} \dfrac{d}{dx}\{a_i(x)\Psi_i(z; x) + b_i(x)\Psi_{i-1}(z; x)\}, & x \in \sigma_i, i \geqslant 2, \\ \Psi'_0(z; x), & x \in \sigma_1. \end{cases}$$

Since we want $(p_n z)(y_i) = z_i$ and $(\mathscr{W} p_n z)(y_i) = (W_n z)_i$, it suffices to take

$$(3.21) \qquad \begin{aligned} &a_i(y_i) = 1, \quad a'_i(y_i) = 0, \quad a_i(y_{i-1}) = 0, \quad a'_i(y_{i-1}) = 0, \\ &b_i(y_i) = 0, \quad b'_i(y_i) = 0, \quad b_i(y_{i-1}) = 1, \quad b'_i(y_{i-1}) = 0. \end{aligned}$$

The reader will recognize that we may take a_i and b_i to be Hermite interpolating polynomials, viz. ($i \geqslant 2$)

$$(3.22) \qquad \begin{aligned} a_i(x) &= [n(x - y_{i-1})]^2(1 + 2n(y_i - x)), \\ b_i(x) &= [n(y_i - x)]^2(1 + 2n(x - y_{i-1})). \end{aligned}$$

Now it is readily verified that $p_n z$ indeed satisfies (3.5), and so $\mathscr{S}_n = \operatorname{im} p_n$ is again a subspace of $\mathscr{X}_n$.

With this choice of p_n and $\mathscr{S}_n$, we now prove Theorem 3.1. First, we prove (3.2). Let $0 \leqslant q \leqslant p$, and $f \in C^q[0, 1]$. From (3.20), we get

$$\|f - p_n r_n f\|_{\sigma_i} \leqslant \|a'_i(\Psi_i - \Psi_{i-1})\|_{\sigma_i} + \|a_i(\Psi'_i - f)\|_{\sigma_i} + \|b_i(\Psi'_{i-1} - f)\|_{\sigma_i},$$

with $\Psi_i = \Psi_i(r_n f; \cdot)$ as before. Here we used that $a_i + b_i \equiv 1$, $a'_i = -b'_i$. Now Lemmas 3.5 and 3.4 give the estimate

$$(3.23) \qquad \|f - p_n r_n f\| \leqslant c\|f^{(q)}\|n^{-q},$$

so (3.2) follows.

The regularity property (3.1) is (again) a consequence of (3.23) by taking f to be the piecewise linear interpolant on z, and $q = 0$ to yield $\|p_n z\| \leqslant \|z\| + c\|z\|$; the inequality $\|z\| \leqslant \|p_n z\|$ follows from $z = r_n p_n z$.

The proof of Lemma 3.2 follows from (3.20) and the inequality

$$\|a_i \Psi_i + b_i \Psi_{i-1} - \mathscr{W} f\|_{\sigma_i} \leqslant \|a_i(\Psi_i - \mathscr{W} f)\|_{\sigma_i} + \|b_i(\Psi_{i-1} - \mathscr{W} f)\|_{\sigma_i},$$

and then applying Lemma 3.6.

This completes the construction of the spaces $\mathscr{S}_n$, and shows the equivalence of (1.4) and (1.11) if $\Phi(x, y, z) = z$.

4. The Collocation-Projection Method and the Quadrature Method. In the previous section, we showed that the system (1.4) for the case $\Phi(x, y, z) \equiv z$ is the *collocation-projection* method (1.11). In this section, we write it first as a projection method, and exhibit the projectors involved. We show that the projectors are uniformly bounded in their natural setting if the quadrature method is zero-stable. Finally, we interpret the system (1.4) in case $\Phi(x, y, z) \equiv K(x, y)z$, with $K \in \mathscr{K}_0$, as a fully discretized version of the collocation-projection method and prove its optimal convergence properties.

Throughout this section, we assume that the quadrature method is zero-stable (1.15), in addition to (1.14).

Consider the system (1.4) if $\Phi(x, y, z) \equiv z$,

$$W_n r_n f_n = \rho_n g. \tag{4.1}$$

By the construction of p_n, this is equivalent to

$$\rho_n \mathscr{W} p_n r_n f_n = \rho_n g,$$

and since $p_n r_n f_n \in \mathscr{S}_n$, this is equivalent to (1.11), i.e. (4.1) is the system of linear equations for the collocation-projection method (1.11). Since W_n is nonsingular, the solution $r_n f_n \in \mathbf{R}^{n+d}$ of (3.1) is given by $W_n^{-1}\rho_n g$, hence $f_n = p_n W_n^{-1}\rho_n g$. Consequently, we may write (1.11) equivalently as

$$\mathscr{W} f_n = \mathscr{P}_n g, \tag{4.2}$$

and its solution f_n as

$$f_n = \mathscr{Q}_n \mathscr{W}^{-1} g, \tag{4.3}$$

with

$$\begin{aligned} \mathscr{P}_n &= \mathscr{W} p_n W_n^{-1}\rho_n \colon C_0^1[0,1] \to \mathscr{W}(\mathscr{S}_n) \subset C_0^1[0,1], \\ \mathscr{Q}_n &= p_n W_n^{-1}\rho_n \mathscr{W} \colon C[0,1] \to \mathscr{S}_n \subset C[0,1]. \end{aligned} \tag{4.4}$$

It is easily verified that $\mathscr{P}_n = \mathscr{P}_n^2$ and $\mathscr{Q}_n = \mathscr{Q}_n^2$, hence $\mathscr{P}_n$ and $\mathscr{Q}_n$ are *projectors*. The formulation (4.2) is the standard formulation of a projection method; cf. Phillips [15]. We are interested in the boundedness, uniformly in n, of the projectors $\mathscr{P}_n$ and $\mathscr{Q}_n$ in their natural setting.

LEMMA 4.1. $\sup_n \|\mathscr{Q}_n\| < \infty$.

Proof. Let $f \in C[0,1]$. From (3.9), we obtain for $i > rp^*$ that

$$\Psi_i(x) = \Psi_i(W_n^{-1}\rho_n \mathscr{W} f; x) = n^{-1} \sum_{j=0}^{p^*} (W_n^{-1}\rho_n \mathscr{W} f)_{i-jr} B_j + \sum_{j=0}^{1} (\rho_n \mathscr{W} f)_{i-jr} b_j$$

where we dropped the arguments of B_j and b_j. Hence

$$\|\Psi_i\|_{\sigma_{i+j}} \leqslant c\|\mathscr{W}f\|, \tag{4.5}$$

$j = 0, \pm 1$, where we used the zero-stability. From (4.5), (3.18), (3.20), we then obtain that ($z = W_n^{-1}\rho_n\mathscr{W}f$)

$$\|\mathscr{W}p_n W_n^{-1}\rho_n\mathscr{W}f\| \leqslant c\|\mathscr{W}f\|. \tag{4.6}$$

This says that $\mathscr{P}_n$: $C[0,1] \to C[0,1]$ is bounded, uniformly in n. Now, for every $1 \leqslant i \leqslant n + d$,

$$\|\mathscr{Q}_n f\|_{\sigma_i} = \left\|\frac{d}{dx}\mathscr{P}_n\mathscr{W}f\right\|_{\sigma_i} = \left\|\frac{d}{dx}(\mathscr{P}_n\mathscr{W}f - \kappa_i)\right\|_{\sigma_i}$$

for every function $\kappa_i \in C[0,1]$ which is constant on σ_i. So

$$\begin{aligned}\|\mathscr{Q}_n f\|_{\sigma_i} &\leqslant cn\|\mathscr{P}_n\mathscr{W}f - \kappa_i\|_{\sigma_i} \\ &\leqslant cn\|\mathscr{P}_n(\mathscr{W}f - \kappa_i)\|_{\sigma_i} + cn\|(\mathscr{P}_n - I)\kappa_i\|_{\sigma_i}.\end{aligned} \tag{4.7}$$

The first term on the right may be estimated using (4.6) as

$$cn\|\mathscr{W}f - \kappa_i\|_{\sigma_i},$$

and by setting

$$\kappa_i(x) = \begin{cases} x\mathscr{W}f(y_{i-1}), & 0 \leqslant x < y_{i-1}, \\ \mathscr{W}f(y_{i-1}), & y_{i-1} \leqslant x \leqslant 1, \end{cases} \tag{4.8}$$

this may be estimated as $cn \cdot n^{-1}\|(\mathscr{W}f)'\|_{\sigma_i} \leqslant c\|f\|$. The second term on the right of (4.7) can be written as

$$cn\|(\mathscr{P}_n - I)(\kappa_i - \mathscr{W}\psi)\|_{\sigma_i} \leqslant \tilde{c}n\|\kappa_i - \mathscr{W}\psi\|_{\sigma_i},$$

for every $\psi \in \mathscr{S}_n$. By Lemma 3.2, this may be estimated as $cn \cdot n^{-1}\|\kappa_i'\| \leqslant c\|\mathscr{W}f\| \leqslant c\|f\|$. Consequently, (4.7) yields $\|\mathscr{Q}_n f\|_{\sigma_i} \leqslant c\|f\|$ for each i, n. Piecing together the σ_i then proves the lemma. □

Remark. Following Lemma 3.2, we said we would rather have the stronger inequality (3.4). If (3.4) holds, then the proof of Lemma 4.1 is quite easy: for $f \in C[0,1]$, let $\psi \in \mathscr{S}_n$ such that $\|\psi\| \leqslant c\|f\|$, $\|\mathscr{W}f - \mathscr{W}\psi\| \leqslant cn^{-1}\|f\|$. Then

$$W_n^{-1}\rho_n\mathscr{W}f = W_n^{-1}\rho_n\mathscr{W}\psi + O\left(\|W_n^{-1}\|n^{-1}\|f\|\right). \tag{4.9}$$

By the zero-stability, the big O term is $O(\|f\|)$, and $\|W_n^{-1}\rho_n\mathscr{W}\psi\| = \|r_n\psi\| \leqslant c\|f\|$. So $\|W_n^{-1}\rho_n\mathscr{W}f\| \leqslant c\|f\|$, etc. Now that we have proven Lemma 4.1 via a different route, (3.4) follows, with $\psi = \mathscr{Q}_n f$, but to no apparent purpose. However, it still leaves the conjecture that (3.4) hold without the zero-stability assumption.

Lemma 3.2 says that the projection method (1.11), (4.2) is stable. It is actually possible to prove that the collocation-projection method

$$\begin{cases} \rho_n\mathscr{V}f_n = \rho_n g, \\ f_n \in \mathscr{S}_n, \end{cases} \tag{4.10}$$

is stable as a projection method, i.e. there is a corresponding Lemma 3.2; cf. Eggermont [5], [6]. However, we are more interested in the system (1.4) for $\Phi(x, y, z) \equiv K(x, y)z$, with $K \in \mathscr{K}_p$. Then the system (1.4) may be written as

$$\int_0^{y_i}[p_n r_n K(y_i, \cdot)f_n(\cdot)](y)\,dy = g(y_i), \qquad i = 1,2,\ldots,n + d - 1, \tag{4.11}$$

and a similar equation for $i = 0$ with y_0 replaced by $x_0^{(s)}$. This may be denoted as

$$V_n r_n f_n = \rho_n g. \tag{4.12}$$

LEMMA 4.2. *If $K \in \mathscr{K}_0$, then there exists an $N > 0$ such that*

$$\sup_{n>N} \|V_n^{-1} W_n\| < \infty.$$

Proof. Follows the proof of [5, Lemma 5.1] verbatim, but with different interpretations. □

COROLLARY 4.3. *Under the conditions of Lemma 4.2,*

$$\sup_{n>N} n^{-1} \|V_n^{-1}\| < \infty.$$

THEOREM 4.4. *If $K \in \mathscr{K}_0$, then for n large enough, the solution $r_n f_n$ of (4.12) exists and satisfies*

$$\|r_n(f_n - f)\| \leqslant c \sup_x \sum_{i=0}^{1} \inf \|D^i K(x, \cdot) f(\cdot) - \psi_i\|,$$

where $D^i = d^i/dx^i$, and the infima are taken over $\psi_i \in \mathscr{S}_n$.

Proof. We follow the proof of [5, Theorem 5.2] very closely. From (4.12),

$$V_n r_n (f_n - f) = \rho_n h, \tag{4.13}$$

with

$$h(x) = \int_0^x \{K(x, y) f(y) - [p_n r_n K(x, \cdot) f(\cdot)](y)\}\, dy. \tag{4.14}$$

Observe that $h \in C_0^1[0, 1]$, so $h = \mathscr{W}(h')$. Then from (4.13),

$$r_n(f_n - f) = V_n^{-1} W_n W_n^{-1} \rho_n \mathscr{W}(h')$$

hence

$$\|r_n(f_n - f)\| \leqslant c \|V_n^{-1} W_n\|\, \|r_n\|\, \|\mathscr{Q}_n\|\, \|h'\| \leqslant c\|h'\|,$$

by Lemmas 4.1 and 4.2. For fixed x, $h'(x)$ may be written as

$$\begin{aligned} h'(x) = K(x, x) f(x) - \psi_0(x) - [p_n r_n (K(x, \cdot) f(\cdot) - \psi_0(\cdot))](x) \\ + \int_0^x \{K_x(x, y) f(y) - \psi_1(y) \\ - [p_n r_n (K_x(x, \cdot) f(\cdot) - \psi_1(\cdot))](y)\}\, dy, \end{aligned} \tag{4.15}$$

for every $\psi_0, \psi_1 \in \mathscr{S}_n$ (since $p_n r_n \psi_i = \psi_i$). Consequently,

$$\|h'\| \leqslant c \sup_x \left\{ \inf_{\psi_0} \|K(x, \cdot) f(\cdot) - \psi_0\| + \inf_{\psi_1} \|K_x(x, \cdot) f(\cdot) - \psi_1\| \right\} \tag{4.16}$$

and the theorem follows. □

COROLLARY 4.4. *If $K \in \mathscr{K}_q$, $f \in C^q[0, 1]$, then for n large enough*

$$\|r_n(f_n - f)\| \leqslant c\|K\|_{q+1,T} \|f\|_q n^{-q},$$

for $q = 1, 2, \ldots, p$.

This completes the convergence analysis of the quadrature methods for linear integral equations.

5. The Nonlinear Case. The theory of Section 4 extends to nonlinear equations (1.1) with $\Phi \in \mathcal{N}_0$. Again, we assume throughout that conditions (1.14-15) are satisfied.

LEMMA 5.1. *If $\Phi \in \mathcal{N}_0$, then there exists an N such that*

$$\sup_{n>N} \sup\left\{\left\|\left(U_n'(e_n)\right)^{-1} W_n\right\| : e_n \in \mathbf{R}^{n+d}\right\} < \infty.$$

Here $U_n'(e)$ is the ***Fréchet derivative*** *of U_n at e.*

Proof. The proof follows by inspection of the proof of [5, Lemma 5.1], cf. Lemma 4.2. (The norm $\|(U_n'(e_n))^{-1}W_n\|$ depends only on the quantities $\inf|\Phi_z(x, x, z)|$ and $\|\Phi_{zx}\|_T$.) □

COROLLARY 5.2. *If $\Phi \in \mathcal{N}_0$, then for n large enough, the system* (1.4) *has a unique solution.*

LEMMA 5.3. *If $\Phi \in \mathcal{N}_0$, there exists a constant c such that for all n large enough and for all f, g and $h \in C[0,1]$, the statement*

$$U_n r_n f - U_n r_n g = \rho_n \mathscr{W} h$$

implies $\|r_n f - r_n g\| \leqslant c\|h\|$.

Proof. The proof is a slight variation of the proof of Hadamard's theorem, Berger [3, (5.1.5)]. Let e_n: $[0,1] \to \mathbf{R}^{n+d}$ satisfy

$$U_n(e_n(t)) = tU_n r_n f + (1-t)U_n r_n g. \tag{5.1}$$

Then after differentiation with respect to t, premultiplication with $(U_n'(e_n(t)))^{-1}$ which is possible by Lemma 5.1, and integration over $t \in [0,1]$ we obtain

$$e_n(1) - e_n(0) = \int_0^1 \left(U_n'(e_n(t))\right)^{-1} \rho_n \mathscr{W} h \, dt. \tag{5.2}$$

Apparently, $e_n(1) = r_n f$, and $e_n(0) = r_n g$. From (5.2), we then obtain

$$\|r_n f - r_n g\| \leqslant \sup_t \left\|\left(U_n'(e_n(t))\right)^{-1} W_n\right\| \left\|W_n^{-1}\rho_n \mathscr{W}\right\| \|h\|.$$

By Lemmas 5.1 and 4.1, we then get

$$\|r_n f - r_n g\| \leqslant c\|h\|,$$

and we are done. □

THEOREM 5.4. *If $\Phi \in \mathcal{N}_0$, then for n large enough* (*depending on Φ*) *the solution of* (1.4) *exists and satisfies*

$$\|r_n(f_n - f)\| \leqslant c \sup_x \sum_{i=0}^{1} \inf\left\|D^i\Phi(x, \cdot, f(\cdot)) - \psi_i\right\|$$

where $D^i = d^i/dx^i$, f is the solution of (1.1) *with $g \in C_0^1[0,1]$, and where the infima are over $\psi_i \in \mathscr{S}_n$.*

Proof. We have

$$U_n r_n f_n - U_n r_n f = \rho_n h_n, \tag{5.3}$$

where

$$h_n(x) = \int_0^x \{\Phi(x, y, f(y)) - [p_n r_n \Phi(x, \cdot, f(\cdot))](y)\}\, dy.$$

Note that $h_n \in C_0^1[0,1]$, so $h_n = \mathscr{W}(h'_n)$, whence (5.3) yields

$$U_n r_n f_n - U_n r_n f = \rho_n \mathscr{W} h'_n.$$

Lemma 5.3 then gives for n large enough,

$$\|r_n(f_n - f)\| \leqslant c\|h'_n\|.$$

The theorem follows after a derivation similar to (4.15-16). □

COROLLARY 5.5. *If $\Phi \in \mathscr{N}_q$ and $g \in C_0^{q+1}[0,1]$ then*

$$\|r_n(f_n - f)\| \leqslant c\|\Phi\|_{q+1, T\times \mathbf{R}} \|f\|_q n^{-q},$$

for $q = 1, 2, \ldots, p$.

This completes the analysis of the nonlinear equation.

6. Some Well-Known Methods Fit the Mold. In this section, we show that the following classes of methods satisfy the zero-stability and compositeness conditions (1.13-14):

1. Cyclic linear multistep methods: Holyhead et al. [9], Andrade and McKee [1].
2. Reducible quadrature methods: Wolkenfelt [23], [24], with special cases by Taylor [19], and Gladwin [7], [8].
3. Runge-Kutta methods (but we cannot prove convergence of order higher than two): Keech [10].

6.1. *Cyclic Linear Multistep Methods.* Cyclic linear multistep methods are based on interpolatory quadrature rules used in a cyclic fashion. Consequently, the weights w_{ij} in (1.2) satisfy

$$w_{i+r,j} = w_{i,j}, \qquad j = 0, 1, \ldots, i, \tag{6.1}$$

i.e., r is the *exact* repetition factor, so (1.16) is certainly satisfied, as is (1.14) by the construction of the quadrature rules. The methods contain a number of free parameters which need to be chosen such that the zero-stability condition (1.14) holds. This usually involves the solution of a nonlinear system of equations; Holyhead et al. [9], Andrade and McKee [1].

6.2. *Reducible Quadrature Methods.* These methods were studied extensively by Wolkenfelt [23], [24]. They include those of Taylor [19], and Gladwin [7], [8]. In these methods, the weights w_{ij} are constructed via linear multistep methods, as follows. For the basic linear case $\Phi(x, y, z) \equiv z$, the equation (1.2) is written as

$$\sum_{j=0}^{s} a_j (W_n r_n f)_{i-j} = h \sum_{j=0}^{s} b_j f(x_{i-j}), \qquad i = s, s+1, \ldots, n+d-1, \tag{6.2}$$

where the a_j and b_j form a linear multistep method for the initial value problem for an ordinary differential equation which satisfies the following two conditions.

(i) *Stability requirements*:

$$\rho(\xi) = \sum_{j=0}^{s} a_j \xi^j / (1 - \xi), \tag{6.3}$$

and

$$(6.4)\qquad \sigma(\xi) = \sum_{j=0}^{s} b_j \xi^j,$$

are bounded away from zero on $\mathbf{D} = \{\xi \in \mathbf{C}: |\xi| < 1\}$;

(ii) *The multistep method is convergent* of order p, i.e.,

$$(6.5)\qquad \sum_{j=0}^{s} a_j \mathscr{W} f(x_{i-j}) = h \sum_{j=0}^{k} b_j f(x_{i-j}) + T_i(f), \qquad i \geqslant s,$$

with

$$(6.6)\qquad |T_i(f)| \leqslant c\|f^{(p)}\| n^{-p-1}.$$

One verifies that the first stability requirement guarantees that elements of nW_n are bounded, uniformly in n, and the second one guarantees the zero-stability of W_n, (1.13).

We also claim that (i) and (ii) combined yield the condition (1.14) with $r = 1$. This may be seen as follows. From (6.2) and (6.5),

$$(6.7)\qquad \sum_{j=0}^{k} a_j E_{i-j} = T_i(f), \qquad i \geqslant s,$$

where

$$(6.8)\qquad E_i = \mathscr{W} f(x_i) - (W_n r_n f)_i.$$

We already know (or *assume*) that

$$(6.9)\qquad |E_i| \leqslant c\|f^{(p)}\| n^{-p-1}, \qquad i = 0, 1, \dots, s-1,$$

so, defining $T_i(f)$, $i = 0, 1, \dots, s-1$, by (6.7-9), we have that (6.6) and (6.7) hold for all $i \geqslant 0$. Using generating series, we obtain from (6.7) that for $\xi \in \mathbf{D}$,

$$(6.10)\qquad (1-\xi) \sum_{j=0}^{\infty} E_j \xi^j = \sum_{j=0}^{\infty} T_j(f) \xi^j / \rho(\xi).$$

Now the stability requirement for $\rho(\xi)$ tells us that

$$(6.11)\qquad \{\rho(\xi)\}^{-1} = \sum_{j=0}^{\infty} c_j \xi^j, \qquad \xi \in \mathbf{D},$$

with $c_j = O(R^j)$, $j \to \infty$, for some $0 < R < 1$. Hence

$$(6.12)\qquad \Delta_\infty E_i = \sum_{j=0}^{i} T_j(f) c_{i-j},$$

and so

$$(6.13)\qquad |\Delta_\infty E_i| \leqslant c \max_j |T_j(f)| \leqslant c\|f^{(p)}\| n^{-p-1},$$

where Δ_∞ is the infinite version of Δ_n^1, (1.15). Now (6.13) is exactly (1.14) for $r = 1$, and we have proved our claim. (Observe that, in the process, we have proved that

$$\sup_n n^{-1} \left\| W_n^{-1} (\Delta_n^1)^{-1} \right\| < \infty,$$

cf. Subsection 7.1.)

Since (1.13-14) hold the reducible quadrature methods are covered by the theory of this paper.

A similar analysis leading to (1.13-14) can be done for the *modified multilag methods*, Wolkenfelt [24], applied to Volterra integral equations of the first kind. Here it requires a bit of effort to write the method as a quadrature method. We omit the details.

6.3. *Runge-Kutta methods* fit the mold, too, though not really. The methods we consider are those surveyed by Brunner [4, (3.5)]. For a specific, third order example, see Keech [10]. It is clear that these methods are zero-stable since they provide *local* differentiation formulas when applied to the equation $\mathscr{W}f = g$, but condition (1.14) is not satisfied with the "right" value of p, e.g., for Keech's method [10], (1.14) holds only for $p = 2$ and not for $p = 3$, hence, our method only provides order 2 convergence.

7. Alternative Approaches. Here, we give an outline of two other approaches to optimal error estimates.

7.1. *Suppose that* in addition to zero-stability, we have

$$\sup_n n^{-1}\left\| W_n^{-1}(\Delta_n^r)^{-1}\right\| < \infty. \tag{7.1}$$

We then obtain for the linear equation $\mathscr{W}f = g$ from (1.8) that

$$\|r_n(f_n - f)\| \leq \left\| W_n^{-1}(\Delta_n^r)^{-1}\right\| \, \|\Delta_n^r(\rho_n\mathscr{W} - W_n r_n)f\|$$

and now (7.1) combined with (1.14) yields the optimal estimate. To extend this to the general linear equation $\mathscr{V}f = g$ requires most likely that Lemma 4.2 hold, but this appears to be hard to establish without further assumptions on the structure of W_n (cf. Subsection 7.2). Also (1.14) needs to be established for $\mathscr{V}$ rather than for $\mathscr{W}$. The condition (7.1) appears to be *equivalent* to zero-stability for quadrature methods for which W_n is essentially block-circulant, see below, but without this assumption it appears to be hard to prove.

7.2. *All methods* discussed in Section 6 share one property which we have not used, viz. the matrices W_n are essentially block-circulant, i.e., W_n may be partitioned as

$$W_n = n^{-1}\begin{bmatrix} \tilde{A}_0 & & & & & \\ \tilde{A}_1 & A_0 & & & \bigcirc & \\ \cdot & A_1 & A_0 & & & \\ \cdot & A_2 & A_1 & A_0 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \tilde{A}_m & A_{m-1} & \cdot & A_2 & A_1 & A_0 \end{bmatrix}, \tag{7.2}$$

with $A_i \in \mathbf{R}^{r\times r}$, and $n = mr$ (say). Also, the A_i and $\tilde{A}_i$ are independent of n. Consequently, nW_n is a finite section of an infinite matrix which is independent on n, and power series methods may be used. This way Lemma 4.2 can be proved *without* the use of the uniformly bounded projectors, and optimal error estimates appear to be within reach. It appears that the most successful alternate approach would be a combination of 7.1 and 7.2.

8. Conclusion. We have presented an approach to the analysis of quadrature methods with the following three features: (i) It closes the gap between collocation-projection methods and quadrature methods; (ii) It provides optimal error estimates;

and (iii) There are only minimal assumptions on the particular (algebraic) structure of the quadrature methods.

The analysis does not cover Runge-Kutta methods, and it remains an open question whether they can be treated within our framework. One approach might be to proceed in a block-by-block fashion in the interpolation problems of Section 3.

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