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More Trigonometric Integrals

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Abstract. Integrals of the form

$$\int_0^{\pi/2} e^{ip\theta}\cos^q\theta\,d\theta, \qquad \int_0^{\pi/2} e^{ip\theta}\sin^q\theta\,d\theta$$

(p real, $\mathrm{Re}(q) > -1$) are expressed in terms of Gamma and hypergeometric functions for integer and noninteger values of q and p. The results include those of [2] as special cases.

Introduction. The integrals considered here are

$$\int_0^{\pi/2} e^{ip\theta}\cos^q\theta\,d\theta, \tag{1}$$

$$\int_0^{\pi/2} e^{ip\theta}\sin^q\theta\,d\theta, \tag{2}$$

where p is real, and $\mathrm{Re}(q) > -1$. Values for some of the above integrals are recorded in [1, art. 3.631], but only for special (or integer) values of "q", and not always in closed form. The integrals (1) and (2) are, of course, related since, with the change of variable $\theta \to \pi/2 - \theta$, (2) becomes

$$\int_0^{\pi/2} e^{ip\theta}\sin^q\theta\,d\theta = e^{ip\pi/2}\int_0^{\pi/2} e^{-ip\theta}\cos^q\theta\,d\theta \tag{3}$$

resulting in the following relations:

$$\begin{aligned}\int_0^{\pi/2}\sin^q\theta\cos p\theta\,d\theta &= \sin\frac{p\pi}{2}\int_0^{\pi/2}\cos^q\theta\sin p\theta\,d\theta \\ &\quad+\cos\frac{p\pi}{2}\int_0^{\pi/2}\cos^q\theta\cos p\theta\,d\theta,\end{aligned} \tag{4}$$

$$\begin{aligned}\int_0^{\pi/2}\sin^q\theta\sin p\theta\,d\theta &= \sin\frac{p\pi}{2}\int_0^{\pi/2}\cos^q\theta\cos p\theta\,d\theta \\ &\quad-\cos\frac{p\pi}{2}\int_0^{\pi/2}\cos^q\theta\sin p\theta\,d\theta.\end{aligned} \tag{5}$$

It is evident that, either with the aid of multiple angle formulae, or integration by parts, all of these integrals can be evaluated in finite form if either "p" or "q", or both, are integers, see, e.g., [1, arts. 2.536–8]. For this reason, it will be assumed in the remainder of the paper that "p" and "q" are arbitrary, noninteger quantities, subject to the conditions stated above.

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1. Evaluation by Contour Integration. The integrals in question are evaluated by integrating the function

$$f(z) = (1 + z^2)^q (z^{p-q-1}) \tag{6}$$

in the complex z ($= \rho e^{i\theta} \equiv x + iy$) plane around the contour consisting of the portion of the real axis from $x = 0$ to $x = 1$, the quarter arc of the unit circle from $\theta = 0$ to $\theta = \pi/2$, and the portion of the imaginary axis from $y = 1$ to $y = 0$. The total line integral is zero, provided the contour is modified by small circular arcs about the branch points at $z = 0$ and $z = i$, since there is no contribution to the total value of the integral from these arcs when their radii approach zero, provided $\mathrm{Re}(q) > -1$ and $\mathrm{Re}(p - q) > 0$. The total line integral thus becomes a linear combination of the real integrals

$$\int_0^{\pi/2} \cos^q\theta \sin p\theta \, d\theta, \tag{7}$$

$$\int_0^{\pi/2} \cos^q\theta \cos p\theta \, d\theta, \tag{8}$$

$$\int_0^1 (1 + t)^q t^{(p-q-2)/2} \, dt, \tag{9}$$

$$\int_0^1 (1 - t)^q t^{(p-q-2)/2} \, dt. \tag{10}$$

The integral (10), as is well known, can be expressed by Gamma functions:

$$\int_0^1 (1 - t)^q t^{(p-q-2)/2} \, dt = \frac{\Gamma(1+q)\Gamma\left(\frac{p-q}{2}\right)}{\Gamma\left(\frac{2+p+q}{2}\right)}, \tag{11}$$

while (9) can be evaluated as a hypergeometric function:

$$\int_0^1 (1 + t)^q t^{(p-q-2)/2} \, dt = \frac{2}{p-q} \, {}_2F_1\left(-q, \frac{p-q}{2}; \frac{1+p-q}{2}; -1\right) \tag{12}$$

(see, e.g., [1, art 9.111], or [4, p. 12]).

Equating the real and imaginary parts of the resulting integrals gives

$$\text{(a)} \quad 2^{1+q} \int_0^{\pi/2} \cos^q\theta \cos p\theta \, d\theta = \sin\left(\frac{p-q}{2}\pi\right) \frac{\Gamma(1+q)\Gamma\left(\frac{p-q}{2}\right)}{\Gamma\left(\frac{2+p+q}{2}\right)},$$

$$\text{(b)} \quad 2^{1+q} \int_0^{\pi/2} \cos^q\theta \sin p\theta \, d\theta = \frac{2}{(p-q)} \, {}_2F_1\left(-q, \frac{p-q}{2}; \frac{2+p-q}{2}; -1\right) - \cos\left(\frac{p-q}{2}\pi\right) \frac{\Gamma(1+q)\Gamma\left(\frac{p-q}{2}\right)}{\Gamma\left(\frac{2+p+q}{2}\right)}, \tag{13}$$

and using the relations (4) and (5),

(14)

$$\text{(a)}\quad 2^{1+q}\int_0^{\pi/2} \sin^q\theta\cos p\theta\,d\theta = \frac{2\sin(p\pi/2)}{p-q}\,{}_2F_1\left(-q, \frac{p-q}{2}, \frac{2+p-q}{2}; -1\right) - \sin\left(\frac{q\pi}{2}\right)\cdot\frac{\Gamma(1+q)\Gamma\left(\frac{p-q}{2}\right)}{\Gamma\left(\frac{2+p+q}{2}\right)},$$

$$\text{(b)}\quad 2^{1+q}\int_0^{\pi/2} \sin^q\theta\sin p\theta\,d\theta = \frac{-2\cos(p\pi/2)}{p-q}\,{}_2F_1\left(-q, \frac{p-q}{2}; \frac{2+p-q}{2}; -1\right) + \cos\left(\frac{q\pi}{2}\right)\cdot\frac{\Gamma(1+q)\Gamma\left(\frac{p-q}{2}\right)}{\Gamma\left(\frac{2+p+q}{2}\right)}.$$

(It may be noted that, of the four relations in (13) and (14), only (13a) is given in [1] for arbitrary values of "p" and "q", and also that, although these relations were established under the assumption that $\mathrm{Re}(p-q) > 0$, the various functions on the right of (13) and (14) are defined for all values of "p" and "q", provided $(p-q) \neq 0, -2, -4, \ldots, -2n$, and therefore the results may be extended to the wider range $\mathrm{Re}(q) > -1$ by analytic continuation.)

2. Expressions When $\frac{1}{2}(p-q)$ is a Negative Integer or Zero. From principles of continuity, it follows that all of the results given by Eqs. (13) and (14) must hold if "p" is replaced by "$-p$". In particular, from (13), we get, after some obvious transformations of the Gamma function:

(15)

$$\text{(a)}\quad 2^{1+q}\int_0^{\pi/2} \cos^q\theta\cos p\theta\,d\theta = \pi\frac{\Gamma(1+q)}{\Gamma[1+\frac{1}{2}(p+q)]\Gamma[1-\frac{1}{2}(p-q)]},$$

$$\text{(b)}\quad 2^{1+q}\int_0^{\pi/2} \cos^q\theta\sin p\theta\,d\theta = \frac{2}{p+q}\,{}_2F_1\left(-q, -\tfrac{1}{2}(p+q); 1-\tfrac{1}{2}(p+q); -1\right) - \pi\cot(p+q)\frac{\pi}{2}\,\frac{\Gamma(1+q)}{\Gamma[1+\frac{1}{2}(p+q)]\Gamma[1-\frac{1}{2}(p-q)]},$$

and the above equations now yield determinate expressions when $\frac{1}{2}(p-q)$ is a negative integer or zero. Specifically if we set $p = q + 2k$, and make use of the relation

$${}_2F_1(a, b; c; z) = (1-z)^{-b}\,{}_2F_1\left(c-a, b; c; \frac{z}{z-1}\right),$$

(15b) becomes

$$2^{1+q}\int_0^{\pi/2} \cos^q\theta \sin(q+2k)\theta\, d\theta = \frac{2^{k+q}}{k+q}\, {}_2F_1(1-k, -k-q; 1-k-q; \tfrac{1}{2}) - \pi \cot q\pi \frac{\Gamma(1+q)}{\Gamma(1+k+q)\Gamma(1-k)}, \tag{16}$$

and if "k" is a positive integer, the second term vanishes, while the hypergeometric series terminates. The result then becomes equivalent to formula 3.632(3) of [1]. On the other hand, if "k" is a negative integer or zero, the second term remains, and the series is no longer finite.

As a special result in the second case, we obtain the following closed form expression for the integral

$$\int_0^{\pi/2} \cos^q\theta \sin q\theta\, d\theta = \frac{1}{2q}\, {}_2F_1(1, -q; 1-q; \tfrac{1}{2}) - \frac{\pi}{2^{q+1}} \cot q\pi, \tag{17}$$

which has not been given previously, except for integer "q", in which case the above expression is indeterminate [see Eq. (20) below].

3. Alternative Integral Representations When $p - q$ is an Even Integer. A different, and much simpler representation for the integrals in (16) and (17) can be obtained by re-expressing (13b) in terms of (11) and (12):

$$\begin{aligned} &2^{1+q}\int_0^{\pi/2} \cos^q\theta \sin p\theta\, d\theta \\ &= \int_0^1 [(1+t)^q - 1]\, t^{(p-q)/2-1}\, dt \\ &+ 2\frac{1 - \cos\frac{\pi}{2}(p-q)}{p-q} + \cos\frac{\pi}{2}(p-q)\int_0^1 [1-(1-t)^q]\, t^{(p-q)/2-1}\, dt. \end{aligned} \tag{18}$$

In the limit, as $p \to q$, this gives, for arbitrary "q" > -1,

$$\begin{aligned} 2^{1+q}\int_0^{\pi/2} \cos^q\theta \sin q\theta\, d\theta &= \int_0^1 \frac{(1+t)^q - 1}{t}\, dt + \int_0^1 \frac{1-(1-t)^q}{t}\, dt \\ &= \int_0^2 \frac{u^q - 1}{u-1}\, du \end{aligned} \tag{19}$$

and if "q" is a positive integer, "n", the following result quoted in [1, formula 3.631(16)]:

$$2^{1+n}\int_0^{\pi/2} \cos^n\theta \sin n\theta\, d\theta = \int_0^2 \frac{u^n - 1}{u-1}\, du = 2 + \frac{2^2}{2} + \frac{2^3}{3} + \cdots + \frac{2^n}{n}. \tag{20}$$

The result of Eq. (19) is significant in that it shows that the integral on the left can be evaluated in terms of elementary functions whenever "q" is a rational number, (n/m), since then, with the change of variable

$$u = v^n$$

on the right-hand side, the integrand becomes the quotient of two polynomials. For example,

$$\int_0^{\pi/2} \cos^{1/2}\theta \sin\tfrac{1}{2}\theta\, d\theta = 2^{-1/2}\int_0^{2^{+1/2}} \frac{v\,dv}{1+v} = 1 - 2^{-1/2}\ln(2^{1/2}+1);$$

$$(21)\quad \int_0^{\pi/2} \cos^{1/4}\theta \sin\tfrac{1}{4}\theta\, d\theta = 2^{+3/4}\int_0^{2^{1/4}} \frac{v^3\,dv}{(1+v)(1+v^2)}$$

$$= 2 - 2^{-1/4}\left[\ln(2^{1/4}+1) = \tfrac{1}{2}\ln(2^{1/2}+1) + \tan^{-1}(2^{1/4})\right].$$

Application of the same transformation to the integral in (16) gives, for $k > 0$,

$$(22)\qquad 2^{1+q}\int_0^{\pi/2} \cos^q\theta \sin(q+2k)\theta\, d\theta = \int_0^2 u^q(u-1)^{k-1}\,du,$$

and after expanding the term $(u-1)^{k-1}$ by the binomial theorem and integrating term-by-term, we obtain the expansion of formula 3.632(3) of [1]. On the other hand, successive integrations by parts applied to the right-hand side of (22) leads to a different representation:

$$(23)\quad \begin{aligned}&\int_0^{\pi/2} \cos^q\theta \sin(q+2k)\theta\, d\theta \\ &\quad = \frac{1}{q+1}\left[1 - \frac{k-1}{q+2}\cdot 2 + \frac{(k-1)(k-2)}{(q+2)(q+3)}\cdot 2^2 + \cdots\right].\end{aligned}$$

When "k" is negative, similar, but more complex, expressions can be obtained for the integral in (16), by first subtracting from the integrands on the right side of (18) the first $(|k|+1)$ terms of the Taylor series for $(1+t)^q$ and $(1-t)^q$. For example, if $k = -1$, we get

$$(24)\quad \begin{aligned}&2^{1+q}\int_0^{\pi/2} \cos^q\theta \sin(q-2)\theta\, d\theta \\ &= \lim_{p-q\to -2}\left\{\int_0^1 \left[(1+t)^q - 1 - qt\right]t^{(p-q)/2-1}\,dt \right.\\ &\qquad + \cos\frac{\pi}{2}(p-q)\int_0^1 \left[1 - qt - (1-t)^q\right]t^{(p-q)/2-1}\,dt \\ &\qquad \left. + 2\frac{1-\cos\frac{\pi}{2}(p-q)}{p-q} + 2q\frac{1+\cos\frac{\pi}{2}(p-q)}{p-q+2}\right\} \\ &= \int_0^1 \frac{(1+t)^q - 1 - qt}{t^2}\,dt - \int_0^1 \frac{(1 - qt - (1-t)^q)}{t^2}\,dt - 2 \\ &= \int_0^2 \frac{u^q - 1 - q(u-1)}{(u-1)^2}\,du - 2.\end{aligned}$$

For integer values of $q = n$, this again becomes a terminating series, since

$$\begin{aligned}\frac{u^n - 1 - n(u-1)}{(u-1)^2} &= 0; \qquad n = 0, 1, \\ &= u^{n-2} + 2u^{n-3} + \cdots + (n-1); \qquad n > 1,\end{aligned}$$

and hence

$$2^{1+n}\int_0^{\pi/2}\cos^n\theta\sin(n-2)\theta\,d\theta \tag{25}$$
$$=\left[\frac{2^{n-1}}{n-1}+2\cdot\frac{2^{n-2}}{n-2}+\cdots+(n-1)\cdot 2\right]-2;\qquad n>1,$$
$$=-2;\qquad n=0,1.$$

Similarly,

$$2^{1+q}\int_0^{\pi/2}\cos^q\theta\sin(q-4)\theta\,d\theta \tag{26}$$
$$=\int_0^2\frac{u^q-1-q(u-1)-\tfrac{1}{2}q(q-1)(u-1)^2}{(u-1)^3}du-2q$$

and

$$2^{1+n}\int_0^{\pi/2}\cos^n\theta\sin(n-4)\theta\,d\theta \tag{27}$$
$$=\left[\frac{2^{n-2}}{n-2}+3\frac{2^{n-3}}{n-3}+6\frac{2^{n-4}}{n-4}+\cdots+\frac{(n-1)(n-2)}{2}\cdot 2\right]$$
$$-2n;\qquad n>2,$$
$$=-2n;\qquad n=0,1,2.$$

Alternatively, integration by parts leads to

$$\int_0^{\pi/2}\cos^q\theta\sin p\theta\,d\theta=\frac{1}{q-p}\left\{1-q\int_0^{\pi/2}\cos^{q-1}\theta\sin(p+1)\theta\,d\theta\right\}, \tag{28}$$

which, when applied successively k times gives, for $k>0$, $q>k-1$,

$$\int_0^{\pi/2}\cos^q\theta\sin(q-2k)\theta\,d\theta \tag{29}$$
$$=\frac{q(q-1)\cdots(q-k+1)}{2^k k!}$$
$$\times\left\{\int_0^{\pi/2}\cos^{q-k}\theta\sin(q-k)\theta\,d\theta\right.$$
$$-\left[\frac{1}{(q-k+1)}+\frac{2\cdot 1!}{(q-k+1)(q-k+2)}\right.$$
$$\left.\left.+\cdots+\frac{(2)^{k-1}(k-1)!}{(q-k+1)(q-k+2)\cdots q}\right]\right\},$$

with the second integral in (29) given by (19). If $q=k$, the above result reduces to that of (20).

4. The Special Case: $q=1$. In the case of Eq. (14b), the condition

$$\mathrm{Re}(q)>-1$$

may be modified to

$$\mathrm{Re}(q)\geqslant -1.$$

For $q = -1$, the first term on the right becomes

$$(30)\quad \begin{aligned} &-2\frac{\cos\frac{1}{2}p\pi}{p+1}\,{}_2F_1(1,\tfrac{1}{2}(1+p);\tfrac{1}{2}(3+p);-1)\\ &\quad= -\cos\tfrac{1}{2}p\pi\int_0^1 (1+t)^{-1}t^{(p-1)/2}dt = -\cos\tfrac{1}{2}p\pi\beta\left(\frac{1+p}{2}\right),\end{aligned}$$

where

$$\beta(x) = \frac{1}{2}\left[\Psi\left(\frac{1+x}{2}\right) - \Psi\left(\frac{x}{2}\right)\right]$$

(see [1, formula 8.371(1)], or [3, formula 15.1.23]), while the second term reduces to $\pi/2$. Hence we obtain

$$(31)\quad \int_0^{\pi/2}\frac{\sin p\theta}{\sin\theta}d\theta = \frac{\pi}{2} - \frac{1}{2}\cos\frac{1}{2}p\pi\left[\Psi\left(\frac{3+p}{4}\right) - \Psi\left(\frac{1+p}{4}\right)\right],$$

a result given previously by the author [2, Eq. (11)]. In a similar manner, by subtracting the result of Eq. (14a) from the corresponding one when $p = 0$, and taking the limit as $q \to -1$, we get the expression for

$$(32)\quad \begin{aligned}\int_0^{\pi/2}\frac{1-\cos p\theta}{\sin\theta}d\theta &= \Psi\left(\frac{1+p}{2}\right) - \Psi\left(\frac{1}{2}\right)\\ &\quad -\frac{1}{2}\sin\frac{1}{2}p\pi\left[\Psi\left(\frac{3+p}{4}\right) - \Psi\left(\frac{1+p}{4}\right)\right],\end{aligned}$$

as given by Eq. (11) of the above reference.

5. Watson's Integral. In [5, p. 313], the following value is given for the integral

$$(33)\quad \int_0^{\pi}\cos^m\theta\cos p\theta\,d\theta = \frac{(-)^m\sin p\pi}{2^m(p+m)}\,{}_2F_1\left(-m,-\frac{p+m}{2};1-\frac{p+m}{2};-1\right),$$

where the sign of the second parameter has been corrected (cf. *Math. Comp.*, v. 14, 1960, p. 221). The above result has been reproduced with the incorrect sign in [4, p. 16], as well as in [1, art. 3.631(18)], with reference to the wrong page of [5], but is given correctly in art. 9.114 of [1].

The relation (33) is easily derived from (13a) and (14b) by bisecting the range of integration, and it can also be shown that the restriction $p \neq 0, \pm n$ is not necessary, since from elementary considerations, the integral is, in this case, zero except when $m \geqslant n$ and $m - n$ is even, in which instance it is equal to

$$(34)\quad \frac{\pi}{2^m}\binom{m}{\frac{m-n}{2}}.$$

On the other hand, after replacing p by $-p$ in the right side of (33), the latter expression becomes

$$(35)\quad \frac{(-)^m\sin p\pi}{2^m(p-m)}\,{}_2F_1\left(-m,\frac{p-m}{2};1+\frac{p-m}{2};-1\right),$$

while the hypergeometric series may be written in the form

$$\left(\frac{1}{p-m}\right){}_2F_1\left(-m, \frac{p-m}{2}; 1+\frac{p-m}{2}; -1\right) = \sum_{k=0}^{m} \frac{\binom{m}{k}}{p-m+2k}, \tag{36}$$

which will be finite unless $(p-m)$ is a negative even integer, or zero. It follows that, since

$$\lim_{p \to m-2k} \left(\frac{\sin p\pi}{p-m+2k}\right) = \pi(-1)^m,$$

the expression (35) will attain the value given by (34) when p is an integer: $n \leqslant m$, and $m-n$ is even, but will vanish otherwise.

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