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Generation of Permutations Following Lehmer and Howell*

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Abstract. This paper proves two formulas used to obtain, by an arithmetic method, both the next permutation with respect to a given one and the mth permutation, in lexicographic direct ordering.

1. Introduction. Almost all the methods used to generate permutations are based on exchanges and/or cycles and/or recursive techniques (see [4] and [5]).

Lehmer [3] developed a method to produce permutations of k marks $0, 1, \ldots, k-1$ in lexicographic ordering, based on the consideration that any permutation can be thought of as a base k integer.

In [1] Howell improved the above result of Lehmer, showing that all permutations of the k marks $0, 1, \ldots, k-1$ may be obtained in the direct lexicographic ordering, from the permutation $0\ 1 \cdots (k-2)\ (k-1)$, regarded as a base k integer, successively adding the number $k-1$ radix k, and rejecting all results which contain repeated digits.

In the present note this result is further improved upon, by computing the difference between two consecutive permutations in the direct lexicographic ordering. Moreover, an arithmetic method to find the mth permutation (in the direct lexicographic ordering) is provided.

To avoid ambiguities in the following, we denote the mth permutation by P_m and the same permutation, considered as a base k integer, by $\sigma_k P_m$.

2. Generation of the Next Permutation in the Direct Lexicographic Ordering. Let P_α be a permutation of the k marks $0, 1, \ldots, k-1$. P_α can be represented as a sequence $a_k\ a_{k-1} \cdots a_2\ a_1$ of k numbers a_i with $0 \leqslant a_i \leqslant k-1$ and $a_i \neq a_s$ for every $i \neq s$ $(1 \leqslant i \leqslant k, 1 \leqslant s \leqslant k)$.

Lemma 1. *Let $a_k \cdots a_h \cdots a_j \cdots a_1$ be the permutation P_α of the k marks $0, 1, \ldots, k-1$ and suppose $a_1 < \cdots < a_j < \cdots < a_{h-1} \not< a_h$ and $a_{j-1} < a_h < a_j$. Then $\sigma_k P_{\alpha+1}$ may be obtained by adding*

$$\Delta_\alpha = \sum_{i=1,[(h-1)/2]} k^{i-1}(1 - k^{h-2i})(a_{h-i} - a_i) + k^{h-j-1}(1 - k^j)(a_h - a_j)$$

radix k, to $\sigma_k P_\alpha$.

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Proof. It is well known that, in our hypotheses, $P_{\alpha+1}$ may be obtained from P_α exchanging a_h with a_j and reversing the order of the k marks $a_{h-1} \cdots a_{j+1}\, a_h\, a_{j-1} \cdots a_1$ (see [5]). Then, computing $\Delta_\alpha = \sigma_k P_{\alpha+1} - \sigma_k P_\alpha$, we obtain

$$\begin{aligned}\Delta_\alpha &= \sum_{i=1,h-1} k^{i-1}(a_{h-i} - a_i) + k^{h-j-1}(a_h - a_j) + k^{h-1}(a_j - a_h)\\ &= \sum_{i=1,[(h-1)/2]} k^{i-1}(a_{h-i} - a_i) + \sum_{i=1,[(h-1)/2]} k^{h-i-1}(a_i - a_{h-i})\\ &\quad + (k^{h-j-1} - k^{h-1})(a_h - a_j)\\ &= \sum_{i=1,[(h-1)/2]} k^{i-1}(1 - k^{h-2i})(a_{h-i} - a_i) + k^{h-j-1}(1 - k^j)(a_h - a_j).\end{aligned}$$

We wish to point out that, while the procedure described in [1] provides

$$\sum_{i=1,[k/2]} k^{i-1}(k - 2i + 1)(k^{k-2i} + k^{k-2i-1} + \cdots + 1) - k! + 1$$

integers which must be rejected in order to construct all permutations of the k marks $0, 1, 2, \ldots, k-1$, by our method we obtain the only integers which are associated with some permutation.

Remark 2. By an analogous procedure, we may easily verify that, starting at the αth permutation $a_k \cdots a_h \cdots a_j \cdots a_1$ with $a_1 > \cdots > a_j > \cdots > a_{h-1} \not> a_h$ and $a_{j-1} > a_h > a_j$, the previous permutation $P_{\alpha-1}$ may be obtained by adding to $\sigma_k P_\alpha$ the integer

$$\Phi_\alpha = \sum_{i=1,[(h-1)/2]} k^{i-1}(1 - k^{h-2i})(a_{h-i} - a_i) + k^{h-j-1}(1 - k^j)(a_h - a_j)$$

radix k.

3. Generation of the mth Permutation in the Direct Lexicographic Ordering.

THEOREM 3. *Let P_1 be the permutation $0\ 1\ 2 \cdots (k-1)$ of the k marks $0, 1, \ldots, k-1$; let m be an integer such that $m - 1 = \sum_{i=1,l} b_i i!$ with $l < k$ and $0 \leqslant b_i \leqslant i$. Then the permutation P_m may be obtained by adding, radix k, to $\sigma_k P_1$ the integer*

$$\begin{aligned}\Delta^m &= \sum_{i=1,l} \left\{ b_i k^i - (k^i - k^{i-b_i})/(k-1) \right\}\\ &\quad + \sum_{j=0,l-2}\ \sum_{i=j+1,l-1} \left\{ \delta_{l-j,l-i}\left(k^{l-i} - k^{l-j-b_{l-j}-\sum_{h=l-i,l-j-1}\delta_{l-j,h}}\right)\right\}\end{aligned}$$

where $\delta_{v,v-1} = \delta(b_{v-1} - b_v + 1)$ *and, if* $v - w > 1$,

$$\delta_{v,w} = \delta\left(v - b_v - \sum_{i=w+1,v-1} \delta_{v,i} - w + b_w\right) \quad \text{with } \delta(x) = \begin{cases} 1 & \text{if } x > 0,\\ 0 & \text{if } x \leqslant 0.\end{cases}$$

Proof. We will proceed by induction on m. For $m = 1$ one has $\Delta^1 = 0$. Now suppose that the statement is true for m, some positive integer: we shall prove that it is true for $m + 1$. To this end, in view of Lemma 1, it suffices to prove that

$$\sigma_k P_1 + \Delta^m + \Delta_m = \sigma_k P_1 + \Delta^{m+1},$$

or, equivalently,

$$\Delta^{m+1} - \Delta^m = \Delta_m.$$

Let $m - 1 = \Sigma_{i=2,l} b_i i! + b_1 1!$ ($l < k$) be the factorial representation of $m - 1$. We shall distinguish two cases: $b_1 = 0$ and $b_1 = 1$.

Case 1. Let $b_1 = 0$. Then we have $m = \Sigma_{i=2,l} b_i i! + 1$. Denoting by δ^* the δ's computed for Δ^{m+1}, it follows that $\delta_{i,j} = \delta^*_{i,j}$ for every $j > 1$. Moreover, it is easily verifiable that

$$\delta^*_{i,1} = 0 \quad \text{implies} \quad \delta_{i,1} = 0 \qquad \text{and} \qquad \delta_{i,1} = 1 \quad \text{implies} \quad \delta^*_{i,1} = 1$$

and we have

$$\delta_{i,1} = 0 \quad \text{and} \quad \delta^*_{i,1} = 1 \quad \text{if and only if } i - b_i - \sum_{j=2,i-1} \delta_{i,j} = 1.$$

That being stated, we have

$$\begin{aligned} \Delta^{m+1} - \Delta^m = \sum_{i=2,l} & \left[\delta^*_{i,1}\left(k - k^{i-b_i-\Sigma_{h=2,i-1}\delta_{i,h}-\delta^*_{i,1}}\right) \right. \\ & \left. -\delta_{i,1}\left(k - k^{i-b_i-\Sigma_{h=2,i-1}\delta_{i,h}-\delta_{i,1}}\right)\right] + (k-1) \\ = \sum_{i\in I} & (k-1) + (k-1) = (1 + |I|)(k-1), \end{aligned}$$

where $I = \{i | \delta^*_{i,1} = 1 \text{ and } \delta_{i,1} = 0\}$, i.e., $I = \{i | i - b_i - \Sigma_{j=2,i-1}\delta_{i,j} = 1\}$.

We remark that for $b_1 = 0$ in the mth permutation $a_k \cdots a_2\, a_1$ we have $a_2 < a_1$; therefore the $(m + 1)$st permutation may be obtained by adding to $\sigma_k P_m$ the integer $(a_2 - a_1)(1 - k)$, radix k (Lemma 1). Then, it will suffice to verify that

$$a_1 - a_2 = 1 + |I|.$$

To this end we note that a_2 and a_1, by the action of Δ^m, are respectively:

$$a_2 = k - 2 + |N| - |M| \quad \text{where } N = \{i | \delta_{i,1} = 1\} \text{ and}$$

$$M = \left\{ i | i - b_i - \sum_{h=1,i-1} \delta_{i,h} \leqslant 1 \right\},$$

$$a_1 = k - 1 - |L| \qquad \text{where } L = \left\{ i | i - b_i - \sum_{h=1,i-1} \delta_{i,h} = 0 \right\}.$$

Thus $a_1 - a_2 = 1 - |L| + |M| - |N|$. That being stated, let us show that $\delta_{i,1} = 1$ implies $i - b_i - \Sigma_{h=1,i-1}\delta_{i,h} \leqslant 1$. In fact, let us suppose

$$(1) \qquad i - b_i - \sum_{h=1,i-1} \delta_{i,h} \geqslant 2 \quad \text{and} \quad \delta_{i,1} = 1.$$

Then $\delta_{i,1} = \delta(i - b_i - \Sigma_{h=2,i-1}\delta_{i,h} - 1) = 1$ and, by (1), it results that $i - b - \Sigma_{h=2,i-1}\delta_{i,h} \geqslant 3$. Hence $\delta_{i,2} = 1$ and $i - b_i - \Sigma_{h=3,i-1}\delta_{i,h} \geqslant 4$. At this point it is clear that $\delta_{i,j} = 1$ for every $j < i$. Replacing this result in the first relation of (1), we obtain the contradiction $i - b_i - i + 1 \geqslant 2$. Then it follows that

$$|M| - |N| = \left| \left\{ i | i - b_i - \sum_{h=1,i-1} \delta_{i,h} \leqslant 1 \text{ and } \delta_{i,1} = 0 \right\} \right|.$$

Moreover, from $i - b_i - \Sigma_{h=1,i-1}\delta_{i,h} = 0$ it follows that $\delta_{i,1} = 0$; thus $|M| - |N| - |L| = |I|$ where $I = \{i | i - b_i - \Sigma_{h=1,i-1}\delta_{i,h} = 1 \text{ and } \delta_{i,1} = 0\}$, i.e., $I = \{i | i - b_i - \Sigma_{h=2,i-1}\delta_{i,h} = 1\}$. Thus, in this case the statement is proved.

Case 2. Let $b_1 = 1$. Then we have $m = \Sigma_{i=r+1,l} b_i i! + (b_r + 1)r!$, where r is the least positive integer such that $b_r < r$. Then it follows that $r \geqslant 2$ and $b_{r-i} = r - i$ for every i with $0 < i < r$. As in the preceding case, we shall denote by δ^* the δ's computed for Δ^{m+1}. We state now some propositions which shall be used in the sequel, and which may be deduced by examining the $\delta_{i,j}$'s and $\delta^*_{i,j}$'s.

(2) If $j > r$, then $\delta_{i,j} = \delta^*_{i,j}$.

(3) $\delta^*_{i,r} = 0 \to \delta_{i,r} = 0, \quad \delta_{i,r} = 1 \to \delta^*_{i,r} = 1,$

$$\delta^*_{i,r} = 1 \quad \text{and} \quad \delta_{i,r} = 0 \Leftrightarrow i - b_i - \sum_{h=r+1,i-1} \delta_{i,h} - r + b_r = 0.$$

(4) If $i > r$, then, for every j with $1 \leqslant j < r$, we have

$\delta^*_{i,r-1} = 1 \to \delta^*_{i,r-j} = 1$ and $\delta_{i,r-j} = 1$,

$\delta_{i,r-1} = 0 \to \delta_{i,r-j} = 0$ and $\delta^*_{i,r-j} = 0$.

(5) If $i > r$ and $j < r$ then it follows that, for every s with $j < s < r$,

$\delta_{i,j} = 1 \to \delta_{i,s} = 1, \quad \delta^*_{i,j} = 0 \to \delta^*_{i,s} = 0.$

(6) $\delta_{r,j} = 1 \Leftrightarrow j \geqslant b_r, \quad \delta^*_{r,j} = 1 \Leftrightarrow j < r - b_r - 1.$

(7) If $i < r$, then we have $\delta_{i,j} = 0$ and $\delta^*_{i,j} = 1$.

(8) $\delta_{i,r} = 1$ implies $\delta_{i,r-1} = 1$.

(9) $\delta^*_{i,r-1} = 1$ implies $\delta^*_{i,r} = 1$.

(10) Let $i > r$. If there exists an integer $j < r$ such that $\delta_{i,j} = 1$ ($\delta^*_{i,j} = 0$), then, by proposition (5), there is an integer j_i ($\hat{j}_i$) so that $\delta_{i,j} = 1$ ($\delta^*_{i,j} = 0$) for every j with $j_i \leqslant j < r$ ($\hat{j}_i \leqslant j < r$), while we have $\delta_{i,j} = 0$ ($\delta^*_{i,j} = 1$) for every j with $0 < j < j_i$ ($0 < j < \hat{j}_i$). Moreover, we obtain $i - b_i - \Sigma_{j=r,i-1}\delta_{i,j} = r - j_i$ ($i - b_i - \Sigma_{j=r,i-1}\delta^*_{i,j} = \hat{j}_i$).

That being stated, let us compute $\Delta^{m+1} - \Delta^m$. In view of (2) and (7), since (7) implies $i - \Sigma_{h=j,i-1}\delta^*_{i,h} = j$ for every i with $2 \leqslant i < r$, we have

$$\begin{aligned}(11)\quad \Delta^{m+1} - \Delta^m = &\left\{ \sum_{i=1,r-1} \left[(i-r)k^{r-i} + k^{r-i-1} + \cdots + 1\right]\right\} \\ &+\left\{k^r - k^{r-b_r-1} + \sum_{j=1,r-1}\left[\delta^*_{r,j}\left(k^j - k^{r-b_r-1-\Sigma_{h=j,r-1}\delta^*_{r,h}}\right)\right.\right. \\ &\left.\left.-\delta_{r,j}\left(k^j - k^{r-b_r-\Sigma_{h=j,r-1}\delta_{r,h}}\right)\right]\right\} \\ &+\left\{\sum_{h=1,l-r}\left[\delta^*_{r+h,r}\left(k^r - k^{r+h-b_{r+h}-\Sigma_{l=r,r+h-1}\delta^*_{r+h,l}}\right)\right.\right. \\ &\left.\left.-\delta_{r+h,r}\left(k^r - k^{r+h-b_{r+h}-\Sigma_{l=r,r+h-1}\delta_{r+h,l}}\right)\right]\right\} \\ &+\left\{\sum_{i=1,l-r}\ \sum_{j=1,r-1}\left[\delta^*_{r+h,j}\left(k^j - k^{r+h-b_{r+h}-\Sigma_{l=j,r+h-1}\delta^*_{r+h,l}}\right)\right.\right. \\ &\left.\left.-\delta_{r+h,j}\left(k^j - k^{r+h-b_{r+h}-\Sigma_{l=j,r+h-1}\delta_{r+h,l}}\right)\right]\right\}.\end{aligned}$$

In view of (6), the second term of sum (11) becomes

$$(12)\qquad k^r - \sum_{j=b_r,r-1} k^j + \sum_{j=0,r-b_r-2} k^j.$$

Now, we shall examine the third term; if $\delta^*_{r+h,r} = \delta_{r+h,r}$, making use of the proposition (2), we obtain

$$\delta^*_{r+h,r}\left(k^r - k^{r+h-b_{r+h}-\sum_{i=r+1,r+h-1}\delta_{r+h,i}-\delta^*_{r+h}}\right) \\ -\delta_{r+h,r}\left(k^r - k^{r+h-b_{r+h}-\sum_{i=r+1,r+h-1}\delta_{r+h,i}-\delta_{r+h,r}}\right) = 0.$$

If, on the contrary, we have $\delta^*_{r+h,r} \neq \delta_{r+h,r}$, then, by the third relation of proposition (3), we have $\delta^*_{r+h,r} = 1$ and $\delta_{r+h,r} = 0$, or equivalently,

$$r + h - b_{r+h} - \sum_{i=r+1,r+h-1} \delta_{r+h,i} - r + b_r = 0. \tag{13}$$

That being stated, the third term of the sum (11) becomes

$$|M|\left(k^r - k^{r-b_r-1}\right), \tag{14}$$

where M is the set of the h indices for which the equality (13) holds. Now, let us consider the fourth term of the sum and start by examining

$$\sum_{j=1,r-1} \left[\delta^*_{r+h,j}\left(k^j - k^{r+h-b_{r+h}-\sum_{i=r+1,r+h-1}\delta_{r+h,i}-\sum_{i=j,r}\delta^*_{r+h,i}}\right) \right. \\ \left. -\delta_{r+h,j}\left(k^j - k^{r+h-b_{r+h}-\sum_{i=r+1,r+h-1}\delta_{r+h,i}-\sum_{i=j,r}\delta_{r+h,i}}\right)\right]. \tag{15}$$

If $\delta^*_{r+h,r-1} = 1$, from (4) and (9) we obtain $\delta^*_{r+h,r-j} = 1$, $\delta_{r+h,r-j} = 1$ and $\delta^*_{r+h,r} = 1$. Thus the term (15) is zero. If $\delta_{r+h,r-1} = 0$, making use of (4), we have $\delta^*_{r+h,r-j} = 0$ and $\delta_{r+h,r-j} = 0$, and (15) is again zero.

It now remains to examine the case $\delta^*_{r+h,r-1} = 0$ and $\delta_{r+h,r-1} = 1$. By (10), denoting by i_{r+h} and $\hat{i}_{r+h}$ respectively the least positive integers such that $\delta_{r+h,i_{r+h}} = 1$ and $\delta^*_{r+h,\hat{i}_{r+h}} = 0$, (15) becomes, in view of (5),

$$\sum_{j=1,\hat{i}_{r+h}-1} \left(k^j - k^{\hat{i}_{r+h}-\sum_{i=j,\hat{i}_{r+h}-1}\delta^*_{r+h,i}}\right) \\ - \sum_{j=i_{r+h},r-1} \left(k^j - k^{r-i_{r+h}-\sum_{i=j,r-1}\delta_{r+h,i}}\right) = \sum_{j=i_{r+h},r-1} \left(k^{j-i_{r+h}} - k^j\right).$$

Then the fourth term of the sum becomes

$$\sum_{h\in H} \sum_{j=i_{r+h},r-1} \left(k^{j-i_{r+h}} - k^j\right), \tag{16}$$

where $H = \{h \mid 1 \leqslant h \leqslant l - r,\ \delta^*_{r-h,r-1} = 0 \text{ and } \delta_{r+h,r-1} = 1\}$. In conclusion, from (11), (12), (14) and (16) it follows that

$$\Delta^{m+1} - \Delta^m = \sum_{i=1,r-1} \left[(i-r)k^{r-i} + k^{r-i-1} + \cdots + 1\right] + k^r - \sum_{j=b_r,r-1} k^j \\ + \sum_{j=0,r-b_r-2} k^j + |M|\left(k^r - k^{r-b_r-1}\right) \\ + \sum_{h\in H} \sum_{j=i_{r+h},r-1} \left(k^{j-i_{r+h}} - k^j\right),$$

that is,

$$\Delta^{m+1} - \Delta^m = \sum_{i=1,r+1} c_i k^{i-1},$$

where

$$
(17)\qquad
\begin{aligned}
c_i &= r - 2i + 1 + \delta(r - b_r - i) - \delta(i - b_r) \\
&\quad + |H_{i-1}| - |\overline{H}_{i-1}| \quad \text{if } i \neq r - b_r, i \leqslant r, \\
c_{r+1} &= 1 + |M|, \\
c_{r-b_r} &= 2b_r - r + 1 - \delta(r - 2b_r) + |H_{r-b_r}| - |\overline{H}_{r-b_r}| - |M|,
\end{aligned}
$$

and $H_j = \{h \in H \mid j \leqslant r - 1 - i_{r+h}\}$, $\overline{H}_j = \{h \in H \mid j \geqslant i_{r+h}\}$. Note that for $j = 0$ it follows that $H_0 = H$ and $\overline{H}_0 = \varnothing$.

That being stated, let us remark that $m = \sum_{i=r+1,l} b_i i! + (b_r + 1)r!$, and therefore for the last $r + 1$ digits of the $(m + 1)$st permutation we have the relations

$$a_1 > a_2 > \cdots > a_r \not> a_{r+1} \quad \text{and} \quad a_{r-b_r} < a_{r+1} < a_{r-b_r-1}.$$

Then, the mth permutation becomes $\alpha_k \cdots \alpha_{r+1}\alpha_r \cdots \alpha_1$ with $\alpha_i = a_i$ (if $i > r + 1$), $\alpha_{r+1} = a_{r-b_r}$, $\alpha_r = a_1, \ldots, \alpha_{b_r} = a_{r-b_r+1}, \alpha_{b_r+1} = a_{r+1}, \ldots,$ $\alpha_1 = a_r$. Therefore we have $\alpha_1 < \alpha_2 < \cdots < \alpha_r \not< \alpha_{r+1}$ and $\alpha_{b_r} < \alpha_{r+1} < \alpha_{b_r+1}$. Then, by Lemma 1, we find

$$\Delta_m = \sum_{i=1,r} k^{i-1}(\alpha_{r+1-i} - \alpha_i) + k^{r-b_r-1}(\alpha_{r+1} - \alpha_{b_r+1}) + k^r(\alpha_{b_r+1} - \alpha_{r+1}).$$

Hence, putting $L_i = \{h \mid h > 0, \delta_{r+h,i} = 1\}$ and

$$N_i = \left\{h \mid h > 0, r + h - b_{r+h} - \sum_{j=1,r+h-1} \delta_{r+h,j} \leqslant i\right\},$$

we obtain

$$
\begin{aligned}
\alpha_{r+1} &= k - r - 1 + b_r + |L_r| - |N_r|, \\
\alpha_{b_r+1} &= k - r + b_r + |L_{b_r}| - |N_{b_r}| \quad \text{if } b_r \neq 0, \\
\alpha_i &= k - r + i - 2 + |L_{i-1}| - |N_{i-1}| + \delta_{r,i-1} \quad \text{if } 1 < i \leqslant r, \\
\alpha_1 &= k - 1 - r - |N_0| \quad \text{if } b_r \neq 0, \\
\alpha_1 &= k - r - |N_0| \quad \text{if } b_r = 0.
\end{aligned}
$$

That being stated, if $b_r \neq 0$, we have

$$\alpha_{b_r+1} - \alpha_{r+1} = 1 + |L_{b_r}| - |L_r| + |N_r| - |N_{b_r}|.$$

Making use of (8) and (10), it is easily shown that $\delta_{r+h,r} = 1$ implies $\delta_{r+h,b_r} = 1$, whence

$$|L_{b_r}| - |L_r| = \left|\{h \mid h > 0, \delta_{r+h,b_r} = 1 \text{ and } \delta_{r+h,r} = 0\}\right|.$$

From $\delta_{r+h,b_r} = 1$, in view of (5), it follows that

$$(18)\qquad r + h - b_{r+h} - \sum_{i=r+1,r+h-1} \delta_{r+h,i} - \delta_{r+h,r} - r + b_r + 1 > 0.$$

Moreover, $\delta_{r+h,r} = 0$ is equivalent to

$$(19)\qquad r + h - b_{r+h} - \sum_{i=r+1,r+h-1} \delta_{r+h,i} - r + b_r \leqslant 0.$$

Hence, (18) and (19) hold together if and only if

$$r + h - b_{r+h} - \sum_{i=r+1,r+h-1} \delta_{r+h,i} = r - b_r$$

whence $|L_{b_r}| - |L_r| = |M|$. Moreover it is easily verifiable that

$$|N_r| - |N_{b_r}| = \left|\left\{h \mid h > 0,\, b_r < r + h - b_{r+h} - \sum_{i=1,r+h-1} \delta_{r+h,i} \leqslant r\right\}\right| = 0.$$

Hence $\alpha_{b_r+1} - \alpha_{r+1} = 1 + |M|$. If $b_r = 0$, we have

$$\alpha_{b_r+1} - \alpha_{r+1} = \alpha_1 - \alpha_{r+1} = 1 - |L_r| + |N_r| - |N_0|.$$

After remarking that $r + h - b_{r+h} - \sum_{i=r+1,r+h-1} \delta_{r+h,i} > r$ implies $r + h - b_{r+h} - \sum_{i=1,r+h-1} \delta_{r+h,i} \leqslant r$, it is easy to verify that $|N_r| - |L_r| - |N_0| = |M|$, whence we obtain again $\alpha_{b_r+1} - \alpha_{r+1} = 1 + |M|$. Let $1 < i < r$, and consider

$$\alpha_{r+1-i} - \alpha_i = r - 2i + 1 + |L_{r-i}| - |L_{i-1}| + |N_{i-1}| - |N_{r-i}| + \delta_{r,r-i} - \delta_{r,i-1}.$$

Let us suppose $r + 1 - i > i$. Then it follows that $|L_{r-i}| - |L_{i-1}| = |P_i|$ where

$$P_i = \left\{h \mid h > 0,\, r + h - b_{r+h} - \sum_{j=r+1-i,r+h-1} \delta_{r+h,j} > 0 \text{ and } r + h - b_{r+h} - \sum_{j=i,r+h-1} \delta_{r+h,j} \leqslant 0\right\}.$$

Denoting by i_{r+h} the least positive integer such that $\delta_{r+h,i_{r+h}} = 1$, we have $i - 1 < i_{r+h} \leqslant r - i$, that is, $i - 1 < i_{r+h}$ and $i - 1 \leqslant r - i_{r+h} - 1$. Moreover, for $h \in P_i$ we have $\delta_{r+h,r-1} = 1$ and $r + h - b_{r+h} - \sum_{j=r,r+h-1} \delta_{r+h,j} < r - 1$, whence $h \in H$. Thus

$$|L_{r-i}| - |L_i| = |\{h \in H \mid i - 1 < i_{r+h},\, i - 1 \leqslant r - i_{r+h} - 1\}| = |H_{i-1}| - |\overline{H}_{i-1}|.$$

Since it is immediate that $|N_{i-1}| - |N_{r-i}| = 0$, it follows that

(20) $$\alpha_{r+1-i} - \alpha_i = r - 2i + 1 + |H_{i-1}| - |\overline{H}_{i-1}| + \delta_{r,r-i} - \delta_{r,i-1}.$$

In an analogous way we find the same result if $i > r - i + 1$. Now, let $i = 1$. It follows that

(21) $$\alpha_r - \alpha_1 = r + |L_{r-1}| - |N_{r-1}| + |N_0|.$$

Let us compute $|L_{r-1}| - |N_{r-1}| + |N_0|$; we have $N_{r-1} - N_0 = N^*$, where

$$N^* = \left\{h \mid h > 0,\, 0 < r + h - b_{r+h} - \sum_{j=1,r+h-1} \delta_{r+h,j} \leqslant r - 1\right\}.$$

Moreover, $h \in N^*$ implies $\delta_{r+h,r-1} = 1$, whence

$$|L_{r-1}| - |N_{r-1}| + |N_0| = |\{h \mid h > 0,\, \delta_{r+h,r-1} = 1 \text{ and } h \notin N^*\}|.$$

But $h \notin N^*$ and $\delta_{r+h,r-1} = 1$ imply $r + h - b_{r+h} - \sum_{j=1,r+h-1} \delta_{r+h,j} = 0$ whence $|L_{r-1}| - |N_{r-1}| + |N_0| = |H|$. Therefore, by an examination of (17), (20) and (21) it easily follows that $\alpha_{r-i+1} - \alpha_i = c_i$ for every $i \neq r - b_r$. If $i = r - b_r$, the coefficient of k^{r-b_r-1} in Δ_m is

$$\alpha_{b_r+1} - \alpha_{r-b_r} + \alpha_{r+1} - \alpha_{b_r+1} = -1 - |M| + r - 2r + 2b_r + 1 + |H_{r-b_r-1}| - |\overline{H}_{r-b_r-1}| + 1 - \delta_{r,r-b_r-1} = c_{r-b_r}$$

and the statement is proved.

4. Adaptation to a Computer. These arithmetic methods for the generation of permutations have been shown to be convenient because they allow one to perform the computation of Δ_m and Δ^m in the most convenient base for the computer used; the specific base k intervenes only in the final passage in which the wanted permutation is generated.

Obviously one must see that Δ_m and/or Δ^m are smaller than the maximum available integer of the standard software. In the opposite case one must use a procedure which performs the operations on the integers in multiple precision.

Since Δ_m is smaller than k^{k-1} and since 10^9 is a normally used integer in the scientific languages of programming, permutations of 10 marks can also be generated by this method with the usual instructions.

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