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Supplement to Semidiscrete and Single Step Fully Discrete Approximations for Second Order Hyperbolic Equations With Time-Dependent Coefficients

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8. Proofs of Theorem 5.1 and Theorem 5.2.

The following four lemmas will be used to prove Theorem 5.1. Throughout these proofs the general positive constant C will be independent of α.

Lemma 8.1. For any nonnegative integer m and any $\phi \in L^2$

(8.1) $$\|T^{1/2} L^{(m)} T^{1/2} \phi\| \leq C\|\phi\|$$

and

(8.2) $$\|L^{1/2} T^{(m)} L^{1/2} \phi\| \leq C\|\phi\| .$$

Proof. Assume ϕ is smooth so that $T^{1/2} L^{(m)} T^{1/2} \phi$ is in L^2. For any $\psi \in L^2$

(8.3) $$\begin{aligned}(T^{1/2} L^{(m)} T^{1/2} \phi,\psi) &= (L^{(m)} T^{1/2} \phi ,T^{1/2} \psi) \\ &= a^{(m)}(T^{1/2} \phi,T^{1/2} \psi) .\end{aligned}$$

Therefore,

(8.4) $$(T^{1/2}L^{(m)}T^{1/2}\phi,\psi) \leq C||T^{1/2}\phi||_1||T^{1/2}\psi||_1 .$$

Since $||T^{1/2}\phi||_1 \leq C||\phi||$ and $||T^{1/2}\psi||_1 \leq C||\psi||$, it follows that

(8.5) $$(T^{1/2}L^{(m)}T^{1/2}\phi,\psi) \leq C||\phi|| \; ||\psi|| .$$

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(8.5) proves that

$$||T^{1/2}L^{(m)}T^{1/2}\phi|| = \sup_{\psi\, L_2} \frac{(T^{1/2}L^{(m)}T^{1/2}\phi,\psi)}{||\psi||} \le C||\phi|| .$$

Since smooth functions are dense in L^2, (8.1) holds for all $\phi \in L^2$. (8.2) is proved using (8.1) and induction on m since

$$T^{(m)} = -\sum_{j=0}^{m-1} \binom{m}{j} TL^{(m-j)}T^{(j)}$$

and

$$L^{1/2}T^{(m)}L^{1/2} = -\sum_{j=0}^{m-1} \binom{m}{j} T^{1/2}L^{(m-j)}T^{1/2}L^{1/2}T^{(j)}L^{1/2} .$$

The next three lemmas contain bounds for terms in (5.5). These bounds will be given in the following special norm on $L^2 \times L^2$.

$$|||\phi||| \equiv (||\phi_1||^2 + (T\phi_2,\phi_2))^{1/2}$$

for $\phi = \binom{\phi_1}{\phi_2} \in L^2 \times L^2$.

<u>Lemma 8.2.</u> For any positive integer m

(8.6) $$|||\hat{\mathcal{L}}\hat{\mathcal{T}}^{(m)}\phi||| \le (C_1 + C_2\alpha)|||\phi||| ,$$

where C_1 and C_2 are constants which are independent of α.

<u>Proof</u>. Define $\hat{L} = L + \alpha^2 I$ and $\hat{T}^{(m)} = (\hat{L}^{-1})^{(m)}$. The following two estimates are from Sammon [20] and [21]. For integer $m \ge 0$ and $f \in L^2$

(8.7) $$||\hat{T}^{(m)}f|| \le \frac{C}{\alpha^2}||f|| ,$$

and

(8.8) $$||\hat{L}\hat{T}^{(m)}f|| \le C||f||$$

where the constants are independent of α. The following proofs of (8.7) and (8.8) are from Sammon [20] and [21]. Let $\{\phi_i\}_{i=1}^{\infty}$ and $\{\lambda_i\}_{i=1}^{\infty}$ be the eigenfunctions and eigenvalues of L. Since $\hat{T}f = \sum_{i=1}^{\infty}(\lambda_i + \alpha^2)^{-1}(f,\phi_i)\phi_i$,

(8.9) $$||\hat{T}f|| \le \frac{1}{\alpha^2}||f||$$

and

(8.10) $$||L\hat{T}f|| \le C||f||$$

where C is independent of α. Now since

$$||\hat{L}\hat{T}^{(m)}f|| = ||\sum_{\ell=0}^{m-1} \binom{m}{\ell} L^{(m-\ell)}\hat{T}^{(\ell)}f||$$

it follows by induction that

$$||\hat{L}\hat{T}^{(m)}f|| \le C \sum_{\ell=0}^{m-1} ||L\hat{T}^{(\ell)}f|| \le C||f||$$

which is (8.8). The estimate

$$||\hat{T}^{(m)}f|| = ||\hat{T}\hat{L}\hat{T}^{(m)}f|| \le \frac{C}{\alpha^2}\,||f||$$

proves (8.7). In addition to (8.7) and (8.8) the following two estimates will be needed.

(8.11) $\quad ||T^{1/2}\hat{T}^{(m)}L^{1/2}f|| \le \frac{C}{\alpha^2}\,||f||$ and

(8.12) $\quad ||T^{1/2}\hat{L}\hat{T}^{(m)}L^{1/2}f|| \le C||f||$

where the constant is independent of α. (8.12) is proved by induction on m. For $m = 1$,

$$T^{1/2}\hat{L}\hat{T}^{(1)}L^{1/2} = -\,T^{1/2}L^{(1)}\hat{T}L^{1/2} = -\,T^{1/2}L^{(1)}T^{1/2}L\hat{T}$$

Using Lemma 8.1 and (8.10), it follows that

$$||T^{1/2}\hat{L}\hat{T}^{(1)}L^{1/2}f|| \le C||f|| \quad .$$

Now assume (8.12) for $m \le n-1$. Since

$$\hat{L}\hat{T}^{(n)} = -\sum_{\ell=0}^{n-1}\binom{n}{\ell}\hat{L}^{(n-\ell)}\hat{T}^{(\ell)}$$

and

$$T^{1/2}\hat{L}\hat{T}^{(n)}L^{1/2} = -\sum_{\ell=0}^{n-1}\binom{n}{\ell}(T^{1/2}L^{(n-\ell)}T^{1/2}(L\hat{T})(T^{1/2}\hat{L}\hat{T}^{(\ell)}L^{1/2} \,,$$

Lemma 8.1, (8.10) and the induction hypothesis imply (8.12).

(8.11) follows from (8.9) and (8.12) since

$$||T^{1/2}\hat{T}^{(m)}L^{1/2}f|| = ||\hat{T}T^{1/2}\hat{L}\hat{T}^{(m)}L^{1/2}f|| \le \frac{C}{\alpha^2}\,||f|| \quad .$$

The estimates (8.8) and (8.12) are used to prove the lemma. Since

$$\hat{\mathcal{L}} = \begin{pmatrix} -\alpha I & I \\ -L & -\alpha I \end{pmatrix} \text{ and}$$

$$\hat{\mathcal{T}}^{(m)} = \begin{pmatrix} -\alpha\hat{T}^{(m)} & -\hat{T}^{(m)} \\ -\alpha^2\hat{T}^{(m)} & -\alpha\hat{T}^{(m)} \end{pmatrix} \,,$$

it follows that

$$\hat{\mathcal{L}}\hat{\mathcal{T}}^{(m)} = \begin{pmatrix} 0 & 0 \\ \alpha\hat{L}\hat{T}^{(m)} & \hat{L}\hat{T}^{(m)} \end{pmatrix} \,,$$

so that for $\Phi \in L^2 \times L^2$

$$|||\hat{\mathcal{L}}\hat{\mathcal{T}}^{(m)}\Phi||| = ||T^{1/2}(\alpha\hat{L}\hat{T}^{(m)}\Phi_1 + \hat{L}\hat{T}^{(m)}\Phi_2)||$$

$$\leq \alpha||T^{1/2}\hat{L}\hat{T}^{(m)}\phi_1|| + ||T^{1/2}\hat{L}\hat{T}^{(m)}\phi_2|| .$$

Since $T^{1/2}$ is a bounded operator on L^2 and $T^{1/2}\hat{L}\hat{T}^{(m)} = T^{1/2}\hat{L}\hat{T}^{(m)}L^{1/2}T^{1/2}$, (8.8) and (8.12) imply that

$$|||\hat{\mathcal{F}}^{(m)}\phi||| \leq C\alpha||\phi_1|| + C||T^{1/2}\phi_2||$$

(8.6) follows from this estimate.

<u>Lemma 8.3</u>. For α sufficiently large $(I-m\,\hat{\mathcal{F}}^{(1)})$ is invertible on $L^2 \times L^2$ and

$$\text{(8.13)} \qquad |||(I-m\,\hat{\mathcal{F}}^{(1)})^{-1}\phi||| \leq C|||\phi||| ,$$

where C is independent of α.

<u>Proof</u>. Since $\hat{\mathcal{F}}^{(1)} = \begin{pmatrix} -\alpha\hat{T}^{(1)} & \hat{T}^{(1)} \\ -\alpha^2\hat{T}^{(1)} & -\alpha\hat{T}^{(1)} \end{pmatrix}$, if $(I + m\alpha\hat{T}^{(1)})$ is invertible, it follows that

$$(I - m\hat{\mathcal{F}}^{(1)})^{-1} = (I + 2m\alpha\hat{T}^{(1)})^{-1}\begin{pmatrix} I+m\alpha\hat{T}^{(1)} & -m\hat{T}^{(1)} \\ -m\alpha^2\hat{T}^{(1)} & I+m\alpha\hat{T}^{(1)} \end{pmatrix}$$

so that

$$\text{(8.14)} \qquad |||(I + m\hat{\mathcal{F}}^{(1)})^{-1}\phi|||^2$$

$$= ||(I + 2m\alpha\hat{T}^{(1)})^{-1}((I + m\alpha\hat{T}^{(1)})\phi_1 - m\hat{T}^{(1)}\phi_2)||^2$$

$$+ ||T^{1/2}(I + 2m\alpha\hat{T}^{(1)})^{-1}(-m\alpha^2\hat{T}^{(1)}\phi_1 + (I + m\alpha\hat{T}^{(1)})\phi_2)||^2 .$$

(8.7) states that $||\hat{T}^{(1)}f|| \leq \frac{C}{\alpha^2}||f||$ so that if α is large enough $||2m\alpha\hat{T}^{(1)}f|| \leq \gamma_1||f||$ where $\gamma_1 < 1$ and for α large can be chosen independent of α. Writing $(I+2m\alpha\hat{T}^{(1)})^{-1} = I-2m\alpha\hat{T}^{(1)} + (2m\alpha\hat{T}^{(1)})^2 - \ldots$ gives

$$\text{(8.15)} \qquad ||(I+2m\alpha\hat{T}^{(1)})^{-1}f|| \leq (1+\gamma_1+\gamma_1^2+\ldots)||f|| \leq \frac{1}{1-\gamma_1}||f|| .$$

Also,

$$\text{(8.16)} \qquad ||T^{1/2}(I+2m\alpha\hat{T}^{(1)})^{-1}L^{1/2}f||$$

$$= ||(I+2m\alpha T^{1/2}\hat{T}^{(1)}L^{1/2})^{-1}f|| \leq \frac{1}{1-\gamma_2}||f||$$

since $||2m\alpha T^{1/2}\hat{T}^{(1)}L^{1/2}f|| \leq \frac{C}{\alpha}||f||$ (from (8.11)), where $\gamma_2 < 1$ and for α large can be chosen independent of α.

Using (8.15) and (8.16) in (8.14) gives

$$\text{(8.17)} \qquad |||(I+m\hat{\mathcal{F}}^{(1)})^{-1}\phi|||^2 \leq C(||(I+m\alpha\hat{T}^{(1)})\phi_1 - m\hat{T}^{(1)}\phi_2||^2$$

$$+ ||T^{1/2}(-m\alpha^2\hat{T}^{(1)}\phi_1 + (I+m\alpha\hat{T}^{(1)})\phi_2)||^2)$$

where the constant C is independent of α. Since

$$\hat{T}^{(1)}L^{1/2} = T^{1/2}(L\hat{T})(T^{1/2}\hat{L}\hat{T}^{(1)}L^{1/2}) ,$$

(8.10) and (8.12) imply that

$$\text{(8.18)} \qquad ||\hat{T}^{(1)}L^{1/2}g|| \leq C||g|| ,$$

where the constant C is independent of α. Now using (8.7), (8.18), the fact that $T^{1/2}$ is a bounded operator on L^2, and (8.11) in (8.17) gives

$$|||(I+m\hat{\mathcal{T}}(1))^{-1}\phi|||^2 \le C(||\phi_1||^2 + ||T^{1/2}\phi_2||^2)$$

for α large, where C can be chosen indpendent of α. This completes the proof of the lemma.

Lemma 8.4 For any $\phi \in L^2 \times L^2$

$$(8.19) \qquad |||\hat{\mathcal{T}}\phi||| \le \frac{1}{\alpha}|||\phi||| .$$

Proof. Let $\{\phi_j\}_{j=1}^{\infty}$ and $\{\lambda_j\}_{j=1}^{\infty}$ be the orthonormal eigenfunctions and eigenvalues of L. $\hat{\mathcal{L}}$ has a complete orthonormal set of eigenfunctions given by $\phi_{\pm j} = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_j \\ \pm i\sqrt{\lambda_j}\,\phi_j \end{pmatrix}$ with eigenvalues $-\alpha \pm i\sqrt{\lambda_j}$, for $j = 1, \dots, \infty$. Let $\phi = \sum_{j=-\infty}^{\infty} c_j\phi_j$. Then

$$\hat{\mathcal{T}}\phi = \sum_{j=-\infty}^{\infty} \frac{c_j}{(\mathrm{sgn}\ j)\, i\sqrt{\lambda_{|j|}} - \alpha}\,\phi_j$$

where $(\mathrm{sgn}\ j)$ is the sign of j. Since $\frac{1}{|(\mathrm{sgn}\ j)\, i\sqrt{\lambda_{|j|}} - \alpha|} \le \frac{1}{\alpha}$, it follows that

$$|||\hat{\mathcal{T}}\phi|||^2 = \sum_{j=-\infty}^{\infty} \frac{|c_j|^2}{|(\mathrm{sgn}\ j)\, i\sqrt{\lambda_{|j|}} - \alpha|^2} \le \frac{1}{\alpha^2}|||\phi|||^2 ,$$

This gives (8.19).

Lemmas 8.1, 8.2, 8.3, and 8.4 are now used to prove Theorem 5.1.

Proof of Theorem 5.1. The theorem is proved by induction on m. Assume that it is true for $\hat{E}_i$, $i = 1, \dots, m$, $(m \ge 2)$. Also, as part of the induction hypothesis, assume that each of the four components of $\hat{E}_i^{-1}$, $i = 1, \dots, m$, is a bounded operator on H^{ℓ} for $\ell \ge 0$ (i.e. writing $\hat{E}_i^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, the operators A_{11}, A_{12}, A_{21} and A_{22} are assumed to be bounded where the bound can depend on α and ℓ) and that $|||\hat{E}_i^{-1}\phi||| \le \frac{C}{\alpha}|||\phi|||$, for $i = 1, \dots, m$ and for any $\phi \in H^1 \times L^2$ where the constant C is independent of α. (It is straightforward using Lemmas 8.3 and 8.4 and by writing out formulas for $\hat{E}_1^{-1}$ and $\hat{E}_2^{-1}$ to see that the induction hypothesis is true for $\hat{E}_1$ and $\hat{E}_2$.)

From the induction hypothesis it is easy to see that (1) in Theorem 5.1 is satisfied for $\hat{E}_{m+1}$. Let $\phi \in (H^2 \cap H_0^1) \times H^1$. In order to prove (2) in Theorem 5.1 we first show that

$$(8.20) \qquad \frac{\alpha}{C}|||\phi||| \le |||\hat{E}_{m+1}\phi|||$$

where the constant C is independent of α when α is sufficiently large. From Lemma 8.2 it follows that

$$\sum_{\ell=0}^{m-2} \binom{m}{\ell} |||\hat{\mathcal{L}}\hat{\mathcal{T}}^{(m-1)}\hat{E}_{\ell+2}^{-1}, \dots, \hat{E}_m^{-1}\phi|||$$

$$\le \sum_{\ell=0}^{m-2} C\alpha|||\hat{E}_{\ell+2}^{-1} \cdots \hat{E}_m^{-1}\phi||| .$$

The induction hypothesis implies that

$$|||\hat{E}_{\ell+2}^{-1} \cdots \hat{E}_m^{-1}\phi||| \le \frac{C}{\alpha}|||\phi||| ,$$

so that

$$(8.21)\qquad \left|\left|\left|\sum_{\ell=0}^{m-2} \binom{m}{\ell}\hat{\mathcal{L}}\hat{\mathcal{T}}^{(m-1)}\hat{E}_{\ell+2}^{-1}\cdots\hat{E}_m^{-1}\phi\right|\right|\right| \le C_1|||\phi|||$$

where C_1 is independent of α. Lemma 5.3 and Lemma 5.4 imply that

$$C_2\alpha|||\phi||| \le \alpha|||(I-m\hat{\mathcal{T}}^{(1)})\phi||| \le |||\hat{\mathcal{L}}(I-m\hat{\mathcal{T}}^{(1)})\phi||| .$$

This estimate used with (8.21) and (5.5) proves that

$$(C_2\alpha - C_1)|||\phi||| \le |||\hat{E}_{m+1}\phi||| .$$

Choosing $\alpha \ge \frac{2C_1}{C_2}$, gives $C_2\alpha - C_1 \ge C_2\alpha - \frac{C_2}{2}\alpha \ge \frac{C_2}{2}\alpha$.

So defining a new constant to be $\frac{C_2}{2}$ completes the proof of (8.20).

(8.20) shows that if $F \in H^1 \times L^2$ is given, then the equation $\hat{E}_{m+1}\phi = F$ can have at most one solution in $(H^2 \cap H_0^1) \times H^1$. We now prove the existence of a solution of $\hat{E}_{m+1}\phi = F$ for $F \in H^1 \times L^2$. Since

$$\hat{\mathcal{L}}\hat{\mathcal{T}}^{(j)} = \begin{pmatrix} 0 & 0 \\ \alpha L\hat{T}^{(j)} & L\hat{T}^{(j)} \end{pmatrix}$$

for any positive integer j,

$$\hat{E}_{m+1} = \begin{pmatrix} -\alpha I & I \\ -L & -\alpha I \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ B_1 & B_2 \end{pmatrix}$$

where (using the induction hypothesis) B_1 and B_2 are bounded operators on H^ℓ (the bound depending on α), for any integer $\ell \ge 0$. Solving $\hat{E}_{m+1}\,\phi = F$ is equivalent to solving

$$\begin{pmatrix} -\alpha I & I \\ -L+B_1 & -\alpha I+B_2 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

which is equivalent to solving

$$(8.22)\qquad \begin{pmatrix} -\alpha I & I \\ -L+B_1-\alpha^2 I+\alpha B_2 & 0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2+(\alpha I-B_2)f_1 \end{pmatrix} .$$

(8.22) can be solved if $-L+B_1-\alpha^2 I+\alpha B_2$ is invertible. This is equivalent to the invertibility of $I-T(B_1-\alpha^2 I+\alpha B_2)$. Since T is a compact operator and $B_1-\alpha^2 I+\alpha B_2$ is a bounded operator on L^2 (where the bound can depend on α), either the operator $I-T(B_1-\alpha^2 I+\alpha B_2)$ is invertible or there exists a nonzero solution ψ_1 of $(I-T(B_1-\alpha^2 I+\alpha B_2))\psi_1 = 0$. If ψ_1 exists and is nonzero, then a nonzero solution of $E_{m+1}\psi = 0$ in $(H^2 \cap H_0^1) \times H^1$ can be constructed using (8.22) (with $f_1 = 0$ and $f_2 = 0$ in (8.22)). However, this contradicts (8.20) (for α large). Therefore the operator $-L+B_1-\alpha^2 I+\alpha B_2$ in (8.22) must be invertible.

If (8.22) is solved for $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$, then

$$(8.23)\qquad \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} (-L+B_1-\alpha^2 I+\alpha B_2)^{-1}(f_2+(\alpha I-B_2)f_1) \\ f_1+\alpha(-L+B_1-\alpha^2 I+\alpha B_2)^{-1}(f_2+(\alpha I-B_2)f_1) \end{pmatrix}$$

$$= \hat{E}_{m+1}^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

where

$$\hat{E}_{m+1}^{-1} \equiv (-L+B_1-\alpha^2 I+\alpha B_2)^{-1}\begin{pmatrix}\alpha I-B_2 & I\\ -L+B_1 & \alpha I\end{pmatrix} .$$

From the regularity properties of elliptic operators it follows that the components of $\hat{E}_{m+1}^{-1}$ are bounded operators on H^{ℓ} for any fixed integer $\ell \geq 0$ (this bound can depend on α) . From (8.20) it follows that

$$|||\hat{E}_{m+1}^{-1}F||| \leq \frac{C}{\alpha}\, |||F||| \quad .$$

Also, the regularity properties of the operator $(-L+B_1-\alpha^2 I+\alpha B_2)^{-1}$ and (8.23) imply that

$$||\phi_1||_{\ell+2} \leq C(\alpha)||f_2+\alpha f_1-B_2 f_1||_{\ell} \leq C(\alpha)(||f_1||_{\ell+1} + ||f_2||_{\ell})$$

and

$$||\phi_2||_{\ell+1} \leq ||f_1||_{\ell+1} + \alpha||\phi_1||_{\ell+1} \leq C(\alpha)(||f_1||_{\ell+1} + ||f_2||_{\ell}) .$$

These inequalities imply that (2) in Theorem 5.1 is satisfied for $\hat{E}_{m+1}^{-1}$.

Proof of (3) in Theorem 5.1 will complete the proof of the theorem. The proof is formally the same as in Sammon [20] and [21]. Assume $\hat{A}_{\ell} = \hat{E}_{\ell}\hat{A}_{\ell-1}$ for $0 \leq \ell \leq m$. Then

$$\hat{E}_{m+1}\hat{A}_m = \hat{\mathcal{L}}(I-m\hat{\mathcal{F}}(1) - \sum_{\ell=0}^{m-2}\binom{m}{\ell}\hat{\mathcal{F}}^{(m-\ell)}\hat{E}_{\ell+2}^{-1}\cdots\hat{E}_m^{-1})\hat{A}_m$$

$$= \hat{\mathcal{L}}(\hat{A}_m - m\hat{\mathcal{F}}^{(1)}\hat{A}_m - \sum_{\ell=0}^{m-2}\binom{m}{\ell}\hat{\mathcal{F}}^{(m-\ell)}\hat{A}^{(\ell+1)})$$

$$= \hat{\mathcal{L}}(\hat{A}_m - \sum_{\ell=0}^{m-1}\binom{m}{\ell}\hat{\mathcal{F}}^{(m-\ell)}\hat{A}_{\ell+1}) \quad .$$

Using (5.4)

$$\hat{E}_{m+1}\hat{A}_m = \hat{\mathcal{L}}(\hat{A}_m - \sum_{\ell=0}^{m-1}\binom{m}{\ell}\hat{\mathcal{F}}^{(m-\ell)}\sum_{j=0}^{\ell}\binom{\ell}{j}\hat{\mathcal{L}}^{(\ell-j)}\hat{A}_j)$$

$$= \hat{\mathcal{L}}(\hat{A}_m - \sum_{j=0}^{m-1}\sum_{\ell=j}^{m-1}\binom{m}{\ell}\binom{\ell}{j}\hat{\mathcal{F}}^{(m-\ell)}\hat{\mathcal{L}}^{(\ell-j)}\hat{A}_j)$$

$$= \hat{\mathcal{L}}(\hat{A}_m - \sum_{j=0}^{m-1}\sum_{k=0}^{m-1-j}\binom{m}{k+j}\binom{k+j}{j}\hat{\mathcal{F}}^{(m-k-j)}\hat{\mathcal{L}}^{(k)}\hat{A}_j)$$

$$= \hat{\mathcal{L}}(\hat{A}_m - \sum_{j=0}^{m-1}\binom{m}{j}(\sum_{k=0}^{m-1-j}\binom{m-j}{k}\hat{\mathcal{F}}^{(m-k-j)}\hat{\mathcal{L}}^{(k)})\hat{A}_j)$$

$$= \hat{\mathcal{L}}(\hat{A}_m - \sum_{j=0}^{m-1}\binom{m}{j}((\hat{\mathcal{F}}\hat{\mathcal{L}})^{(m-j)} - \hat{\mathcal{F}}\hat{\mathcal{L}}^{(m-j)})\hat{A}_j)$$

$$= \hat{\mathcal{L}}\hat{A}_m + \sum_{j=0}^{m-1}\binom{m}{j}\hat{\mathcal{L}}^{(m-j)}\hat{A}_j \quad .$$

Using (5.4) again ($j = m$ in (5.4)) gives $\hat{E}_{m+1}\hat{A}_m = \hat{A}_{m+1}$. This completes the proof of Theorem 5.1.

The proof of the invertibility of the operators $\hat{E}_{1,h},\ldots,\hat{E}_{m,h}$ is similar to the proof of the invertibility of $\hat{E}_1,\ldots,\hat{E}_m$. The analogue of Lemma 8.1 is Lemma 3.1. The following lemma is the counterpart of Lemma 8.2.

<u>Lemma 8.5</u>. For any positive integer m and all $\Phi \in S_h \times S_h$

$$(8.24) \qquad |||\hat{\mathcal{L}}_h\hat{\mathcal{F}}_h^{(m)}\Phi|||_0 \leq (C_1 + C_2\alpha)\ |||\Phi|||_0 ,$$

where C_1 and C_2 are constants which are independent of α .

Proof. Define $\hat{L}_h = L_h + \alpha^2 I$ and $\hat{T}_h^{(m)} = (\hat{L}_m^{-1})^{(m)}$. Let $\{\phi_i\}_{i=1}^M$ and $\{\lambda_i\}_{i=1}^M$ be the eigenfunctions and eigenvalues of L_h. Since $\hat{T}_h f = \sum_{i=1}^{M} (\lambda_i + \alpha^2)^{-1}(f,\phi_i)\phi_i$, it follows that

$$(8.25) \qquad ||\hat{T}_h f|| \le \frac{1}{\alpha^2} ||f||$$

and

$$(8.26) \qquad ||L_h \hat{T}_h f|| \le C||f||$$

where C is independent of α.

Also $\hat{L}_h \hat{T}_h^{(m)} = - \sum_{j=0}^{m-1} \binom{m}{j} \hat{L}_h^{(m-j)} \hat{T}_h^{(j)}$, so that

$$T_h^{1/2} \hat{L}_h \hat{T}_h^{(m)} L_h^{1/2} = - \sum_{j=0}^{m-1} \binom{m}{j} (T_h^{1/2} L_h^{(m-j)} T_h^{1/2})(L_h \hat{T}_h)(T_h^{1/2} \hat{L}_h \hat{T}_h^{(j)} L_h^{1/2}) .$$

By induction (using Lemma 3.1 and (8.26)),

$$(8.27) \qquad ||T_h^{1/2} \hat{L}_h \hat{T}_h^{(m)} L_h^{1/2} f|| \le C||f||$$

where the constant C is independent of α. Since

$$\hat{\mathcal{L}}_h \hat{\mathcal{T}}_h^{(m)} = \begin{pmatrix} 0 & 0 \\ \alpha \hat{L}_h \hat{T}_h^{(m)} & \hat{L}_h \hat{T}_h^{(m)} \end{pmatrix} ,$$

$$|||\hat{\mathcal{L}}_h \hat{\mathcal{T}}_h^{(m)} \Phi|||_0 = ||T_h^{1/2}(\alpha \hat{L}_h \hat{T}_h^{(m)} \phi_1 + \hat{L}_h \hat{T}_h^{(m)} \phi_2)||$$

and

$$|||\hat{\mathcal{L}}_h \hat{\mathcal{T}}_h^{(m)} \Phi|||_0 \le \alpha ||T_h^{1/2} \hat{L}_h \hat{T}_h^{(m)} \phi_1|| + ||T_h^{1/2} \hat{L}_h \hat{T}_h^{(m)} \phi_2|| .$$

Using (8.27) and the fact that $T_h^{1/2}$ is a bounded operator gives

$$|||\hat{\mathcal{L}}_h \hat{\mathcal{T}}_h^{(m)} \Phi|||_0 \le C\alpha ||\phi_1|| + C||T_h^{1/2} \phi_2||$$

where C is independent of α. (8.24) follows from this estimate.

The next lemma is the discrete counterpart of Lemma 8.3.

Lemma 8.6. For α sufficiently large $(I - m\hat{\mathcal{T}}_h^{(1)})$ is invertible on $S_h \times S_h$ and

$$(8.28) \qquad |||(I - m\hat{\mathcal{T}}_h^{(1)})^{-1} \Phi|||_0 \le C|||\Phi|||_0 ,$$

where the constant C is independent of α.

Proof. If $(I + 2m\alpha \hat{T}_h^{(1)})$ is invertible, then

$$(I - m\hat{\mathcal{T}}_h^{(1)})^{-1} = (I + 2m\alpha \hat{T}_h^{(1)})^{-1} \begin{pmatrix} I + m\alpha \hat{T}_h^{(1)} & -m\hat{T}_h^{(1)} \\ -m\alpha^2 \hat{T}_h^{(1)} & I + m\alpha \hat{T}_h^{(1)} \end{pmatrix}$$

so that

(8.29) $$|||(I+m\hat{\mathcal{T}}_h^{(1)})^{-1}\phi|||_0^2$$

$$= ||(I+2m\alpha\hat{T}_h^{(1)})^{-1}((I+m\alpha\hat{T}_h^{(1)})\phi_1 - m\hat{T}_h^{(1)}\phi_2)||^2$$

$$+ ||T_h^{1/2}(I+2m\alpha\hat{T}_h^{(1)})^{-1}(-m\alpha^2\hat{T}_h^{(1)}\phi_1 + (I+m\alpha\hat{T}_h^{(1)})\phi_2)||^2 \ .$$

Since $\hat{T}_h f = \sum_{i=1}^{M}(\lambda_i + \alpha^2)^{-1}(f,\phi_i)\phi_i$, it follows that

$$L_h^{1/2}\hat{T}_h f = \sum_{i=1}^{M}\lambda_i^{1/2}(\lambda_i + \alpha^2)^{-1}(f,\phi_i)\phi_i \ .$$

This and the equality $\dfrac{\sqrt{\lambda_i}}{\lambda_i+\alpha^2} = \dfrac{\sqrt{\lambda_i}}{(\lambda_i+\alpha^2)^{1/2}(\lambda_i+\alpha^2)^{1/2}}$ imply that

(8.30) $$||L_h^{1/2}\hat{T}_h f|| \le \frac{1}{\alpha}\,||f|| \ .$$

Since

$$\hat{T}_h^{(1)} = -\hat{T}_h L_h^{(1)}\hat{T}_h = -\hat{T}_h L_h^{1/2}(T_h^{1/2}L_h^{(1)}T_h^{1/2})L_h^{1/2}\hat{T}_h$$

and $T_h^{1/2}\hat{T}_h^{(1)}L_h^{1/2} = -\hat{T}_h(T_h^{1/2}L_h^{(1)}T_h^{1/2})(L_h\hat{T}_h)$, Lemma 3.1, (8.30) , (8.25) and (8.26) prove that

(8.31) $$||\hat{T}_h^{(1)}f|| \le \frac{C}{\alpha^2}\,||f||$$

and

(8.32) $$||T_h^{1/2}\hat{T}_h^{(1)}L_h^{1/2}f|| \le \frac{C}{\alpha^2}\,||f||$$

where the constant C is independent of α . As in the proof of Lemma 8.3, (8.29), (8.31) and (8.32) show that for α large

(8.33) $$|||(I+m\hat{\mathcal{T}}_h^{(1)})^{-1}\phi|||_0^2 \le C(||(I+m\alpha\hat{T}_h^{(1)})\phi_1 - \hat{T}_h^{(1)}\phi_2||^2$$

$$+ ||T_h^{1/2}(-m\alpha^2\hat{T}_h^{(1)}\phi_1 + (I+m\alpha\hat{T}_h^{(1)}))\phi_2||^2) \ .$$

Since $\hat{T}_h^{(1)}L_h^{1/2} = T_h^{1/2}(L_h\hat{T}_h)(T_h^{1/2}\hat{L}_h\hat{T}_h^{(1)}L_h^{1/2})$,

(8.26) and (8.27) show that

(8.34) $$||\hat{T}_h^{(1)}L_h^{1/2}g|| \le C||g|| \ .$$

Using (8.31), (8.34) and (8.32) in (8.33) gives

$$|||(I+m\hat{\mathcal{T}}_h^{(1)})^{-1}\phi|||_0^2 \le C(||\phi_1||^2 + ||T_h^{1/2}\phi_2||^2)$$

for α large, where C can be chosen independent of α . This completes the proof of the lemma.

The next lemma is the discrete counterpart of Lemma 8.4.

<u>Lemma 8.7</u>. For all $\phi \in S_h \times S_h$

(8.35) $$||\hat{\mathcal{T}}_h\phi|||_0 \le \frac{C}{\alpha}\,|||\phi|||_0$$

where the constant C is independent of α :

Proof. $\hat{\mathcal{L}}_h$ has a complete orthonormal set of eigenfunctions given by

$$\Phi_{\pm j} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_j \\ \pm i\sqrt{\lambda_j}\,\phi_j \end{pmatrix}$$

with eigenvalues $-\alpha \pm i\sqrt{\lambda_j}$, for $j = 1, \ldots, M$. Let

$$\Phi = \sum_{j=-M}^{M} C_j \Phi_j .$$

Then

$$\hat{\mathcal{T}}_h \Phi = \sum_{j=-M}^{M} \frac{C_j}{(\text{sgn } j) i\sqrt{\lambda_{|j|}} - \alpha} \Phi_j$$

where sgn j is the sign of j . Since

$$\frac{1}{|(\text{sgn } j) i\sqrt{\lambda_{|j|}} - \alpha|} \le \frac{1}{\alpha}$$

(8.35) follows (as in the proof of Lemma 8.4).

Proof of Theorem 5.2. Using Lemmas 8.5, 8.6, and 8.7 and the definition of $\hat{E}_{m,h}$ it follows that

(8.36) $$(\alpha C_1 - C_2)\, |||\Phi|||_0 \le |||\hat{E}_{m,h}\Phi|||_0$$

for all $\Phi \in S_h \times S_h$. (8.36) implies that $\hat{E}_{m,h}$ is invertible on $S_h \times S_h$ for sufficiently large α . (5.8) follows from exactly the same computation given at the end of the proof of Theorem 5.1.

The proof of (5.9) will complete the proof of Theorem 5.2. Assume

(8.37) $$|||(\hat{E}_i^{-1} - \hat{E}_{i,h}^{-1})F|||_0 \le Ch^2(||f_1||_{s-2} + ||f_2||_{s-2})$$

for $i = 1, \ldots, m$. Let $W = \hat{E}_{m+1}^{-1} F$, $W_h = \hat{E}_{m+1,h}^{-1} F$,

$$\hat{R} = \sum_{\ell=0}^{m-2} \binom{m}{\ell} \hat{\mathcal{L}} \hat{\mathcal{T}}^{(m-\ell)} \hat{E}_{\ell+2}^{-1} \cdots \hat{E}_m^{-1}$$ and

$$\hat{R}_h = \sum_{\ell=0}^{m-2} \binom{m}{\ell} \hat{\mathcal{L}}_h \hat{\mathcal{T}}_h^{(m-\ell)} \hat{E}_{\ell+2,h}^{-1} \cdots \hat{E}_{m,h}^{-1} .$$ Since

$(\hat{\mathcal{L}}(I - m\hat{\mathcal{T}}^{(1)}) - \hat{R})W = F$ and $(\hat{\mathcal{L}}_h(I - m\hat{\mathcal{T}}_h^{(1)}) - \hat{R}_h)W_h = F$,

$W = (I - m\hat{\mathcal{T}}^{(1)})^{-1}\hat{\mathcal{T}}(F + \hat{R}W)$ and $W_h = (I - m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h(F + \hat{R}_h W_h)$.

These equations imply that

(8.38) $$\begin{aligned} W - W_h &= [(I - m\hat{\mathcal{T}}^{(1)})^{-1}\hat{\mathcal{T}} - (I - m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h]F \\ &+ [(I - m\hat{\mathcal{T}}^{(1)})^{-1}\hat{\mathcal{T}}\hat{R} - (I - m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h\hat{R}_h]W \\ &+ [(I - m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h\hat{R}_h](W - W_h) . \end{aligned}$$

Using Lemmas 8.5, 8.6 and 8.7 and (8.36), it follows that

(8.39) $$\begin{aligned} |||(I - m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h\hat{R}_h(W - W_h)|||_0 &\le \frac{C}{\alpha}\, |||W - W_h|||_0 \\ &\le \frac{1}{2}\, |||W - W_h|||_0 \end{aligned}$$

for $\alpha \ge 2C$.

(8.40) $$(I-m\hat{\mathcal{T}}^{(1)})^{-1}\hat{\mathcal{T}} = (I+2m\alpha\hat{T}^{(1)})^{-1}\begin{pmatrix} -\alpha\hat{T}-m\hat{T}^{(1)} & -\hat{T} \\ I+\alpha m\hat{T}^{(1)} - \alpha^2\hat{T} & -\alpha\hat{T} \end{pmatrix}$$

and

(8.41) $$(I-m\hat{\mathcal{T}}^{(1)})^{-1}\hat{\mathcal{T}}^{(m-\ell)} = (I+2m\alpha\hat{T}^{(1)})^{-1}\begin{pmatrix} -\alpha\hat{T}^{(m-\ell)} & -\hat{T}^{(m-\ell)} \\ -\alpha^2\hat{T}^{(m-\ell)} & -\alpha\hat{T}^{(m-\ell)} \end{pmatrix}$$

In order to estimate the terms in (8.38) we show that for $j \geq 0$

(8.42) $$||((I+2m\alpha\hat{T}^{(1)})^{-1}\hat{T}^{(j)} - (I+2m\alpha\hat{T}_h^{(1)})^{-1}\hat{T}_h^{(j)})\phi|| \leq C(\alpha)h^2||\phi||_{s-2}$$

and

(8.43) $$||T_h^{1/2}((I+2m\alpha\hat{T}^{(1)})^{-1} - (I+2m\alpha\hat{T}_h^{(1)})^{-1}P)\phi|| \leq C(\alpha)h^2||\phi||_{s-2}$$

where P is the L^2 orthogonal projection onto S_h . To show (8.42) write

$$(I+2m\alpha\hat{T}^{(1)})^{-1}\hat{T}^{(j)} - (I+2m\alpha\hat{T}_h^{(1)})^{-1}\hat{T}_h^{(j)}$$

$$= ((I+2m\alpha\hat{T}^{(1)})^{-1} - (I+2m\alpha\hat{T}_h^{(1)})^{-1}P)\hat{T}^{(j)}$$

$$+ (I+2m\alpha\hat{T}_h^{(1)})^{-1}P(\hat{T}^{(j)} - \hat{T}_h^{(j)})$$

$$= (I+2m\alpha\hat{T}_h^{(1)})^{-1}P(P+2m\alpha\hat{T}_h^{(1)} - I-2m\alpha\hat{T}^{(1)})(I+2m\alpha\hat{T}^{(1)})^{-1}\hat{T}^{(j)}$$

$$+ (I-P)(I+2m\alpha\hat{T}^{(1)})^{-1}\hat{T}^{(j)}$$

$$+ (I+2m\alpha\hat{T}_h^{(1)})^{-1}P(\hat{T}^{(j)} - \hat{T}_h^{(j)}) .$$

This identity and $||(I-P)\psi|| \leq ||(TL-T_hL)\psi|| \leq Ch^s||L\psi||_{s-2}$ imply that

$$||((I+2m\alpha\hat{T}^{(1)})^{-1}\hat{T}^{(j)} - (I+2m\alpha\hat{T}_h^{(1)})^{-1}\hat{T}^{(j)})\phi||$$

$$\leq C(\alpha)h^s(||L(I+2m\alpha\hat{T}^{(1)})^{-1}\hat{T}^{(j)}\phi||_{s-2} + ||\phi||_{s-2})$$

$$\leq C(\alpha)h^s(||L(\hat{L}+2m\alpha\hat{L}\hat{T}^{(1)})^{-1}\hat{L}\hat{T}^{(j)}\phi||_{s-2} + ||\phi||_{s-2})$$

$$\leq C(\alpha)h^s||\phi||_{s-2} .$$

This is (8.42). The proof of (8.43) is similar.

$$T_h^{1/2}((I+2m\alpha\hat{T}^{(1)})^{-1} - (I+2m\alpha\hat{T}_h^{(1)})^{-1}P)$$

$$= T_h^{1/2}(I+2m\alpha\hat{T}_h^{(1)})^{-1}P(P+2m\alpha\hat{T}_h^{(1)} - I-2ma\alpha\hat{T}^{(1)})(I+2m\alpha\hat{T}^{(1)})^{-1}$$

Therefore,

$$||T_h^{1/2}((I+2m\alpha\hat{T}^{(1)})^{-1} - (I+2m\alpha\hat{T}_h^{(1)})P)\phi||_{s-2}$$

$$\leq C(\alpha)h^s||(I+2m\alpha\hat{T}^{(1)})^{-1}\phi||_{s-2}$$

$$\le C(\alpha)h^s \| (\hat{L}+2m\alpha\hat{L}\hat{T}^{(1)})^{-1}\hat{L}\phi\|_{s-2}$$

$$\le C(\alpha)h^s\|\phi\|_{s-2} \,.$$

This is (8.43). Using (8.42) and (8.43) to estimate the difference between (8.40) and the corresponding formula for $(I-m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h$ gives

$$(8.44)\qquad |||((I-m\hat{\mathcal{T}}^{(1)})^{-1}\hat{\mathcal{T}} - (I-m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h)F|||$$

$$\le C(\alpha)h^s(\|f_1\|_{s-2} + \|f_2\|_{s-2}) \,.$$

Estimating (8.38) with (8.39) and (8.44) gives

$$(8.45)\qquad |||W-W_h|||_0 \le C(\alpha)h^s(\|f_1\|_{s-2} + \|f_2\|_{s-2})$$

$$+ 2|||((I-m\hat{\mathcal{T}}^{(1)})^{-1}\hat{\mathcal{T}}\hat{\mathcal{R}} - (I-m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h\hat{R}_h)W|||_0 \,.$$

Since

$$(I-m\hat{\mathcal{T}}^{(1)})^{-1}\hat{\mathcal{T}}\hat{\mathcal{R}} = \sum_{\ell=0}^{m-2}\binom{m}{\ell}(I-m\hat{\mathcal{T}}^{(1)})^{-1}\hat{\mathcal{T}}^{(m-\ell)}\hat{E}_{\ell+2}^{-1}\cdots\hat{E}_m^{-1}$$

and

$$(I-m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h\hat{R}_h = \sum_{\ell=0}^{m-2}\binom{m}{\ell}(I-m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h^{(m-\ell)}\hat{E}_{\ell+2,h}^{-1}\cdots\hat{E}_{m,h}^{-1} \,, \text{ it}$$

follows that

$$|||((I-m\hat{\mathcal{T}}^{(1)})^{-1}\hat{\mathcal{T}}\hat{\mathcal{R}} - (I-m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h\hat{R}_h)W|||_0$$

$$\le \sum_{\ell=0}^{m-2}\binom{m}{\ell}|||((I-m\hat{\mathcal{T}}^{(1)})^{-1}\hat{\mathcal{T}}^{(m-\ell)} - (I-m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h^{(m-\ell)})$$

$$\circ\, \hat{E}_{\ell+2}^{-1}\cdots\hat{E}_m^{-1}W|||_0$$

$$+ \sum_{\ell=0}^{m-2}\binom{m}{\ell}|||((I-m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h^{(m-\ell)})(\hat{E}_{\ell+2}^{-1}\cdots\hat{E}_m^{-1} - \hat{E}_{\ell+2,h}^{-1}$$

$$\cdots\hat{E}_{m,h}^{-1})W|||_0 \,.$$

Using (8.42) and (8.43) to estimate the difference between (8.41) and the corresponding formula for $(I-m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h^{(m-\ell)}$ gives

$$|||W-W_h|||_0 \le C(\alpha)(h^s(\|f_1\|_{s-2} + \|f_2\|_{s-2})$$

$$+ \sum_{\ell=0}^{m-2}|||(\hat{E}_{\ell+2}^{-1}\cdots\hat{E}_m^{-1} - \hat{E}_{\ell+2,h}^{-1}\cdots\hat{E}_{m,h}^{-1})W|||_0) \,.$$

Since $\hat{E}_{\ell+2}^{-1}\cdots\hat{E}_m^{-1} - \hat{E}_{\ell+2,h}^{-1}\cdots\hat{E}_{m,h}^{-1}$

$$= \sum_{j=\ell+2}^{m}\hat{E}_{\ell+2,h}^{-1}\cdots\hat{E}_{j-1,h}^{-1}(\hat{E}_j^{-1} - \hat{E}_{j,h}^{-1})\hat{E}_{j+1}^{-1}\cdots\hat{E}_m^{-1} \,,$$

(8.36), (8.37) and (2) in Theorem 5.1 imply that

$$|||W-W_h|||_0 \le C(\alpha)h^s(\|f_1\|_{s-2} + \|f_2\|_{s-2}) \,.$$

This completes the proof of Theorem 5.2.