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Estimates for the Chebyshev Function $\psi(x) - \theta(x)$

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Abstract. A simple approximation for the difference $\psi(x) - \theta(x)$ is established by elementary methods. This approximation is used to obtain several estimates for $\psi(x) - \theta(x)$ which are sharper than those previously given in the literature.

Let ψ and θ be defined, as usual, by

$$\psi(x) = \sum_{p^\alpha \leqslant x} \log p; \qquad \theta(x) = \sum_{p \leqslant x} \log p$$

with $x > 0$, p prime and α a positive integer. In this paper we show that it is possible to approximate the difference $\psi - \theta$ in terms of ψ in quite a simple form. As consequences we deduce some estimates for $\psi - \theta$ which improve those given in [3], [4], and [5].

THEOREM 1. *For every $x > 0$ we have*

$$\psi(x) - \theta(x) \leqslant \psi(x^{1/2}) + \psi(x^{1/3}) + \psi(x^{1/5}), \tag{1}$$

$$\psi(x) - \theta(x) \geqslant \psi(x^{1/2}) + \psi(x^{1/3}) + \psi(x^{1/7}). \tag{2}$$

Proof. From the well-known identity

$$\psi(x) = \sum_{k \geqslant 1} \theta(x^{1/k}), \tag{3}$$

we deduce

$$\psi(x^{1/2}) = \sum_{k \geqslant 1} \theta(x^{1/2k}). \tag{4}$$

Substituting (4) in (3) we get

$$\psi(x) = \theta(x) + \psi(x^{1/2}) + \sum_{k \geqslant 1} \theta(x^{1/2k+1})$$

or

$$\psi(x) - \theta(x) = \psi(x^{1/2}) + \sum_{k \geqslant 1} \theta(x^{1/6k-3}) + \sum_{k \geqslant 1} \theta(x^{1/6k-1}) + \sum_{k \geqslant 1} \theta(x^{1/6k+1}). \tag{5}$$

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From (3) we can still deduce

$$\psi(x^{1/3}) = \sum_{k\geqslant 1} \theta(x^{1/3k}) = \sum_{k\geqslant 1} \theta(x^{1/6k-3}) + \sum_{k\geqslant 1} \theta(x^{1/6k})$$

and (5) is transformed into

$$\psi(x) - \theta(x) = \psi(x^{1/2}) + \psi(x^{1/3}) + \sum_{k\geqslant 1} \theta(x^{1/6k-1}) - \sum_{k\geqslant 1} \theta(x^{1/6k}) + \sum_{k\geqslant 1} \theta(x^{1/6k+1}). \tag{6}$$

Since θ is an increasing function, from (6) we get the inequalities

$$\begin{aligned} \psi(x) - \theta(x) &\leqslant \psi(x^{1/2}) + \psi(x^{1/3}) + \sum_{k\geqslant 1} \theta(x^{1/6k-1}) \\ &\leqslant \psi(x^{1/2}) + \psi(x^{1/3}) + \sum_{k\geqslant 1} \theta(x^{1/5k}) \end{aligned} \tag{7}$$

and

$$\begin{aligned} \psi(x) - \theta(x) &\geqslant \psi(x^{1/2}) + \psi(x^{1/3}) + \sum_{k\geqslant 1} \theta(x^{1/6k+1}) \\ &\geqslant \psi(x^{1/2}) + \psi(x^{1/3}) + \sum_{k\geqslant 1} \theta(x^{1/7k}). \end{aligned} \tag{8}$$

Using (3) again, we obtain

$$\psi(x^{1/5}) = \sum_{k\geqslant 1} \theta(x^{1/5k}); \qquad \psi(x^{1/7}) = \sum_{k\geqslant 1} \theta(x^{1/7k}).$$

Substituting these identities respectively in (7) and (8), we get (1) and (2), which proves the theorem.

Other estimates for $\psi - \theta$ could be obtained with the methods used in the preceding proof. As an example, observing that

$$\begin{aligned} &\sum_{k\geqslant 1} \theta(x^{1/6k-1}) + \sum_{k\geqslant 1} \theta(x^{1/6k+1}) \\ &\qquad \geqslant \sum_{k\geqslant 1} \theta(x^{1/10k-5}) + \sum_{k\geqslant 1} \theta(x^{1/10k}) = \psi(x^{1/5}), \end{aligned}$$

we get from (6),

$$\psi(x) - \theta(x) \geqslant \psi(x^{1/2}) + \psi(x^{1/3}) + \psi(x^{1/5}) - \psi(x^{1/6}),$$

which is sharper than (2) for large values of x. Combining this inequality with (1), we have

$$\psi(x) - \theta(x) = \psi(x^{1/2}) + \psi(x^{1/3}) + \psi(x^{1/5}) - \xi\psi(x^{1/6})$$

with $0 \leqslant \xi \leqslant 1$.

The inequalities given by Theorem 1 are, however, sharp enough for our present purposes. In order to apply Theorem 1, we have first constructed a table of $\psi(x) - \theta(x)$ in the range $0 < x \leqslant 3 \cdot 10^6$ with an accuracy of five decimals (Table I). Since $\psi(x) - \theta(x)$ is a step function that only jumps where $x = p^\alpha$ with p prime and α an integer greater than one, we only need to tabulate $\psi(x) - \theta(x)$ for those values of x.

We will also use two simple lemmas.

LEMMA 1. *If an interval I is the union of a finite collection of intervals* $I_k = (m_k, n_k)$, *and there are positive constants* a_k, b_k, c_k, L^+ *and a constant* $\varepsilon \geqslant 0$, *such that* $a_k > 1 + \varepsilon$ *and*

$$\psi(x^{1/2}) < a_k x^{1/2}; \qquad \psi(x^{1/3}) < b_k x^{1/3}, \tag{9}$$
$$\psi(x^{1/5}) < c_k x^{1/5} \quad \textit{for } x \in I_k,$$

$$m_k \geqslant \left(4c_k/5(a_k - 1 - \varepsilon)\right)^{10/3}, \tag{10}$$

$$L^+ \geqslant \sup_k \left\{(a_k - 1 - \varepsilon)n_k^{1/6} + b_k + c_k n_k^{-2/15}\right\}, \tag{11}$$

we have

$$\psi(x) - \theta(x) < (1 + \varepsilon)x^{1/2} + L^+ x^{1/3} \quad \textit{for } x \in I.$$

Proof. If $x \in I_k$, from Theorem 1 and (9) it follows that

$$\psi(x) - \theta(x) < a_k x^{1/2} + b_k x^{1/3} + c_k x^{1/5} \quad \text{if } x \in I_k,$$

or,

$$\psi(x) - \theta(x) < (1 + \varepsilon)x^{1/2}$$
$$\left((a_k - 1 - \varepsilon)x^{1/6} + b_k + c_k x^{-2/15}\right)x^{1/3} \quad \text{if } x \in I_k.$$

The coefficient of $x^{1/3}$ in this last expression increases monotonically in I_k provided that

$$x \geqslant \left(4c_k/5(a_k - 1 - \varepsilon)\right)^{10/3} \quad \text{for } x \in I_k$$

and this condition is implied by (10). Thus, we get

$$\psi(x) - \theta(x) < (1 + \varepsilon)x^{1/2} + \left((a_k - 1 - \varepsilon)n_k^{1/6} + b_k + c_k n_k^{-2/15}\right)x^{1/3}$$

if $x \in I_k$ and the lemma follows from (11).

LEMMA 2. *If an interval J is the union of a finite collection of intervals* $J_k = (m_k, n_k)$, *and there are positive constants* a'_k, b'_k, d_k, L^- *and a constant* $\varepsilon \geqslant 0$ *such that* $a'_k < 1 - \varepsilon$ *and*

$$\psi(x^{1/2}) > a'_k x^{1/2}; \qquad \psi(x^{1/3}) > b'_k x^{1/3}, \tag{12}$$
$$\psi(x^{1/7}) > d_k x^{1/7} \quad \textit{for } x \in J_k,$$

$$L^- \leqslant \inf_k \left\{(a'_k - 1 + \varepsilon)n_k^{1/6} + b'_k + d_k n_k^{-4/21}\right\}, \tag{13}$$

we have

$$\psi(x) - \theta(x) > (1 - \varepsilon)x^{1/2} + L^- x^{1/3} \quad \textit{for } x \in J.$$

Proof. If $x \in J_k$, from Theorem 1 and (12), we have

$$\psi(x) - \theta(x) > a'_k x^{1/2} + b'_k x^{1/3} + d_k x^{1/7}$$

or

$$\psi(x) - \theta(x) > (1 - \varepsilon)x^{1/2}$$
$$+ \left((a'_k - 1 + \varepsilon)x^{1/6} + b'_k + d_k x^{-4/21}\right)x^{1/3} \quad \text{if } x \in J_k.$$

The coefficient of $x^{1/3}$ in this last expression clearly decreases when x increases and we get, if $x \in J_k$,

$$\psi(x) - \theta(x) > (1-\varepsilon)x^{1/2} + \left((a'_k - 1 + \varepsilon)n_k^{1/6} + b'_k + d_k n_k^{-4/21}\right)x^{1/3}.$$

Now the lemma follows from (13).

It is now possible to show the following

THEOREM 2. *We have*

$$\psi(x) - \theta(x) < \sqrt{x} + \frac{4}{3}\sqrt[3]{x} \quad \text{if } 0 < x \leqslant 10^8, \tag{14}$$

$$\psi(x) - \theta(x) > \sqrt{x} + \frac{2}{3}\sqrt[3]{x} \quad \text{if } 2187 \leqslant x \leqslant 10^8. \tag{15}$$

Proof. Inequalities (14) and (15) can be verified directly from our Table I for $x \leqslant 3 \cdot 10^6$. To prove (14) in this range we start with $x > 0$ and we get

$$\sqrt{x} + \frac{4}{3}\sqrt[3]{x} > 0 = \lim_{t \to 4^-} \psi(t) - \theta(t) = \psi(x) - \theta(x) \quad \text{if } 0 < x < 4.$$

Next, we take $x \geqslant 4$ and Table I gives

$$\sqrt{x} + \frac{4}{3}\sqrt[3]{x} > 4.116 > \lim_{t \to 25^-} \psi(t) - \theta(t) \geqslant \psi(x) - \theta(x) \quad \text{if } 4 \leqslant x < 25.$$

Continuing in the same way and noticing that $\psi(3 \cdot 10^6) - \theta(3 \cdot 10^6)$ is given by the last entry in the table, we establish (14) for $x \leqslant 3 \cdot 10^6$.

To verify (15) in the range $2187 \leqslant x \leqslant 3 \cdot 10^6$ we start at the end of Table I. Taking $x \leqslant 3 \cdot 10^6$, we have

$$\sqrt{x} + \frac{2}{3}\sqrt[3]{x} < 1828.201 < \psi(x_1) - \theta(x_1),$$

with $x_1 = 2765569$, and this gives (15) for $x_1 \leqslant x \leqslant 3 \cdot 10^6$. Next, taking $x < x_1$ we have

$$\sqrt{x} + \frac{2}{3}\sqrt[3]{x} < 1756.578 < \psi(x_2) - \theta(x_2),$$

with $x_2 = 2571353$, and this gives (15) for $x_2 \leqslant x < x_1$. This process can be continued down to $x = 2187$ and (15) is established for $x \leqslant 3 \cdot 10^6$.

If $3 \cdot 10^6 < x \leqslant 10^8$ we have

$$1732 < x^{1/2} \leqslant 10^4; \quad 144 < x^{1/3} < 465; \quad 19 < x^{1/5} < 40; \quad 8 < x^{1/7} < 14.$$

From the Appel and Rosser [1] table of $\theta(x)$, Lehmer's [2] table of primes, and Table I, it follows that we can apply Lemma 2 to the interval $J = J_1 = (3 \cdot 10^6, 10^8]$ with

$$\varepsilon = 0; \quad L^- = 2/3; \quad a'_1 = 0.98708; \quad b'_1 = 0.94842; \quad d_1 = 0.71200,$$

and this proves (15).

To prove (14) in the interval $I = (3 \cdot 10^6, 10^8]$, we consider the intervals $I_1 = (3 \cdot 10^6, 83.5 \cdot 10^6]$ and $I_2 = (83.5 \cdot 10^6, 10^8]$. Using the same tables it is possible to apply Lemma 2 with

$$\varepsilon = 0; \quad L^+ = 4/3; \quad a_1 = 1.00990; \quad a_2 = 1.00463;$$
$$b_1 = b_2 = 1.03591; \quad c_1 = c_2 = 1.02938,$$

and this establishes (14).

From Theorem 2, we can deduce the following estimates for $\psi(x)$ in the range $0 < x \leqslant 10^8$:

THEOREM 3. *We have*

$$\psi(x) < x + 0.656\sqrt{x} + \frac{4}{3}\sqrt[3]{x} \quad \textit{if } 0 < x \leqslant 10^8, \tag{16}$$

$$\psi(x) > x - 0.833\sqrt{x} + \frac{2}{3}\sqrt[3]{x} \tag{17}$$
$$\textit{if } 1427 \leqslant x \leqslant 3298,\ 3299 \leqslant x \leqslant 19371 \textit{ or } 19373 \leqslant x \leqslant 10^8.$$

Proof. From the Appel and Rosser [1] tables of $\theta(x)$ and $(x - \theta(x))/\sqrt{x}$, we have

$$\frac{x - \theta(x)}{\sqrt{x}} > 0.344 \quad \text{if } 0 < x \leqslant 10^8,$$

and

$$\frac{x - \theta(x)}{\sqrt{x}} < 1.833 \quad \text{if } 19801 \leqslant x \leqslant 10^8.$$

These inequalities, together with (14) and (15), prove (16) for $0 < x \leqslant 10^8$ and (17) for $19801 \leqslant x \leqslant 10^8$. Directly from [1], [2] and Table I, we can verify that (17) still holds when $1427 \leqslant x \leqslant 3298$, $3299 \leqslant x \leqslant 19371$ and $19373 \leqslant x < 19801$.

We have applied Theorem 3 together with the tables [1], [2] and Table I to get some close estimates for $\psi(x)$ in the range $0 < x \leqslant 10^8$. In Table II, we have listed triplets $(n, \lambda^-, \lambda^+)$ such that

$$\lambda^- x < \psi(x) < \lambda^+ x \quad \text{if } n \leqslant x \leqslant 10^8.$$

To evaluate the lower bounds λ^-, we observe that

$$\psi(x)/x > \psi(n_k)/n_{k+1} \quad \text{for } n_k \leqslant x < n_{k+1},$$

where n_k and n_{k+1} are two consecutive prime powers. Then, if n_k is the largest prime power (not exceeding 10^8) such that $\psi(n_{k-1})/n_k < \lambda^-$, it follows that $\psi(x) > \lambda^- x$ for $n_k \leqslant x \leqslant 10^8$; it also follows that this inequality fails immediately below n_k.

Now, in the range $10^8 \leqslant x \leqslant 10^{16}$, we can get better estimates for $\psi(x) - \theta(x)$ than those given by (14) and (15).

THEOREM 4. *We have*

$$\psi(x) - \theta(x) < \sqrt{x} + \frac{6}{5}\sqrt[3]{x} \quad \textit{if } 10^8 \leqslant x \leqslant 10^{16}, \tag{18}$$

$$\psi(x) - \theta(x) > \sqrt{x} + \frac{6}{7}\sqrt[3]{x} \quad \textit{if } 10^8 \leqslant x \leqslant 10^{16}. \tag{19}$$

Proof. To prove (18) we divide $I = [10^8, 10^{16}]$ into intervals $I_k = [m_k, n_k)$ for $k = 1, \ldots, 8$ and $I_9 = [m_9, 10^{16}]$. Using the tables [1], [2] together with (16) and Tables I and II, it is now easy to verify that we can apply Lemma 1 with $\varepsilon = 0$, $L^+ = 6/5$, and the following constants m_k, n_k, a_k, b_k, c_k.

k	m_k	n_k	a_k	b_k	c_k
1	10^8	10273^2	1.00347	1.02117	0.90967
2	10273^2	41^5	1.00517	1.00913	0.90967
3	41^5	661^3	1.00421	1.01130	1.01604
4	661^3	24103^2	1.00347	1.01802	1.01604
5	24103^2	839^3	1.00458	0.99829	0.94445
6	839^3	10^{10}	1.00297	1.01364	1.02871
7	10^{10}	$16 \cdot 10^{11}$	1.00153	1.00990	1.03883
8	$16 \cdot 10^{11}$	$2 \cdot 10^{15}$	1.00049	1.00990	1.02728
9	$2 \cdot 10^{15}$	10^{16}	1.00037	1.00990	1.01364

Similarly, to prove (19), we divide $J = [10^8, 10^{16}]$ into intervals $J_k = [m_k, n_k)$ for $k = 1,\dots,7$ and $J_8 = [m_8, 10^{16}]$. Again using [1], [2] together with (17) and Tables I and II, we can apply Lemma 2 with $\varepsilon = 0$, $L^- = 6/7$ and the following constants $m_k, n_k, a'_k, b'_k, d_k$.

k	m_k	n_k	a'_k	b'_k	d_k
1	10^8	11801^2	0.99343	0.99189	0.87827
2	11801^2	18500^2	0.99486	0.97494	0.79967
3	18500^2	20000^2	0.99486	0.97870	0.79631
4	20000^2	53000^2	0.99643	0.97870	0.79341
5	53000^2	180000^2	0.99770	0.97870	0.83763
6	180000^2	$121 \cdot 10^{10}$	0.99870	0.98828	0.88117
7	$121 \cdot 10^{10}$	$15 \cdot 10^{14}$	0.99960	0.99343	0.90602
8	$15 \cdot 10^{14}$	10^{16}	0.99987	0.99851	0.94842

With the aid of a computer, it can be easily verified that inequalities (18) and (19) still hold, respectively, for $8236167 \leqslant x \leqslant 10^8$ and $2036329 \leqslant x \leqslant 10^8$ but fail immediately below these bounds. We conclude

$$\psi(x) - \theta(x) < \sqrt{x} + \frac{6}{5}\sqrt[3]{x} \quad \text{if } 8236167 \leqslant x \leqslant 10^{16}, \tag{20}$$

and

$$\psi(x) - \theta(x) > \sqrt{x} + \frac{6}{7}\sqrt[3]{x} \quad \text{if } 2036329 \leqslant x \leqslant 10^{16}. \tag{21}$$

The results we have proved so far are strictly elementary. However, in order to estimate $\psi(x) - \theta(x)$ for $x > 10^{16}$, we need the following bounds for ψ which were recently deduced by Schoenfeld [5], using powerful analytical methods.

(22) $|\psi(x) - x| < 0.00119721x$ if $10^8 \leqslant x < e^{18.43}$,

(23) $|\psi(x) - x| < 0.0011930x$ if $e^{18.43} \leqslant x < e^{18.44}$,

(24) $|\psi(x) - x| < 0.0011885x$ if $e^{18.44} \leqslant x < e^{18.45}$,

(25) $|\psi(x) - x| < 0.0011839x$ if $e^{18.45} \leqslant x < e^{18.5}$,

(26) $|\psi(x) - x| < 0.0011615x$ if $e^{18.5} \leqslant x < e^{18.7}$,

(27) $|\psi(x) - x| < 0.0010765x$ if $e^{18.7} \leqslant x < e^{19}$,

(28) $|\psi(x) - x| < 0.00096161x$ if $x \geqslant e^{19}$.

We can now prove the following

THEOREM 5. *We have*

(29) $\psi(x) - \theta(x) < 1.001\sqrt{x} + 1.1\sqrt[3]{x}$ if $x \geqslant 10^{16}$,

(30) $\psi(x) - \theta(x) < 1.001\sqrt{x} + \sqrt[3]{x}$ if $x \geqslant e^{38}$,

(31) $\psi(x) - \theta(x) > 0.999\sqrt{x} + 0.9\sqrt[3]{x}$ if $x \geqslant 10^{16}$,

(32) $\psi(x) - \theta(x) > 0.999\sqrt{x} + \sqrt[3]{x}$ if $x \geqslant e^{38}$.

Proof. To prove (29) we divide $I = [10^{16}, e^{38})$ into intervals $I_1 = [10^{16}, e^{36.88})$, $I_2 = [e^{36.88}, e^{37})$, $I_3 = [e^{37}, e^{38})$. Using [1], [2] and Tables I and II together with estimates (22) to (27), we see that it is possible to apply Lemma 1 with $\varepsilon = 0.001$, $L^+ = 1.1$, and

$$a_1 = 1.00119721, \quad a_2 = 1.0011885, \quad a_3 = 1.0011615,$$
$$b_1 = b_2 = 1.00052, \quad b_3 = 1.00121,$$
$$c_1 = 1.00450, \quad c_2 = c_3 = 1.01364.$$

Now, if $x \geqslant e^{38}$, it follows from (1), (22) to (28) and Table II that

$$\psi(x) - \theta(x) < 1.001x^{1/2} + (-0.00003839x^{1/6} + 1.00115 + 1.00990x^{-2/15})x^{1/3}$$
$$< 1.001x^{1/2} + x^{1/3},$$

and this proves (29) and (30).

To prove (31), we divide $J = [10^{16}, e^{38})$ into intervals $J_1 = [10^{16}, e^{37})$, $J_2 = [e^{37}, e^{38})$. Using inequalities (22) to (27) together with Table II, it is easily seen that we can apply Lemma 2 with $\varepsilon = 0.001$, $L^- = 0.9$, and

$$a_1' = 0.99880279, \quad a_2' = 0.9988385, \quad b_1' = b_2' = 0.99870, \quad d_1 = d_2 = 0.94842.$$

Finally, if $x \geqslant e^{38}$, it follows from (2), (22) to (28) and Table II that

$$\psi(x) - \theta(x) > 0.999x^{1/2} + (0.00003839x^{1/6} + 0.99880)x^{1/3}$$
$$> 0.999x^{1/2} + x^{1/3},$$

and this proves (31) and (32).

TABLE I

x	ψ(x)−θ(x)	x	ψ(x)−θ(x)	x	ψ(x)−θ(x)	x	ψ(x)−θ(x)
4	0.69315	10201	122.02043	80089	340.44853	292681	606.67063
8	1.38629	10609	126.65516	83521	343.28174	299209	612.97508
9	2.48491	11449	131.32799	85849	348.96191	300763	617.17977
16	3.17805	11881	136.01934	94249	354.68876	310249	623.50234
25	4.78749	12167	139.15483	96721	360.42855	316969	629.83562
27	5.88610	12769	143.88222	97969	366.17476	323761	636.17950
32	6.57925	14641	146.28012	100489	371.93366	326041	642.52689
49	8.52516	15625	147.88956	103823	375.78381	332929	648.88473
64	9.21831	16129	152.73374	109561	381.58592	344569	655.25975
81	10.31692	16384	153.42689	113569	387.40601	351649	661.64495
121	12.71482	16807	155.37280	117649	389.35192	357911	665.90763
125	14.32425	17161	160.24800	120409	395.20124	358801	672.30289
128	15.01740	18769	163.16790	121301	401.05631	361201	678.70248
169	17.58235	19321	170.10245	124609	406.92278	368449	685.11001
243	18.68096	19683	171.20106	128881	412.80611	371293	687.67496
256	19.37411	22201	176.20501	130321	415.75054	375769	694.09333
289	22.20732	22801	181.22229	131072	416.44369	380689	700.51820
343	24.15323	24389	184.58959	134689	422.34905	383161	706.94630
361	27.09767	24649	189.64583	139129	428.27063	389017	711.23676
512	27.79082	26569	194.73958	143641	434.20817	390625	712.84620
529	30.92631	27889	199.85758	146689	440.15620	398161	719.29350
625	32.53575	28561	202.42253	148877	444.12649	410881	725.75653
729	33.63436	29791	205.85651	151321	450.09007	413449	732.22268
841	37.00166	29929	211.00980	157609	456.07401	418609	738.69503
961	40.43565	32041	216.19719	160801	462.06797	426409	745.17660
1024	41.12879	32761	221.39569	161051	464.46587	434281	751.66733
1331	43.52669	32768	222.08883	167281	470.47958	436921	758.16108
1369	47.13761	36481	227.34111	175561	476.51745	452929	764.67283
1681	50.85118	37249	232.60380	177147	477.61607	458329	771.19050
1849	54.61238	38809	237.88700	177241	483.65870	466489	777.71699
2048	55.30553	39601	243.18031	185761	489.72481	477481	784.25513
2187	56.40414	44521	248.53216	187489	495.79554	491401	790.80764
2197	58.96909	49729	253.93934	192721	501.88004	493039	795.17709
2209	62.81924	50653	257.55025	196249	507.97361	502681	801.74094
2401	64.76515	51529	262.97520	201601	514.08064	516961	808.31880
2809	68.73544	52441	268.40893	205379	518.15817	524288	809.01195
3125	70.34488	54289	273.85996	208849	524.28286	528529	815.60088
3481	74.42241	57121	279.33643	212521	530.41625	531441	816.69949
3721	78.53329	58081	284.82123	214369	536.55398	537289	823.29664
4096	79.22643	59049	285.91984	218089	542.70031	546121	829.90193
4489	83.43113	63001	291.44529	226981	546.81119	552049	836.51263
4913	86.26434	65536	292.13844	229441	552.98289	564001	843.13403
5041	90.52702	66049	297.68751	237169	559.17115	571787	847.55288
5329	94.81748	68921	301.40109	241081	565.36759	573049	854.18224
6241	99.18693	69169	306.97324	249001	571.58020	579121	860.81687
6561	100.28554	72361	312.56795	253009	577.80079	591361	867.46196
6859	103.22998	73441	318.17007	259081	584.03324	597529	874.11224
6889	107.64882	76729	323.79409	262144	584.72639	619369	880.78047
7921	112.13746	78125	325.40353	271441	590.98214	635209	887.46133
8192	112.83060	78961	331.04188	273529	597.24172	654481	894.15712
9409	117.40531	79507	334.80308	279841	600.37721	657721	900.85539

TABLE I (*continued*)

x	ψ(x)−θ(x)	x	ψ(x)−θ(x)	x	ψ(x)−θ(x)
674041	907.56592	1225043	1235.36754	2093809	1575.77332
677329	914.27887	1229881	1242.37875	2097152	1576.46646
683929	920.99668	1247689	1249.39715	2105401	1583.74647
687241	927.71690	1261129	1256.42091	2111209	1591.02786
703921	934.44911	1274641	1263.45000	2128681	1598.31336
704969	938.93774	1295029	1268.14135	2163841	1605.60706
707281	942.30504	1324801	1275.18973	2193361	1612.90753
727609	949.05380	1329409	1282.23986	2199289	1620.20936
734449	955.80724	1352569	1289.29861	2211169	1627.51387
737881	962.56301	1371241	1296.36423	2217121	1634.81973
744769	969.32342	1394761	1303.43834	2229049	1642.12828
769129	976.09993	1408969	1310.51753	2247001	1649.44083
776161	982.88098	1419857	1313.35074	2248091	1654.31603
779689	989.66431	1423249	1320.43497	2283121	1661.63655
786769	996.45215	1442401	1327.52588	2319529	1668.96499
822649	1003.26230	1442897	1332.25327	2343961	1676.29867
823543	1005.20821	1471369	1339.35412	2380849	1683.64015
829921	1012.02275	1481089	1346.45826	2399401	1690.98552
844561	1018.84604	1495729	1353.56732	2411809	1698.33346
863041	1025.68015	1510441	1360.68128	2430481	1705.68526
877969	1032.52283	1515361	1367.79686	2455489	1713.04218
885481	1039.36977	1530169	1374.91731	2468041	1720.40165
896809	1046.22307	1560001	1382.04740	2476099	1723.34608
908209	1053.08269	1585081	1389.18548	2493241	1730.71063
912673	1057.65740	1594323	1390.28409	2505889	1738.07771
923521	1061.09138	1630729	1397.43636	2550409	1745.45359
935089	1067.96558	1635841	1404.59019	2563201	1752.83197
942841	1074.84391	1646089	1411.74715	2571353	1757.75196
954529	1081.72840	1661521	1418.90877	2582449	1765.13408
966289	1088.61900	1666681	1426.07194	2588881	1772.51745
982081	1095.51772	1682209	1433.23975	2601769	1779.90330
994009	1102.42247	1692601	1440.41064	2621161	1787.29286
1018081	1109.33919	1697809	1447.58307	2627641	1794.68366
1026169	1116.25986	1708249	1454.75856	2647129	1802.07815
1030301	1120.87498	1739761	1461.94318	2679769	1809.47877
1038361	1127.80155	1745041	1469.12933	2685619	1814.41325
1042441	1134.73009	1760929	1476.32000	2745649	1821.82601
1048576	1135.42324	1771561	1478.71790	2765569	1829.24239
1062961	1142.36152	1852321	1485.93388	2778889	1836.66117
1067089	1149.30175	1868689	1493.15425	2785561	1844.08115
1079521	1156.24776	1874161	1496.76517	2825761	1847.79472
1092727	1160.88249	1885129	1503.98992	2866249	1855.22898
1100401	1167.83808	1907161	1511.22048	2879809	1862.66560
1104601	1174.79558	1953125	1512.82992	2886601	1870.10339
1125721	1181.76255	1957201	1520.07343	2920681	1877.54706
1129969	1188.73140	1985281	1527.32407	2961841	1884.99772
1142761	1195.70588	2024929	1534.58459	2968729	1892.44954
1181569	1202.69705	2036329	1541.84792		
1190281	1209.69190	2042041	1549.11265		
1194649	1216.68858	2048383	1553.95684		
1203409	1223.68892	2053489	1561.22436		
1216609	1230.69471	2070721	1568.49607		

TABLE II

$$\lambda^{-}x < \psi(x) < \lambda^{+}x \text{ if } n \leqslant x \leqslant 10^{8}$$

n	λ^-	λ^+	n	λ^-	λ^+
23	0.86583	1.03883	19379	0.99547	1.00458
41	0.90602	1.03883	19423	0.99643	1.00458
59	0.92237	1.03883	24281	0.99643	1.00361
101	0.94842	1.03883	24297	0.99643	1.00297
114	0.94842	1.03591	32059	0.99703	1.00297
201	0.94842	1.02728	32321	0.99770	1.00297
227	0.96764	1.02728	43068	0.99770	1.00291
285	0.96764	1.02117	59851	0.99770	1.00237
347	0.97494	1.02117	60356	0.99770	1.00182
469	0.97494	1.01802	60976	0.99770	1.00157
569	0.97870	1.01802	69997	0.99787	1.00157
664	0.97870	1.01386	70843	0.99816	1.00157
684	0.97870	1.01364	88807	0.99851	1.00157
1429	0.98708	1.01364	96020	0.99851	1.00153
1630	0.98708	1.01196	102688	0.99851	1.00144
1670	0.98708	1.00990	155941	0.99851	1.00121
2657	0.98828	1.00990	175939	0.99870	1.00121
2868	0.98828	1.00744	230481	0.99870	1.00115
2974	0.98828	1.00662	303283	0.99891	1.00115
3299	0.99002	1.00662	312229	0.99906	1.00115
3461	0.99227	1.00662	356184	0.99906	1.00089
3511	0.99237	1.00662	359808	0.99906	1.00079
3948	0.99237	1.00649	445206	0.99906	1.00072
5387	0.99330	1.00649	463447	0.99914	1.00072
6385	0.99330	1.00640	467867	0.99938	1.00072
6404	0.99330	1.00543	618736	0.99938	1.00057
7045	0.99330	1.00517	1092893	0.99960	1.00057
7459	0.99343	1.00517	1198538	0.99960	1.00049
10359	0.99343	1.00458	1520786	0.99960	1.00037
11801	0.99486	1.00458	1790309	0.99967	1.00037

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