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On the Diophantine Equation $1 + 2^a = 3^b5^c + 2^d3^e5^f$

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Abstract. In this paper the Diophantine equation $1 + 2^a = 3^b5^c + 2^d3^e5^f$, where a, b, c, d, e and f are nonnegative integers, is solved. The related equations $1 + 3^a = 2^b5^c + 2^d3^e5^f$ and $1 + 5^a = 2^b3^c + 2^d3^e5^f$ are also solved. This work is related to and extends recent work of L. L. Foster, J. L. Brenner, and the author.

1. Introduction. In this paper we consider equations of the form

$$1 + p^a = q^br^c + p^dq^er^f, \tag{1}$$

where p, q, r are the primes 2, 3, and 5 in some order. These equations are exponential Diophantine equations, as it is the nonnegative integer exponents a, b, c, d, e, f which are to be found.

Equation (1) is a special case of the general equation $\Sigma x_i = 0$, $i = 1, 2, \ldots, m$, where the primes dividing $x_1 \cdot x_2 \cdots x_m$ are specified. There has been very little work done in general to solve such equations. It is unknown whether such equations always have a finite number of nonobvious solutions. Equation (1) has an infinite number of obvious solutions of the form $(a, b, c, d, e, f) = (t, 0, 0, t, 0, 0)$.

It follows from the work of Dubois and Rhin [6] and Schlickewei [7] that the related equation $p^a \pm q^b \pm r^c \pm s^d = 0$ has only finitely many solutions when p, q, r and s are distinct primes. Also, a result of Senge and Straus [8] implies that equations of the form $\Sigma m^{a_i} = \Sigma n^{b_j}$, where m and n are distinct positive integers, have only finitely many solutions. However, their results do not seem to apply to more general exponential equations. Also, their results do not determine the solutions.

The author, L. L. Foster, and J. L. Brenner [1], [2], [4], [5], have recently developed techniques which solve such equations in many cases. These techniques involve careful consideration of the equation modulo a series of primes and prime powers. Recently, Yen [10] has applied these techniques to solve several exponential Diophantine equations including a special case of Eq. (1).

It turns out that similar equations are not equally amenable to solution using modular arithmetic. For example, the equation

$$1 + 3^a = 2^b + 2^c3^d$$

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is easily solved with modular arithmetic techniques while the similar equation

$$1 + 3^a = 5^b + 3^c5^d$$

cannot be solved using these techniques alone.

Here, with computer assistance, these techniques of modular arithmetic are used together with some recent results of Tijdeman [9] on exponential Diophantine inequalities.

Equations of the type considered in this paper arise quite naturally in the character theory of finite groups. If G is a finite simple group and p is a prime dividing the order of G to the first power only, then the degrees $x_1, x_2, \ldots, x_m$ of the ordinary irreducible characters in the principal p-block of G satisfy an equation of the form $\sum \delta_i x_i = 0$, $\delta_i = \pm 1$, where the primes dividing $x_1 x_2 \cdots x_m$ are those in $|G|/p$. Much information concerning the group G can be obtained from the solutions to this degree equation. For example, the author in [3] has used solutions to the equation $1 + 2^a = 3^b5^c + 2^d3^e5^f$ to characterize the simple groups $L(2,7)$, $U(3,3)$, $L(3,4)$ and A_8.

In Sections 2, 3 and 4 the equations $1 + 2^a = 3^b5^c + 2^d3^e5^f$, $1 + 3^a = 2^b5^c + 2^d3^e5^f$, and $1 + 5^a = 2^b3^c + 2^b3^e5^f$, respectively, are solved.

2. The Equation $1 + 2^a = 3^b5^c + 2^d3^e5^f$. Here we consider the equation

$$1 + 2^a = 3^b5^c + 2^d3^e5^f, \tag{2.1}$$

where a, b, c, d, e, f are nonnegative integers.

The first step in solving Eq. (2.1) is to test the equation modulo a sequence of primes and prime powers in order to determine information regarding the exponents a, b, c, d, e and f. The equation is tested by computer modulo 7, 13, 19, 37 and 73 in that order. The computer used for this purpose was the CDC 6600 at the University of Minnesota Computer Center. These tests yield sets of congruences on the exponents a, b, d, and e modulo 36, and on the exponents c and f, the congruences are modulo 72. This is due to the fact that the exponents of 2, 3, and 5 modulo $7 \cdot 13 \cdot 19 \cdot 37 \cdot 73$ are 36, 36, and 72 respectively. Next, the equation is tested modulo 5, 3, 9, 27, 4, 8, and 16. These tests yield that for a solution other than the trivial solutions $(a, b, c, d, e, f) = (t, 0, 0, t, 0, 0)$, the exponents must satisfy one of the 44 sets of congruences listed in Table 2.1.

Before we consider the sets of congruences listed in Table 2.1 further, we list several useful lemmas. The first lemma is due to R. Tijdeman. A proof appears in [8] with computations due to P. L. Cijsouw and J. Korlaar.

LEMMA 2.1. *The only solutions to the inequality* $0 < |p^x - q^y| < p^{x/2}$ *in primes* p, q *with* $1 < p < q < 20$ *are* $(p, q, x, y) = (2,3,1,1)$, $(2,3,2,1)$, $(2,3,3,2)$, $(2,3,5,3)$, $(2,3,8,5)$, $(2,5,2,1)$, $(2,5,7,3)$, $(2,7,3,1)$, $(2,11,7,2)$, $(2,13,4,1)$, $(2,17,4,1)$, $(2,19,4,1)$, $(3,5,3,2)$, $(3,7,2,1)$, $(3,11,2,1)$, $(3,13,7,3)$, $(5,7,1,1)$, $(5,11,3,2)$, $(7,19,3,2)$, $(11,13,1,1)$, *and* $(17,19,1,1)$.

Our next two lemmas deal with two special cases of Eq. (2.1).

LEMMA 2.2. *The only nonnegative integral solutions to the equation* $1 + 2^a = 5^c + 2^d5^f$ *are* $(a, c, d, f) = (3,1,2,0)$, $(5,2,3,0)$, $(6,2,3,1)$, $(7,3,2,0)$, $(10,4,4,2)$, $(10,2,3,3)$, *and* $(t,0,t,0)$, *where t is an arbitrary nonnegative integer.*

TABLE 2.1

	a (mod 36)	b (mod 36)	c (mod 72)	d (mod 36)	e (mod 36)	f (mod 72)
(1)	2	0	0	2	0	0
(2)	6	0	0	6	0	0
(3)	10	0	0	10	0	0
(4)	14	0	0	14	0	0
(5)	18	0	0	18	0	0
(6)	22	0	0	22	0	0
(7)	26	0	0	26	0	0
(8)	30	0	0	30	0	0
(9)	34	0	0	34	0	0
(10)	3	0	1	2	0	0
(11)	5	0	2	3	0	0
(12)	6	0	2	3	0	1
(13)	7	0	3	2	0	0
(14)	10	0	4	4	0	2
(15)	10	0	2	3	0	3
(16)	9	0	0	9	0	0
(17)	27	0	0	27	0	0
(18)	2	1	0	1	0	0
(19)	3	1	0	1	1	0
(20)	4	2	0	3	0	0
(21)	5	3	0	1	1	0
(22)	5	2	0	3	1	0
(23)	7	4	0	4	1	0
(24)	9	4	0	4	3	0
(25)	9	3	0	1	5	0
(26)	6	1	1	1	0	2
(27)	9	5	0	1	3	1
(28)	6	0	1	2	1	1
(29)	12	0	5	2	5	0
(30)	6	2	1	2	0	1
(31)	9	2	2	5	2	0
(32)	9	4	1	2	3	0
(33)	7	2	0	3	1	1
(34)	4	0	1	2	1	0
(35)	10	0	3	2	2	2
(36)	10	2	2	5	0	2
(37)	4	1	1	1	0	0
(38)	7	1	2	1	3	0
(39)	11	4	2	3	1	0
(40)	8	2	2	5	0	0
(41)	5	1	0	1	1	1
(42)	5	1	1	1	2	0
(43)	9	5	0	1	21	37
(44)	11	22	38	3	1	0

Proof. If $c = 0$, then $f = 0$ and $a = d$. Hence, we consider $c > 0$, so that $a > d$. Thus, $5^c \equiv 1 \pmod{2^d}$. This implies that 2^{d-2} divides c, whence $2^d \leqslant 4c$.

Case 1: $c \geqslant f$. Here $2^a \equiv -1 \pmod{5^f}$, hence $2 \cdot 5^{f-1}$ divides a. Thus, $5^f \leqslant (5a)/2$. Hence, $0 < 2^a - 5^c < 2^d 5^f \leqslant 10ac$. But $5^c < 2^a$. Hence, $c < a/2$. Now Lemma 2.1 implies that $|2^a - 5^c| \geqslant 2^{a/2}$ if $(a, c) \neq (2, 1)$ or $(7, 3)$. Thus, $2^{a/2} < 5a^2$ for $(a, c) \neq (2, 1)$ or $(7, 3)$. A short calculation now yields the bounds $a \leqslant 22$, $c \leqslant 11$, $d \leqslant 5$ and $f \leqslant 2$.

Case 2: $c < f$. Here $2^a \equiv -1 \pmod{5^c}$, whence $5^c \leqslant (5a)/2$. Thus, $0 < 2^a - 2^d 5^f < (5a)/2$. Hence, $0 < 2^{a-d} - 5^f < (5a)/2 \cdot 2^d$. Now Lemma 2.1 gives $|2^{a-d} - 5^f| \geqslant 2^{(a-d)/2}$ if $(a - d, f) \neq (2,1)$ or $(7,3)$. Thus, $2^{(a-d)/2} < (5a)/2 \cdot 2^d$ and hence, $2^{a/2} < (5a)/2$, if $(a - d, f) \neq (2,1)$ or $(7,3)$. Thus, $a \leqslant 8$. When $(a - d, f) = (2,1)$, we get $f = 1$ so that c must be 0. But then $2^a = 5 \cdot 2^d$ which is impossible. When $(a - d, f) = (7,3)$, $c \leqslant 2$. Hence, $2^d \leqslant 8$, so that $d \leqslant 3$. Hence, $a \leqslant 10$. Now a direct calculation or consideration of Table 2.1 yields no solutions other than those listed.

LEMMA 2.3. *The only nonnegative integral solutions to the equation* $1 + 2^a = 3^b + 2^d 3^e$ *are* $(a, b, d, e) = (2,1,1,0), (3,1,1,1), (4,2,3,0), (5,3,1,1), (5,2,3,1), (7,4,4,1), (9,4,4,3), (9,3,1,5)$, *and* $(t, 0, t, 0)$, *where t is an arbitrary nonnegative integer.*

Proof. If $b = 0$, then $e = 0$ and $a = d$. Hence, let $b > 0$ and $a > d$. Thus, $3^b \equiv 1 \pmod{2^d}$ so that 2^{d-2} divides b and hence, $2^d \leqslant 4b$.

Case 1: $b \geqslant e$. We have $2^a \equiv -1 \pmod{3^e}$, whence 3^{e-1} divides a. Thus, $3^e \leqslant 3a$. This yields $0 < 2^a - 3^b < 2^d 3^e < 12ab$. But since $2^a > 3^b$, $b < (2a)/3$. So, $0 < 2^a - 3^b < 8a^2$. Now Lemma 2.1 gives $|2^a - 3^b| \geqslant 2^{a/2}$ unless $(a, b) = (1,1), (2,1), (3,2), (5,3)$, or $(8,5)$. Thus, $2^{a/2} < 8a^2$ for $(a, b) \neq (1,1), (2,1), (3,2), (5,3)$, or $(8,5)$. It follows that $a \leqslant 24$, $b \leqslant 16$, $d \leqslant 6$ and $e \leqslant 3$.

Case 2: $b < e$. Here $2^a \equiv -1 \pmod{3^b}$. Thus, 3^{b-1} divides a, whence $3^b \leqslant 3a$. Thus, $0 < 2^a - 2^d 3^e < 3^b \leqslant 3a$. Hence, we have $0 < 2^{a-d} - 3^e < (3a)/2^d$. Now Lemma 2.1 yields $|2^{a-d} - 3^e| \geqslant 2^{(a-d)/2}$ for $(a - d, e) \neq (1,1), (2,1), (3,2), (5,3)$, or $(8,5)$. Thus we obtain $2^{(a-d)/2} < (3a)/2^d$ and hence, $2^{a/2} < 3a$ if $(a - d, e) \neq (1,1), (2,1), (3,2), (5,3)$, or $(8,5)$. Thus, $a \leqslant 9$. When $(a - d, e) = (1,1)$ or $(2,1)$ we have $e = 1$, so that $b = 0$. But then $2^a = 3 \cdot 2^d$, a contradiction. When $(a - d, e) = (3,2)$, we have $b \leqslant 1$ so that $d \leqslant 2$ and hence, $a \leqslant 5$. Similarly, when $(a - d, e) = (5,3), (8,5)$ we obtain $a \leqslant 7$ and $a \leqslant 12$, respectively. Now consideration of Table 2.1 or a direct calculation implies that the listed solutions are the only ones.

We are now in a position to complete the solution of Eq. (2.1). We will do this by consideration of the sets of congruences listed in Table 2.1. For this purpose, we assume (a, b, c, d, e, f) is a solution to Eq. (2.1) other than the trivial solutions $(t, 0, 0, t, 0, 0)$.

LEMMA 2.4. *The only nontrivial solutions to Eq.* (2.1) *with exponents satisfying congruence sets* (1)–(25) *in Table* 2.1 *are* $(a, b, c, d, e, f) = (3,0,1,2,0,0), (5,0,2,3,0,0), (6,0,2,3,0,1), (7,0,3,2,0,0), (10,0,4,4,0,2), (10,0,2,3,0,3), (2,1,0,1,0,0), (3,1,0,1,1,0), (4,2,0,3,0,0), (5,3,0,1,1,0), (5,2,0,3,1,0), (7,4,0,4,1,0), (9,4,0,4,3,0)$ *and* $(9,3,0,1,5,0)$.

Proof. For each of the congruence sets (1)–(9), $b \equiv 0 \pmod{36}$, $e \equiv 0 \pmod{36}$, $c \equiv 0 \pmod{72}$, $f \equiv 0 \pmod{72}$, and $a \equiv d \pmod{36}$. Now, since the exponents of 2 and 5 modulo 27 are both 18, we obtain $1 + 2^a \equiv \{0 \text{ or } 1\} + \{0 \text{ or } 2^a\} \pmod{27}$. Thus, since $a \not\equiv 9 \pmod{18}$ in any of these cases, it must be true that $b = e = 0$. Similarly, consideration of Eq. (2.1) modulo 27 gives $b = e = 0$ for the congruence sets (10)–(15). Now, Lemma 2.2 gives the solutions $(a, b, c, d, e, f) = (3,0,1,2,0,0), (5,0,2,3,0,0), (6,0,2,3,0,1), (7,0,3,2,0,0), (10,0,4,4,0,2)$, and $(10,0,2,3,0,3)$.

TABLE 2.2

Congruence Set	Moduli Used		Result
(26)	3, 9, 4, 31, 25, 17, 97, 128	Solution:	(6, 1, 1, 1, 0, 2)
(27)	5, 4, 32, 31, 25, 11, 17, 97, 128, 512, 257, 1024	"	(9, 5, 0, 1, 3, 1)
(28)	3, 9, 8, 31, 25	"	(6, 0, 1, 2, 1, 1)
(29)	5, 3, 8, 64, 17, 97, 256, 193, 257, 4096, 109, 81, 163, 243, 1459, 729, 65537, 8192	"	(12, 0, 5, 2, 5, 0)
(30)	3, 27, 8, 31, 25	"	(6, 2, 1, 2, 0, 1)
(31)	5, 32, 17, 64, 27, 81, 243, 109, 163, 128, 97, 257, 1024	"	(9, 2, 2, 5, 2, 0)
(32)	5, 8, 31, 25, 109, 81, 163, 243	"	(9, 4, 1, 2, 3, 0)
(33)	5, 9, 27, 16, 31, 25	"	(7, 2, 0, 3, 1, 1)
(34)	5, 3, 9, 8, 32	"	(4, 0, 1, 2, 1, 0)
(35)	3, 27, 8, 31, 61, 125, 101, 256, 193, 65537, 2048	"	(10, 0, 3, 2, 2, 2)
(36)	3, 27, 64, 17, 31, 61, 11, 101, 125	"	(10, 2, 2, 5, 0, 2)
(37)	5, 3, 9, 4, 32	"	(4, 1, 1, 1, 0, 0)
(38)	5, 9, 4, 109, 27, 25, 11, 101, 125	"	(7, 1, 2, 1, 3, 0)
(39)	5, 9, 16, 25, 31, 11, 101, 125, 64, 193, 65537, 4096	"	(11, 4, 2, 3, 1, 0)
(40)	5, 3, 27, 64, 17, 128, 97, 257, 512	"	(8, 2, 2, 5, 0, 0)
(41)	5, 9, 4, 31, 25	"	(5, 1, 0, 1, 1, 1)
(42)	5, 9, 27, 4, 31, 25	"	(5, 1, 1, 1, 2, 0)
(43)	5, 4, 31, 25, 11	"	Contradiction
(44)	5, 9, 32, 17	"	Contradiction

For the congruence sets (16), (17), and (19)–(25), $c \equiv f \equiv 0 \pmod{72}$ and $a \not\equiv 2 \pmod 4$. Thus, since the exponents of 2 and 3 modulo 5 are both 4, consideration modulo 5 yields that $c = f = 0$ in each of these cases. Then Lemma 2.3 provides the solutions $(a, b, c, d, e, f) = (3, 1, 0, 1, 1, 0)$, $(4, 2, 0, 3, 0, 0)$, $(5, 3, 0, 1, 1, 0)$, $(5, 2, 0, 3, 1, 0)$, $(7, 4, 0, 4, 1, 0)$, $(9, 4, 0, 4, 3, 0)$, and $(9, 3, 0, 1, 5, 0)$. Finally, for congruence set (18), consideration of Eq. (2.1) modulo 8 gives $1 + \{4 \text{ or } 0\} \equiv 3 + \{2 \text{ or } 0\} \pmod 8$. Thus $a = 2$, $d = 1$, and the solution $(a, b, c, d, e, f) = (2, 1, 0, 1, 0, 0)$ is determined.

To determine the nontrivial solutions corresponding to the remaining congruence sets (26)–(44) of Table 2.1, more extensive considerations are required. These considerations consist of examination of the given congruence set modulo a carefully chosen sequence of primes and prime powers until a solution to Eq. (2.1) is determined or a contradiction is reached. Moduli sufficient for these determinations are given in Table 2.2.

Next, we will illustrate these procedures by giving the details for the cases of congruence sets (43) and (29).

For congruence set (43) we have $(a, b, c, d, e, f) = (9, 5, 0, 1, 21, 37)$ (mod 36, 36, 72, 36, 36, 72). Consideration of Eq. (2.1) modulo 5 gives $3 \cdot 5^c \equiv 3 \pmod 5$. Thus, $c = 0$. Then consideration modulo 4 gives $3 \cdot 2^d \equiv 2 \pmod 4$, whence $d = 1$. Now we may write Eq. (2.1) as

$$1 + 2^a = 3^b + 2 \cdot 3^e 5^f. \tag{2.2}$$

Next, consideration of Eq. (2.2) modulo 31 yields the six cases summarized in Table 2.3.

Consideration modulo 25 leads to a contradiction in cases (3)–(6), and consideration modulo 11 gives a contradiction in cases (1) and (2). Note here that the exponents of 2, 3, and 5 modulo 31 are 5, 30, 3 respectively; the exponents of 2, 3 modulo 25 are both 20; and the exponents of 2, 3, and 5 modulo 11 are 10, 5, and 5 respectively. This shows there is no solution to Eq. (2.1) corresponding to congruence set (43).

In the case of congruence set (29), considerations of Eq. (2.1) modulo 5, 3, and 8, respectively, yield that $f = 0$, $b = 0$ and $d = 2$. Thus, we may reduce Eq. (2.1) to the form

$$1 + 2^a = 5^c + 4 \cdot 3^e. \tag{2.3}$$

Next, since the exponent of 5 modulo 64 is 16, consideration of Eq. (2.3) modulo 64 gives $5^c \equiv 53 \pmod{64}$, whence $c \equiv 5 \pmod{16}$. Then, consideration modulo 17 and 97 gives $(a, c, e) \equiv (12, 5, 5)$ (mod 48, 96, 48). Next, consideration modulo 256 yields $5^c \equiv 53 \pmod{256}$, thus $c \equiv 5 \pmod{64}$. Here, $c \equiv 5 \pmod{192}$. Now, consideration modulo 193 gives $2^a \equiv 43 \pmod{193}$, so that $a \equiv 12 \pmod{96}$. Next, consideration modulo 257 yields that $c \equiv e \equiv 5 \pmod{256}$. Then, consideration modulo 4096 gives $c \equiv 5 \pmod{1024}$. At this juncture we have determined that a, c, and e satisfy the following congruences:

$$(a, c, e) \equiv (12, 5, 5) \; (\mathrm{mod}\; 2^5 3^2, 2^{10} 3^2, 2^8 3^2). \tag{2.4}$$

Consideration of Eq. (2.3) modulo 109 using congruences (2.4) gives $c \equiv e \equiv 5 \pmod{27}$, whence $c \equiv e \equiv 5 \pmod{54}$. Then, consideration modulo 81 yields $a \equiv 12 \pmod{54}$, and then consideration modulo 163 gives $(a, e) \equiv (12, 5) \pmod{162}$. Next, consideration modulo 243 yields $c \equiv 5 \pmod{81}$. Now, consideration modulo the prime 1459 gives $(a, c, e) \equiv (12, 5, 5)$ (modulo 486, 243, 1458). At this point, consideration of Eq. (2.3) modulo 729 gives $4 \cdot 3^e \equiv 243 \pmod{729}$. Thus, $e = 5$ and we may write Eq. (2.3) as

$$2^a = 5^c + 971. \tag{2.5}$$

TABLE 2.3

	a (mod 5)	b (mod 30)	e (mod 30)	a (mod 20)	b (mod 20)
(1)	1	11	15	1	1
(2)	4	17	21	9	17
(3)	4	5	3	9	5
(4)	0	17	9	5	17
(5)	0	23	3	5	13
(6)	1	29	27	1	9

Then, consideration of Eq. (2.5) modulo 65537 utilizing congruences (2.4) gives $5^c \equiv 3125 \pmod{65537}$, whence $c \equiv 5 \pmod{2^{16}}$. Thus, $2^a \equiv 4096 \pmod{8192}$ which implies $a = 12$. Then, it must be the case that $c = 5$, so that we have shown that the sole solution to Eq. (2.1) corresponding to congruence set (29) of Table 2.1 is $(a, b, c, d, e, f) = (12, 0, 5, 2, 5, 0)$.

We conclude this section by listing the complete set of solutions for Eq. (2.1).

THEOREM 2.5. *The nonnegative integral solutions to the exponential equation* $1 + 2^a = 3^b5^c + 2^d3^e5^f$ *are* $(a, b, c, d, e, f) = (3,0,1,2,0,0)$, $(5,0,2,3,0,0)$, $(6,0,2,3,0,1)$, $(7,0,3,2,0,0)$, $(10,0,4,4,0,2)$, $(10,0,2,3,0,3)$, $(2,1,0,1,0,0)$, $(3,1,0,1,1,0)$, $(4,2,0,3,0,0)$, $(5,3,0,1,0,0)$, $(5,2,0,3,1,0)$, $(7,4,0,4,1,0)$, $(9,4,0,4,3,0)$, $(9,3,0,1,5,0)$, $(6,1,1,1,0,2)$, $(9,5,0,1,3,1)$, $(6,0,1,2,1,1)$, $(12,0,5,2,5,0)$, $(6,2,1,2,0,1)$, $(9,2,2,5,2,0)$, $(9,4,1,2,3,0)$, $(7,2,0,3,1,1)$, $(4,0,1,2,1,0)$, $(10,0,3,2,2,2)$, $(10,2,2,5,0,2)$, $(4,1,1,1,0,0)$, $(7,1,2,1,3,0)$, $(11,4,2,3,1,0)$, $(8,2,2,5,0,0)$, $(5,1,0,1,1,1)$, $(5,1,1,1,2,0)$, *and* $(t,0,0,t,0,0)$, *where* t *is any nonnegative integer.*

3. The Equation $1 + 3^a = 2^b5^c + 2^d3^e5^f$. Here we will find all solutions to the equation

$$(3.1) \qquad 1 + 3^a = 2^b5^c + 2^d3^e5^f$$

in nonnegative integers a, b, c, d, e, and f.

As in Section 2, we begin by examining Eq. (3.1) modulo 7, 13, 19, 37, 73, 5, 3, 9, 27, 4, 8, and 16. These considerations imply that if (a, b, c, d, e, f) is a nontrivial solution to Eq. (3.1), then the exponents must satisfy one of the sets of congruences listed in Table 3.1.

The following lemma deals with the special cases $b = d = 0$ of Eq. (3.1).

LEMMA 3.1. *The only nonnegative integral solutions to the equation* $1 + 3^a = 5^c + 3^e5^f$ *are* $(a, c, e, f) = (2,1,0,1)$, $(3,2,1,0)$, *and* $(t,0,t,0)$, *where* t *is an arbitrary nonnegative integer.*

Proof. If $c = 0$, then $a = e$ and $f = 0$. Thus, we may assume $c > 0$, and hence, $a > e$. We have $5^c \equiv 1 \pmod{3^e}$. Thus, $3^e \leqslant (3c)/2$. We distinguish the cases (1) $c \geqslant f$ and (2) $c < f$. In Case (1) $3^a \equiv -1 \pmod{5^f}$, so that $5^f \leqslant (5a)/2$. Thus, $0 < 3^a - 5^c < 3^e5^f \leqslant (15ac)/4$. Also, $c \leqslant a \log 3/\log 5$ and hence, $0 < 3^a - 5^c \leqslant 2 \cdot 6a^2$. By Lemma 2.1 we see that $|3^a - 5^c| \geqslant 3^{a/2}$ if $(a, c) \neq (3,2)$. Hence, $a \leqslant 10$, $c \leqslant 6$, $e \leqslant 2$, $f \leqslant 2$. In Case (2) we have $3^a \equiv -1 \pmod{5^c}$, whence $5^c \leqslant (5a)/2$. Thus, $0 < 3^a - 3^e5^f < 5^c \leqslant 5a/2$, hence, $0 < 3^{a-e} - 5^f \leqslant (5a)/(2 \cdot 3^e)$. Thus, by Lemma 2.1, $|3^{a-e} - 5^f| > 3^{(a-e)/2}$ if $(a - e, f) \neq (3,2)$. Then, $2 \cdot 3^{a/2} \leqslant 5a$, whence $a \leqslant 4$. Thus, $c \leqslant 1$. When $(a - e, f) = (3,2)$, then $c = 1$, $e = 0$. Thus $3^a = 29$, a contradiction. Consideration of Table 3.1 or a direct calculation now yields the listed solutions.

Now in Lemma 3.2 and Table 3.2 we complete the solution of Eq. (3.1) by consideration of the congruence sets listed in Table 3.1. For this discussion we assume (a, b, c, d, e, f) is a nontrivial solution to Eq. (3.1). Finally, we list all nonnegative integral solutions to Eq. (3.1) in Theorem 3.3.

TABLE 3.1

	a (mod 36)	b (mod 36)	c (mod 72)	d (mod 36)	e (mod 36)	f (mod 72)
(1)	2	0	0	0	2	0
(2)	6	0	0	0	6	0
(3)	10	0	0	0	10	0
(4)	14	0	0	0	14	0
(5)	18	0	0	0	18	0
(6)	22	0	0	0	22	0
(7)	26	0	0	0	26	0
(8)	30	0	0	0	30	0
(9)	34	0	0	0	34	0
(10)	6	0	36	0	6	36
(11)	30	0	36	0	30	36
(12)	2	0	1	0	0	1
(13)	3	0	2	0	1	0
(14)	2	3	0	1	0	0
(15)	4	6	0	1	2	0
(16)	3	3	0	2	0	1
(17)	5	6	0	2	2	1
(18)	4	1	1	3	2	0
(19)	6	1	1	4	2	1
(20)	2	1	0	3	0	0
(21)	6	7	1	1	2	1
(22)	1	1	0	1	0	0
(23)	3	1	1	1	2	0
(24)	4	1	0	4	0	1
(25)	4	4	1	1	0	0
(26)	6	1	3	5	1	1
(27)	3	4	0	2	1	0
(28)	4	1	2	5	0	0
(29)	3	2	1	3	0	0
(30)	3	2	0	3	1	0
(31)	2	2	0	1	1	0
(32)	8	8	2	1	4	0
(33)	4	5	0	1	0	2
(34)	5	2	0	4	1	1
(35)	5	2	2	4	2	0
(36)	6	1	37	4	2	37
(37)	6	7	37	1	2	37
(38)	30	7	13	1	2	13
(39)	30	7	49	1	2	49
(40)	6	1	39	5	1	37
(41)	2	5	71	1	2	71
(42)	2	5	71	1	20	35

LEMMA 3.2. *The only nontrivial solutions to Eq.* (3.1) *with exponents satisfying congruence sets* (1)–(13) *in Table* 3.1 *are* $(a, b, c, d, e, f) = (2, 0, 1, 0, 0, 1)$ *and* $(3, 0, 2, 0, 1, 0)$.

Proof. In each of the congruence sets (1)–(13) consideration of Eq. (3.1) modulo 8 yields that $b = 0$ and $d = 0$. Then Lemma 3.1 gives the listed solutions.

Table 3.2 lists the moduli used to complete consideration of the remaining congruence sets (14)–(42) of Table 3.1.

TABLE 3.2

Congruence Set	Moduli Used		Result
(14)	27	Solution:	(2,3,0,1,0,0)
(15)	5,27,4,81,109,163	"	(4,6,0,1,2,0)
(16)	5,3,8,16,25,11	"	(3,3,0,2,0,1)
(17)	5,27,8,31,25,64,17,97, 257,128	"	(5,6,0,2,2,1)
(18)	5,4,16,27,25,11	"	(4,1,1,3,2,0)
(19)	27,4,31,25,17,32	"	(6,1,1,4,2,1)
(20)	27	"	(2,1,0,3,0,0)
(21)	27,4,31,25,61,11,128,97, 257,256	"	(6,7,1,1,2,1)
(22)	9	"	(1,1,0,1,0,0)
(23)	5,27,8,25,11	"	(3,1,1,1,2,0)
(24)	5,3,4,31,25,17,64	"	(4,1,0,4,0,1)
(25)	5,3,4,31,25,17,64	"	(4,4,1,1,0,0)
(26)	9,4,31,25,17,64,125,101, 251,625	"	(6,1,3,5,1,1)
(27)	5,9,8,32,17	"	(3,4,0,2,1,0)
(28)	5,3,4,32,17,64,25,11,125, 101	"	(4,1,2,5,0,0)
(29)	5,3,8,16,25,11	"	(3,2,1,3,0,0)
(30)	5,9,8,16	"	(3,2,0,3,1,0)
(31)	27	"	(2,2,0,1,1,0)
(32)	5,4,25,31,61,11,101,125 151,32,17,97,257,128, 512,243	"	(8,8,2,1,4,0)
(33)	5,3,4,32,17,64,25,11, 125,101	"	(4,5,0,1,0,2)
(34)	5,9,8,31,25,17,64	"	(5,2,0,4,1,1)
(35)	5,27,8,17,64,31,11, 101,125	"	(5,2,2,4,2,0)
(36)	27,4,31,25	"	Contradiction
(37)	27,4,31,25,61,11	"	Contradiction
(38)	16,25,31,11,61	"	Contradiction
(39)	16,25,31,11,61	"	Contradiction
(40)	9,4,31,25	"	Contradiction
(41)	4,25,31	"	Contradiction
(42)	4,25,31	"	Contradiction

THEOREM 3.3. *The nonnegative integral solutions to the exponential equation* $1 + 3^a = 2^b5^c + 2^d3^e5^f$ *are* $(a, b, c, d, e, f) = (2,0,1,0,0,1), (3,0,2,0,1,0), (2,3,0,1,0,0), (4,6,0,1,2,0), (3,3,0,2,0,1), (5,6,0,2,2,1), (4,1,1,3,2,0), (6,1,1,4,2,1), (2,1,0,3,0,0), (6,7,1,1,2,1), (1,1,0,1,0,0), (3,1,1,1,2,0), (4,1,0,4,0,1), (4,4,1,1,0,0), (6,1,3,5,1,1), (3,4,0,2,1,0), (4,1,2,5,0,0), (3,2,1,3,0,0), (3,2,0,3,1,0), (2,2,0,1,1,0), (8,8,2,1,4,0), (4,5,0,1,0,2), (5,2,0,4,1,1), (5,2,2,4,2,0)$, *and* $(t,0,0,0,t,0)$, *where* t *is an arbitrary nonnegative integer.*

4. The Equation $1 + 5^a = 2^b3^c + 2^d3^e5^f$. Here, we find all solutions to the equation

$$1 + 5^a = 2^b3^c + 2^d3^e5^f, \tag{4.1}$$

in nonnegative integers a, b, c, d, e, and f.

Preliminary examination of Eq. (4.1) modulo 7, 13, 19, 37, 73, 5, 3, 9, 27, 4, 8, and 16 give the possible congruence sets in Table 4.1 for a nontrivial solution (a, b, c, d, e, f). As in previous sections (2) and (3) we first deal with the special case $b = d = 0$ (Lemma 4.1); then we complete the solution of Eq. (4.1) by considering the congruence sets of Table 4.1 (Lemma 4.2 and Table 4.2); finally we list the complete solution set for Eq. (4.1) in Theorem 4.3.

LEMMA 4.1. *The only nonnegative integral solutions to the equation* $1 + 5^a = 3^c + 3^e 5^f$ *are* $(a, c, e, f) = (1,1,1,0)$, $(3,4,2,1)$ *and* $(t,0,0,t)$, *where* t *is an arbitrary nonnegative integer.*

Proof. If $c = 0$, then $e = 0$ and $a = f$. Let $c > 0$, then $a > f$. Hence, $3^c \equiv 1 \pmod{5^f}$, so that $5^f \leqslant (5c)/4$.

Case (1): $c \geqslant e$. Here $5^a \equiv -1 \pmod{3^e}$, whence $3^e \leqslant 3a$. Hence, $|3^c - 5^a| < 3^e 5^f \leqslant (15ac)/4$. Now, since $5^a > 3^c$, $c < (a \log 5)/\log 3$. Thus, $|3^c - 5^c| < 6a^2$. Now Lemma 2.1 yields that $|3^c - 5^a| \geqslant 3^{c/2}$ unless $(c, a) = (3,2)$. So $3^{c/2} < 6a^2$ unless $(c, a) = (3,2)$. Thus, $5^a < 6a^2(6a^2 + 1)$ unless $(c, a) = (3,2)$. Thus, $a \leqslant 7$, $c \leqslant 10$, $e \leqslant 2$, and $f \leqslant 1$.

TABLE 4.1

	a (mod 72)	b (mod 36)	c (mod 36)	d (mod 36)	e (mod 36)	f (mod 72)
(1)	9	0	0	0	0	9
(2)	27	0	0	0	0	27
(3)	45	0	0	0	0	45
(4)	63	0	0	0	0	63
(5)	1	0	1	0	1	0
(6)	3	0	4	0	2	1
(7)	2	3	0	1	2	0
(8)	2	3	1	1	0	0
(9)	4	6	2	1	0	2
(10)	3	3	2	1	3	0
(11)	2	1	0	3	1	0
(12)	3	1	1	3	1	1
(13)	2	1	2	3	0	0
(14)	3	1	3	3	2	0
(15)	2	4	0	1	0	1
(16)	5	10	1	1	3	0
(17)	5	1	3	10	1	0
(18)	1	1	0	2	0	0
(19)	2	1	1	2	0	1
(20)	3	1	2	2	3	0
(21)	1	2	0	1	0	0
(22)	3	5	1	1	1	1
(23)	3	2	2	1	2	1
(24)	3	2	3	1	2	0
(25)	2	4	0	1	18	37
(26)	69	1	17	34	2	66
(27)	9	1	21	26	0	68
(28)	9	1	21	26	18	32

Case (2): $c < e$. Now $5^a \equiv -1 \pmod{3^c}$, hence $3^c \leqslant 3a$. Thus, $|5^a - 3^e5^f| < 3a$. We have $|3^e - 5^{a-f}| < (3a)/5^f$. Then Lemma 2.1 implies that $3^{e/2} < (3a)/5^f$ unless $(e, a - f) = (3,2)$. Thus, we have $5^a < 3^c + 3^e5^f < 3^e + 3^e5^f < 18a^2$. Thus, $a \leqslant 3$ when $(e, a - f) \neq (3,2)$. If $(e, a - f) = (3,2)$, then it must be the case that $c \leqslant 2$, $f = 0$, and $a \leqslant 2$. Now, consideration of Table 4.1 or a direct calculation gives the listed solutions.

LEMMA 4.2. *The only nontrivial solutions to Eq.* (4.1) *with exponents satisfying congruence sets* (1)–(6) *in Table* 4.1 *are* $(a, b, c, d, e, f) = (1,0,1,0,1,0)$ *and* $(3,0,4,0,2,1)$.

Proof. In each of the congruence sets (1)–(6), consideration of Eq. (4.1) modulo 4 yields that $b = 0$ and $d = 0$. Then Lemma 4.1 gives the listed solutions.

THEOREM 4.3. *The nonnegative integral solutions to the exponential equation* $1 + 5^a = 2^b3^c + 2^d3^e5^f$ *are* $(a, b, c, d, e, f) = (1,0,1,0,1,0)$, $(3,0,4,0,2,1)$, $(2,3,0,1,2,0)$, $(2,3,1,1,0,0)$, $(4,6,2,1,0,2)$, $(3,3,2,1,3,0)$, $(2,1,0,3,1,0)$, $(3,1,1,3,1,1)$, $(2,1,2,3,0,0)$, $(3,1,3,3,2,0)$, $(2,4,0,1,0,1)$, $(5,10,1,1,3,0)$, $(5,1,3,10,1,0)$, $(1,1,0,2,0,0)$, $(2,1,1,2,0,1)$, $(3,1,2,2,3,0)$, $(1,2,0,1,0,0)$, $(3,5,1,1,1,1)$, $(3,2,2,1,2,1)$, $(3,2,3,1,2,0)$, *and* $(t,0,0,0,0,t)$, *where t is an arbitrary nonnegative integer*.

TABLE 4.2

Congruence Set	Moduli Used		Result
(7)	4, 16, 3, 27, 5	Solution	(2, 3, 0, 1, 2, 0)
(8)	4, 16, 3, 9, 5	"	(2, 3, 1, 1, 0, 0)
(9)	4, 3, 27, 64, 17, 97, 128, 125	"	(4, 6, 2, 1, 0, 2)
(10)	4, 16, 27, 5, 25, 11, 125, 101, 251, 625	"	(3, 3, 2, 1, 3, 0)
(11)	5, 4, 16, 3, 9	"	(2, 1, 0, 3, 1, 0)
(12)	4, 16, 27, 25	"	(3, 1, 1, 3, 1, 1)
(13)	4, 16, 3, 27, 5	"	(2, 1, 2, 3, 0, 0)
(14)	4, 16, 27, 5, 25, 11, 125, 101, 251, 625	"	(3, 1, 3, 3, 2, 0)
(15)	4, 3, 32, 25	"	(2, 4, 0, 1, 0, 1)
(16)	4, 9, 5, 109, 81, 17, 128, 97, 257, 25, 11, 7681, 65537, 2048	"	(5, 10, 1, 1, 3, 0)
(17)	4, 9, 5, 109, 81, 17, 128, 97, 257, 25, 11, 7681, 65537, 2048	"	(5, 1, 3, 10, 1, 0)
(18)	4, 8, 9, 5	"	(1, 1, 0, 2, 0, 0)
(19)	4, 8, 3, 9, 25	"	(2, 1, 1, 2, 0, 1)
(20)	4, 8, 27, 5, 109, 81	"	(3, 1, 2, 2, 3, 0)
(21)	4, 8, 9, 5	"	(1, 2, 0, 1, 0, 0)
(22)	4, 27, 31, 25, 17, 64	"	(3, 5, 1, 1, 1, 1)
(23)	4, 8, 27, 25	"	(3, 2, 2, 1, 2, 1)
(24)	4, 8, 27, 5, 109, 81	"	(3, 2, 3, 1, 2, 0)
(25)	4, 32, 3, 27	"	Contradiction
(26)	4, 32	"	Contradiction
(27)	4, 25, 31	"	Contradiction
(28)	4, 25, 31	"	Contradiction

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