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Mixed Finite Element Methods for Quasilinear Second-Order Elliptic Problems*

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Abstract. A mixed finite element method is developed to approximate the solution of a quasilinear second-order elliptic partial differential equation. The existence and uniqueness of the approximation are demonstrated and optimal rate error estimates are derived.

1. Introduction. Let $\Omega \subset\subset \mathbf{R}^2$ be a domain with C^2 boundary $\partial\Omega$. We shall assume that for some ε, $0 < \varepsilon < 1$, and for each pair of functions (f, g) in $H^\varepsilon(\Omega) \times H^{3/2+\varepsilon}(\partial\Omega)$ there exists a unique solution $p \in H^{2+\varepsilon}(\Omega)$ of the quasilinear Dirichlet problem

$$(1.1)\qquad \begin{aligned} &\text{(a)}\quad Lp = -\nabla \cdot (a(p)\nabla p + \mathbf{b}(p)) + c(p) = f \quad \text{in } \Omega,\\ &\text{(b)}\quad p = -g \quad \text{on } \partial\Omega, \end{aligned}$$

where ∇w denotes the gradient of a scalar function w, and $\nabla \cdot \mathbf{v}$ denotes the divergence of a vector function $\mathbf{v}$. Note that then p belongs to $W^{1,\infty}(\Omega)$, which will be needed throughout the paper.

We shall also assume that the coefficients a: $\overline{\Omega} \times \mathbf{R} \to \mathbf{R}$, $\mathbf{b}$: $\overline{\Omega} \times \mathbf{R} \to \mathbf{R}^2$, and c: $\overline{\Omega} \times \mathbf{R} \to \mathbf{R}$ are twice continuously differentiable with bounded derivatives through second order; moreover, assume that $a(x, q) \geqslant a_1 > 0$. The variable x will normally be omitted in this notation below.

For $1 \leqslant s < \infty$ and k any nonnegative integer let

$$W^{k,s}(\Omega) = \{ f \in L^s(\Omega) | D^\alpha f \in L^s(\Omega) \text{ if } |\alpha| \leqslant k \}$$

denote the Sobolev space endowed with the norm

$$\|f\|_{k,s;\Omega} = \left(\sum_{|\alpha| \leqslant k} \|D^\alpha f\|^s_{L^s(\Omega)} \right)^{1/s}$$

(the subscript Ω will always be omitted unless necessary to avoid ambiguity). Let $H^k(\Omega) = W^{k,2}(\Omega)$ with norm $\|\cdot\|_k = \|\cdot\|_{k,2}$ (the notation $\|\cdot\|_0$ will mean $\|\cdot\|_{L^2(\Omega)}$ or $\|\cdot\|_{L^2(\Omega)^2}$). For $0 \leqslant r < \infty$ let $W^{r,s}(\Omega)$, $W^{r,s}(\partial\Omega)$, $H^r(\Omega)$, and $H^r(\partial\Omega)$ denote the fractional order Sobolev spaces with norms $\|\cdot\|_{r,s;\Omega}$, $\|\cdot\|_{r,s;\partial\Omega}$, $\|\cdot\|_{r;\Omega}$, and $\|\cdot\|_{r;\partial\Omega}$.

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We shall denote by $(\ ,\ )$ the inner product in either $L^2(\Omega)$ or $L^2(\Omega)^2$, that is,

$$(\eta, \theta) = \int_\Omega \eta\theta\, dx \quad \text{or} \quad \int_\Omega \boldsymbol{\eta} \cdot \boldsymbol{\theta}\, dx.$$

Let $\langle\ ,\ \rangle$ be the L^2-inner product on the boundary of Ω:

$$\langle \lambda, \pi \rangle = \int_{\partial\Omega} \lambda\pi\, ds.$$

We shall use the same notations to indicate the dualities between $W^{r,s}(\Omega)$ and $W^{r,s}(\Omega)'$ and $H^s(\partial\Omega)$ and $H^{-s}(\partial\Omega)$, respectively.

Let

$$\mathbf{V} = \mathbf{H}(\mathrm{div}; \Omega) = \left\{ \mathbf{v} \in L^2(\Omega)^2 \middle| \mathrm{div}\, \mathbf{v} \in L^2(\Omega) \right\},$$

normed by

$$\|\mathbf{v}\|_{\mathbf{V}} = \|\mathbf{v}\|_0 + \|\mathrm{div}\, \mathbf{v}\|_0,$$

and let

$$\mathbf{H}^s(\mathrm{div}; \Omega) = \left\{ \mathbf{v} \in L^2(\Omega)^2 \middle| \mathrm{div}\, \mathbf{v} \in H^s(\Omega) \right\},$$

normed by

$$\|\mathbf{v}\|_{\mathbf{H}^s(\mathrm{div};\Omega)} = \|\mathbf{v}\|_0 + \|\mathrm{div}\, \mathbf{v}\|_s.$$

Let

$$W = L^2(\Omega).$$

If

$$\mathbf{u} = -(a(p)\nabla p + \mathbf{b}(p)), \quad \alpha = 1/a, \quad \boldsymbol{\beta} = \alpha \mathbf{b}, \tag{1.2}$$

then $(\mathbf{u}, p) \in \mathbf{V} \times W$ is a solution of the following weak formulation of (1.1):

$$\begin{aligned} &\text{(a)} \quad (\alpha(p)\mathbf{u}, \mathbf{v}) - (\mathrm{div}\, \mathbf{v}, p) + (\boldsymbol{\beta}(p), \mathbf{v}) = \langle g, \mathbf{v} \cdot \boldsymbol{\nu} \rangle, \qquad \mathbf{v} \in \mathbf{V}, \\ &\text{(b)} \quad (\mathrm{div}\, \mathbf{u}, w) + (c(p), w) = (f, w), \qquad w \in W, \end{aligned} \tag{1.3}$$

where $\boldsymbol{\nu}$ denotes the unit outward normal vector to $\partial\Omega$. Since $\mathbf{v} \cdot \boldsymbol{\nu} \in H^{-1/2}(\partial\Omega)$ (see [12], [16]), the duality $\langle g, \mathbf{v} \cdot \boldsymbol{\nu} \rangle$ is well-defined.

Mixed finite element methods for (1.1) are discrete versions of (1.3) and have been treated for linear operators L by several authors [2], [5]–[7], [9], [12]–[16].

Let $\mathscr{T}_h$ be a quasi-regular polygonalization of Ω (by triangles, rectangles, or possibly parallelograms), with boundary polygons allowed to have one curved side, of characteristic parameter $h \in (0, 1)$, and let

$$\mathbf{V}_h \times W_h \subset \mathbf{V} \times W$$

be the associated Raviart-Thomas-Nedelec space of index $k \geqslant 0$, [11], [12]. To be more explicit, for $E \subset \mathbf{R}^2$ let $P_k(E)$ denote the restrictions of polynomials of total degree k to the set E and let $Q_k(E)$ denote the restriction of $P_k(\mathbf{R}) \otimes P_k(\mathbf{R})$ to E. Then, let $R_k(E) = P_k(E)$ if E is a triangle (interior *or* boundary) and $R_k(E) = Q_k(E)$ if E is a rectangle (interior *or* boundary), and let $\mathbf{R}_k(E) = R_k(E)^2$. For *any* $E \in \mathscr{T}_h$ let

$$\mathbf{V}(E) = \mathbf{R}_k(E) \oplus \mathrm{Span}\{\mathbf{x} R_k(E)\}, \qquad W(E) = R_k(E).$$

Set

$$\mathbf{V}_h = \mathbf{V}(k, \mathcal{T}_h) = \{ \mathbf{v} \in \mathbf{V} | \mathbf{v}|_E \in \mathbf{V}(E), E \in \mathcal{T}_h \}$$
$$= \left\{ \mathbf{v} \in \prod_{E \in \mathcal{T}_h} \mathbf{V}(E) | \mathbf{v}|_{E_i} \cdot \boldsymbol{\nu}_i + \mathbf{v}|_{E_j} \cdot \boldsymbol{\nu}_j = 0 \text{ on } \bar{E}_i \cap \bar{E}_j \right\},$$

where $\boldsymbol{\nu}_l$, $l = i, j$, is the outer normal to ∂E_l on $\bar{E}_i \cap \bar{E}_j$; also, let

$$W_h = W(k, \mathcal{T}_h) = \{ w \in W | w|_E \in W(E), E \in \mathcal{T}_h \}.$$

Let π_h: $\mathbf{V} \to \mathbf{V}_h$ be the Raviart-Thomas projection, [6], [12], which satisfies (see [13] for $q \neq 2$ below)

(1.4) $$(\operatorname{div}[\pi_h \mathbf{v} - \mathbf{v}], w) = 0, \qquad \mathbf{v} \in \mathbf{V}, w \in W_h,$$

(1.5) $$\|\pi_h \mathbf{v} - \mathbf{v}\|_{0,q} \leqslant Q \|\mathbf{v}\|_{s,q} h^s, \qquad 1/q < s \leqslant k+1, \text{ if } \mathbf{v} \in \mathbf{V} \cap W^{s,q}(\Omega)^2,$$

(1.6) $$\|\operatorname{div}(\pi_h \mathbf{v} - \mathbf{v})\|_0 \leqslant Q \|\operatorname{div} \mathbf{v}\|_s h^s, \qquad 0 \leqslant s \leqslant k+1, \text{ if } \mathbf{v} \in \mathbf{V} \cap \mathbf{H}^s(\operatorname{div}; \Omega).$$

Let P_h: $W \to W_h$ be the orthogonal L^2-projection into W_h defined by

(1.7) $$(P_h w - w, \chi) = 0, \qquad w \in W, \chi \in W_h,$$

which satisfies

(1.8) $$\|P_h w - w\|_{0,q} \leqslant Q \|w\|_{s,q} h^s, \qquad 0 \leqslant s \leqslant k+1, \text{ if } w \in W \cap W^{s,q}(\Omega),$$

(1.9) $$\|P_h w - w\|_{-r} \leqslant Q \|w\|_s h^{r+s}, \qquad 0 \leqslant r, s \leqslant k+1, \text{ if } w \in H^s(\Omega),$$

(1.10) $$(\operatorname{div} \mathbf{v}, w - P_h w) = 0, \qquad w \in W, \mathbf{v} \in \mathbf{V}_h.$$

We can now formulate the mixed finite element method to approximate the solution of (1.1):

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ such that

(1.11)
$$\text{(a)} \quad (\alpha(p_h)\mathbf{u}_h, \mathbf{v}) - (\operatorname{div} \mathbf{v}, p_h) + (\boldsymbol{\beta}(p_h), \mathbf{v}) = \langle g, \mathbf{v} \cdot \boldsymbol{\nu} \rangle, \qquad \mathbf{v} \in \mathbf{V}_h,$$
$$\text{(b)} \quad (\operatorname{div} \mathbf{u}_h, w) + (c(p_h), w) = (f, w), \qquad w \in W_h.$$

We shall demonstrate in Section 2 the existence of a solution $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ of the nonlinear algebraic system (1.11) through an adaptation of the method used by Douglas in [4]. In Section 4, we shall establish the uniqueness of that solution inside a certain ball. Furthermore, we shall show that $(\mathbf{u}_h, p_h)$ converges to $(\mathbf{u}, p)$ in $L^2(\Omega)^2 \times L^2(\Omega)$ at an optimal rate as $h \downarrow 0$ (Section 3) and also in $(H^s(\Omega)^2)' \times H^s(\Omega)'$, $0 < s \leqslant k+1$, provided that the boundary of Ω, the coefficients a, $\mathbf{b}$, and c, and the solution p of (1.1) are smooth enough (Section 6). In Section 5, we establish the convergence of p_h to p in $L^q(\Omega)$, $2 \leqslant q \leqslant \infty$, at an optimal rate as $h \to 0$.

2. Solvability of the Discrete Problem. For $\rho \in W_h$, we shall write

(2.1) $$\alpha(\rho) - \alpha(p) = -\tilde{\alpha}_p(\rho)(p - \rho) = -\alpha_p(p)(p - \rho) + \tilde{\alpha}_{pp}(\rho)(p - \rho)^2,$$

where

$$\tilde{\alpha}_p(\rho) = \int_0^1 \alpha_p(\rho + t[p - \rho])\, dt,$$
$$\tilde{\alpha}_{pp}(\rho) = \int_0^1 (1 - t)\alpha_{pp}(p + t[\rho - p])\, dt$$

are bounded functions in $\bar{\Omega}$. Similarly, we can write

(2.2) $$\boldsymbol{\beta}(\rho) - \boldsymbol{\beta}(p) = -\boldsymbol{\beta}_p(p)(p - \rho) + \tilde{\boldsymbol{\beta}}_{pp}(\rho)(p - \rho)^2 = -\tilde{\boldsymbol{\beta}}_p(\rho)(p - \rho),$$

(2.3) $c(\rho) - c(p) = -c_p(p)(p-\rho) + \tilde{c}_{pp}(\rho)(p-\rho)^2 = -\tilde{c}_p(\rho)(p-\rho),$

where $\tilde{\beta}_p(\rho)$, $\tilde{c}_p(\rho)$, $\tilde{\boldsymbol{\beta}}_{pp}(\rho)$, and $\tilde{c}_{pp}(\rho)$ are bounded functions in $\overline{\Omega}$.

If we now subtract (1.11) from (1.3), we obtain the error equations

$$\text{(2.4)}\quad \begin{aligned} &\text{(a)}\quad (\alpha(p)[\mathbf{u}-\mathbf{u}_h],\mathbf{v}) - (\operatorname{div}\mathbf{v}, p-p_h) + (\boldsymbol{\beta}_p(p)[p-p_h],\mathbf{v}) \\ &\qquad = \big([\alpha(p_h)-\alpha(p)]\mathbf{u}_h + \boldsymbol{\beta}(p_h) - \boldsymbol{\beta}(p) + \boldsymbol{\beta}_p(p)[p-p_h],\mathbf{v}\big), \\ &\qquad\qquad \mathbf{v}\in\mathbf{V}_h, \\ &\text{(b)}\quad (\operatorname{div}[\mathbf{u}-\mathbf{u}_h], w) + (c_p(p)[p-p_h], w) \\ &\qquad = (c(p_h) - c(p) + c_p(p)[p-p_h], w), \qquad w\in W_h. \end{aligned}$$

Substituting (2.1), (2.2), and (2.3) into (2.4), we see that (with $\rho = p_h$)

$$\text{(2.5)}\quad \begin{aligned} &\text{(a)}\quad (\alpha(p)[\mathbf{u}-\mathbf{u}_h],\mathbf{v}) - (\operatorname{div}\mathbf{v}, p-p_h) + ([\alpha_p(p)\mathbf{u} + \boldsymbol{\beta}_p(p)](p-p_h),\mathbf{v}) \\ &\qquad = \big([\tilde{\alpha}_{pp}(p_h)\mathbf{u} + \tilde{\boldsymbol{\beta}}_{pp}(p_h)](p-p_h)^2 + \tilde{\alpha}_p(p_h)(p-p_h)(\mathbf{u}-\mathbf{u}_h),\mathbf{v}\big), \\ &\qquad\qquad \mathbf{v}\in\mathbf{V}_h, \\ &\text{(b)}\quad (\operatorname{div}[\mathbf{u}-\mathbf{u}_h], w) + (c_p(p)[p-p_h], w) = (\tilde{c}_{pp}(p_h)[p-p_h]^2, w), \\ &\qquad\qquad w\in W_h. \end{aligned}$$

Set $\boldsymbol{\Gamma} = \alpha_p(p)\mathbf{u} + \boldsymbol{\beta}_p(p) \in C_B^0(\Omega)$ and $\gamma = c_p(p) \in C_B^1(\Omega)$. Let us now replace $\mathbf{u}$ by $\pi_h\mathbf{u}$ and p by $P_h p$ on the left-hand side of (2.5) to obtain (using (1.4) and (1.10)) the relations

$$\text{(2.6)}\quad \begin{aligned} &\text{(a)}\quad (\alpha(p)[\pi_h\mathbf{u}-\mathbf{u}_h],\mathbf{v}) - (\operatorname{div}\mathbf{v}, P_h p - p_h) + (\boldsymbol{\Gamma}[P_h p - p_h],\mathbf{v}) \\ &\qquad = \big(\alpha(p)[\pi_h\mathbf{u}-\mathbf{u}] + \boldsymbol{\Gamma}[P_h p - p] + [\tilde{\alpha}_{pp}(p_h)\mathbf{u} + \tilde{\boldsymbol{\beta}}_{pp}(p_h)](p-p_h)^2 \\ &\qquad\qquad + \tilde{\alpha}_p(p_h)(p-p_h)(\mathbf{u}-\mathbf{u}_h),\mathbf{v}\big), \\ &\qquad\qquad \mathbf{v}\in\mathbf{V}_h, \\ &\text{(b)}\quad (\operatorname{div}[\pi_h\mathbf{u}-\mathbf{u}_h], w) + (\gamma[P_h p - p_h], w) \\ &\qquad = (\gamma[P_h p - p] + \tilde{c}_{pp}(p_h)(p-p_h)^2, w), \qquad w\in W_h. \end{aligned}$$

Now let M: $H^2(\Omega) \to L^2(\Omega)$ be the operator

$$Mw = -\nabla\cdot(a(p)\nabla w + a(p)\boldsymbol{\Gamma} w) + \gamma w,$$

and let M^* be its formal adjoint; that is,

$$M^*\chi = -\nabla\cdot(a(p)\nabla\chi) + a(p)\boldsymbol{\Gamma}\cdot\nabla\chi + \gamma\chi. \tag{2.7}$$

We shall assume that the restrictions of M and M^* to $H^2(\Omega)\cap H_0^1(\Omega)$ have bounded inverses; that is, for any $\psi\in L^2(\Omega)$ there exists a unique $\phi\in H^2(\Omega)\cap H_0^1(\Omega)$ such that $M\phi=\psi$ (respectively, $M^*\phi=\psi$) and $\|\phi\|_2 \leqslant Q\|\psi\|_0$. This would be guaranteed by assuming $c_p \geqslant 0$ (see, for example, [8]). Let

$$\Phi\colon \mathbf{V}_h\times W_h \to \mathbf{V}_h\times W_h$$

be given by $\Phi((\boldsymbol{\mu}, \rho)) = (\mathbf{y}, z)$, $(\mathbf{y}, z)$ being the (unique) solution of the system

$$(2.8)\quad \begin{aligned} &\text{(a)}\quad (\alpha(p)[\pi_h\mathbf{u} - \mathbf{y}], \mathbf{v}) - (\operatorname{div}\mathbf{v}, P_h p - z) + (\Gamma[P_h p - z], \mathbf{v}) \\ &\qquad = \Big(\alpha(p)[\pi_h\mathbf{u} - \mathbf{u}] + \Gamma[P_h p - p] + \big[\tilde{\alpha}_{pp}(\rho)\mathbf{u} + \tilde{\boldsymbol{\beta}}_{pp}(\rho)\big](p-\rho)^2 \\ &\qquad\qquad + \tilde{\alpha}_p(\rho)(p-\rho)(\mathbf{u} - \boldsymbol{\mu}), \mathbf{v}\Big), \\ &\qquad\qquad\qquad \mathbf{v} \in \mathbf{V}_h, \\ &\text{(b)}\quad (\operatorname{div}[\pi_h\mathbf{u} - \mathbf{y}], w) + (\gamma[P_h p - z], w) \\ &\qquad = \Big(\gamma[P_h p - p] + \tilde{c}_{pp}(\rho)(p-\rho)^2, w\Big), \qquad w \in W_h, \end{aligned}$$

the existence of which follows for small h from [5], since the left-hand side of (2.8) corresponds to the mixed method for the operator M. Thus, $(\mathbf{y}, z)$ is the solution of a linear algebraic system of the form

$$(\alpha(p)\boldsymbol{\psi}, \mathbf{v}) - (\operatorname{div}\mathbf{v}, \phi) + (\Gamma\phi, \mathbf{v}) = F(\mathbf{v}), \qquad \mathbf{v} \in \mathbf{V}_h,$$
$$(\operatorname{div}\boldsymbol{\psi}, w) + (\gamma\phi, w) = G(w), \qquad w \in W_h,$$

which for h sufficiently small has a unique solution $(\boldsymbol{\psi}, \phi) \in \mathbf{V}_h \times W_h$ for any $F \in \mathbf{V}'$, $G \in W'$. (Existence follows from uniqueness. Thus it suffices to prove that if $F = 0$ and $G = 0$, then $(\boldsymbol{\psi}, \phi) = (\mathbf{0}, 0)$. This is done in [5] by an argument entirely analogous to that of our Lemma 2.1.) We are taking in (2.8)

$$F(\mathbf{v}) = -(p, \operatorname{div}\mathbf{v}) + \Big(\alpha(p)\mathbf{u} + \Gamma p - \big[\tilde{\alpha}_{pp}(\rho)\mathbf{u} + \tilde{\boldsymbol{\beta}}_{pp}(\rho)\big](p-\rho)^2 + \tilde{\alpha}_p(\rho)(p-\rho)(\mathbf{u} - \boldsymbol{\mu}), \mathbf{v}\Big),$$

$$G(w) = \Big(\operatorname{div}\mathbf{u} + \gamma p - \tilde{c}_{pp}(\rho)(p-\rho)^2, w\Big).$$

The existence of a solution $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ of (1.11) is equivalent to that of the following problem.

Problem 2.1. The map Φ has a fixed point.

The solvability of Problem 2.1 will follow from the Brouwer fixed point theorem if we can prove that Φ maps a ball of $\mathbf{V}_h \times W_h$ into itself.

In order to do that, we shall use the following technical result.

LEMMA 2.1. *Let* $2 \leqslant \theta < \infty$. *Let* $\boldsymbol{\omega} \in \mathbf{V}$, $\mathbf{q} \in L^2(\Omega)^2$, *and* $r \in L^2(\Omega)$. *If* $\tau \in W_h$ *satisfies*

$$(2.9)\quad \begin{cases} (\alpha(p)\boldsymbol{\omega}, \mathbf{v}) - (\operatorname{div}\mathbf{v}, \tau) + (\Gamma\tau, \mathbf{v}) = (\mathbf{q}, \mathbf{v}), & \mathbf{v} \in \mathbf{V}_h, \\ (\operatorname{div}\boldsymbol{\omega}, w) + (\gamma\tau, w) = (r, w), & w \in W_h, \end{cases}$$

then, there exists a constant $C = C(\theta, \alpha, \Gamma, \gamma, \Omega)$ *such that*

$$(2.10)\quad \|\tau\|_{0,\theta} \leqslant C\big[h^{2/\theta}\|\boldsymbol{\omega}\|_0 + h^{1+(2/\theta)(1-\delta_{0k})}\|\operatorname{div}\boldsymbol{\omega}\|_0 + \|\mathbf{q}\|_0 + \|r\|_0\big]$$

for h *sufficiently small. Also, if* $\boldsymbol{\omega} \in W^{0,\theta}(\operatorname{div}; \Omega) = \{\mathbf{v} \in L^\theta(\Omega)^2 | \operatorname{div}\mathbf{v} \in L^\theta(\Omega)\}$, $\mathbf{q} \in L^\theta(\Omega)^2$, *and* $r \in L^\theta(\Omega)$,

$$\|\tau\|_{0,\theta} \leqslant C\big[h\|\boldsymbol{\omega}\|_{0,\theta} + h^{2-\delta_{0k}}\|\operatorname{div}\boldsymbol{\omega}\|_{0,\theta} + h\|\mathbf{q}\|_{0,\theta} + h^{2-\delta_{0k}}\|r\|_{0,\theta} + \|\mathbf{q}\|_{-1,\theta} + \|r\|_{-2,\theta}\big].$$

Here, and throughout the paper, δ_{ij} will denote the Kronecker symbol.

Proof. Let $\theta' = \theta/(\theta - 1)$ be the conjugate exponent of θ. Since

$$\|\tau\|_{0,\theta} = \sup_{\substack{\psi \in L^{\theta'}(\Omega) \\ \psi \neq 0}} \frac{(\tau, \psi)}{\|\psi\|_{0,\theta'}},$$

we wish to bound (τ, ψ) for $\psi \in L^{\theta'}(\Omega)$. Let $\phi \in W^{2,\theta'}(\Omega)$ be the (unique) solution of $M^*\phi = \psi$ in Ω, $\phi = 0$ on $\partial\Omega$. Then [1],

$$\|\phi\|_{2,\theta'} \leqslant K\|\psi\|_{0,\theta'}.$$

Observe that the Sobolev imbedding theorem [17] implies that $W^{2-(2/\theta),\theta'}(\Omega) \subset H^1(\Omega)$ and $W^{1-(2/\theta),\theta'}(\Omega) \subset L^2(\Omega)$; so, $W^{1,\theta'}(\Omega) \subset H^{(2/\theta)}(\Omega)$ with

$$\|\lambda\|_{2/\theta} \leqslant K\|\lambda\|_{1,\theta'}. \tag{2.11}$$

Next, note that for any $E \in \mathscr{T}_h$

$$a(p)\nabla\phi \in W^{1,\theta'}(E)^2 \Rightarrow a(p)\nabla\phi|_{\partial E} \in W^{1-(1/\theta'),\theta'}(\partial E)^2,$$

and therefore the degrees of freedom defining $\pi_h a(p)\nabla\phi$ on E are well-defined, that is, $\pi_h a(p)\nabla\phi$ is well-defined.

Using (1.4), (1.10), (2.9), and integration by parts, we see that

$$\begin{aligned}(\tau, \psi) &= (\tau, M^*\phi) = (\tau, -\nabla\cdot(a(p)\nabla\phi) + a(p)\Gamma\cdot\nabla\phi + \gamma\phi)\\ &= (\tau, -\nabla\cdot(\pi_h a(p)\nabla\phi)) + (\Gamma\tau, \pi_h a(p)\nabla\phi)\\ &\qquad + (\Gamma\tau, a(p)\nabla\phi - \pi_h a(p)\nabla\phi) + (\gamma\tau, \phi)\\ &= (\mathbf{q}, \pi_h a(p)\nabla\phi) - (\alpha(p)\boldsymbol{\omega}, \pi_h a(p)\nabla\phi)\\ &\qquad + (\Gamma\tau, a(p)\nabla\phi - \pi_h a(p)\nabla\phi) + (\gamma\tau, \phi)\\ &= (\mathbf{q}, a(p)\nabla\phi) + (\mathbf{q}, \pi_h a(p)\nabla\phi - a(p)\nabla\phi)\\ &\quad + (\alpha(p)\boldsymbol{\omega} + \Gamma\tau, a(p)\nabla\phi - \pi_h a(p)\nabla\phi) + (\operatorname{div}\boldsymbol{\omega}, \phi) + (\gamma\tau, \phi)\\ &= (\mathbf{q}, a(p)\nabla\phi) + (\mathbf{q}, \pi_h a(p)\nabla\phi - a(p)\nabla\phi)\\ &\quad + (\alpha(p)\boldsymbol{\omega} + \Gamma\tau, a(p)\nabla\phi - \pi_h a(p)\nabla\phi) + (\operatorname{div}\boldsymbol{\omega} + \gamma\tau, \phi - P_h\phi)\\ &\quad + (r, \phi) + (r, P_h\phi - \phi).\end{aligned}$$

First, observe that

$$(\mathbf{q}, a(p)\nabla\phi) \leqslant K\|\mathbf{q}\|_0\|\nabla\phi\|_0 \leqslant K\|\mathbf{q}\|_0\|\phi\|_{2,\theta'}.$$

Furthermore, since $\mathbf{V}(E) \supset P_0(E)^2$, an L^p-version of the Bramble-Hilbert Lemma [3] implies that (using (2.11))

$$\|a(p)\nabla\phi - \pi_h a(p)\nabla\phi\|_0 \leqslant Kh^{2/\theta}\|\nabla\phi\|_{2/\theta} \leqslant Kh^{2/\theta}\|\nabla\phi\|_{1,\theta'} \leqslant Kh^{2/\theta}\|\phi\|_{2,\theta'},$$

(a bound which cannot be obtained from (1.5) if $2/\theta \leqslant 1/2$) and therefore

$$\begin{aligned}(\mathbf{q} - \alpha(p)\boldsymbol{\omega}, \pi_h a(p)\nabla\phi - a(p)\nabla\phi) &\leqslant K(\|\mathbf{q}\|_0 + \|\boldsymbol{\omega}\|_0)\|\pi_h a(p)\nabla\phi - a(p)\nabla\phi\|_0\\ &\leqslant K(\|\mathbf{q}\|_0 + \|\boldsymbol{\omega}\|_0)h^{2/\theta}\|\phi\|_{2,\theta'}.\end{aligned}$$

Also, we see from (1.5) that

$$\begin{aligned}(\Gamma\tau, a(p)\nabla\phi - \pi_h a(p)\nabla\phi) &\leqslant K\|\tau\|_{0,\theta}\|a(p)\nabla\phi - \pi_h a(p)\nabla\phi\|_{0,\theta'}\\ &\leqslant K\|\tau\|_{0,\theta}h\|\nabla\phi\|_{1,\theta'} \leqslant K\|\tau\|_{0,\theta}h\|\phi\|_{2,\theta'}.\end{aligned}$$

Finally, from (1.8) and the Sobolev imbedding theorem we see that

$$\begin{aligned}(\operatorname{div}\boldsymbol{\omega}, \phi - P_h\phi) &\leqslant K\|\operatorname{div}\boldsymbol{\omega}\|_0\|\phi - P_h\phi\|_0\\ &\leqslant K\|\operatorname{div}\boldsymbol{\omega}\|_0 h^{1+(2/\theta)(1-\delta_{0k})}\|\phi\|_{1+(2/\theta)(1-\delta_{0k})}\\ &\leqslant K\|\operatorname{div}\boldsymbol{\omega}\|_0 h^{1+(2/\theta)(1-\delta_{0k})}\|\phi\|_{2,\theta},\end{aligned}$$

$$(\gamma\tau, \phi - P_h\phi) \leqslant K\|\tau\|_{0,\theta}\|\phi - P_h\phi\|_{0,\theta'} \leqslant K\|\tau\|_{0,\theta}h^{2-\delta_{0k}}\|\phi\|_{2,\theta'},$$

$$(r, \phi) + (r, P_h\phi - \phi) \leqslant K\|r\|_0\|\phi\|_0 \leqslant K\|r\|_0\|\phi\|_{1-(2/\theta),\theta'} \leqslant K\|r\|_0\|\phi\|_{2,\theta'}.$$

It then follows that

$$(\tau, \psi) \leqslant K\left[h\|\tau\|_{0,\theta} + h^{2/\theta}\|\boldsymbol{\omega}\|_0 + h^{1+(2/\theta)(1-\delta_{0k})}\|\operatorname{div}\boldsymbol{\omega}\|_0 + \|\mathbf{q}\|_0 + \|r\|_0\right]\|\psi\|_{0,\theta'},$$

and thus, if h is sufficiently small, it suffices to take $C \geqslant (1 - Kh)^{-1}$ in (2.10). The second bound for $\|\tau\|_{0,\theta}$ follows easily from the equation for (τ, ψ) following (2.11), (1.5), and (1.8). Q.E.D.

Let now $\mathscr{V}_h = \mathbf{V}_h$ with the stronger norm $\|\mathbf{v}\|_{\mathscr{V}_h} = \|\mathbf{v}\|_{0,2+\varepsilon} + \|\operatorname{div}\mathbf{v}\|_0$, and let $\mathscr{W}_h = W_h$ with the stronger norm $\|w\|_{\mathscr{W}_h} = \|w\|_{0,(4+2\varepsilon)/\varepsilon}$.

We are now ready to demonstrate the existence of a solution of Problem 2.1.

THEOREM 2.1. *For $\delta > 0$ sufficiently small (dependent on h), Φ maps a ball of radius δ of $\mathscr{V}_h \times \mathscr{W}_h$ into itself.*

Proof. Let $\theta = (4 + 2\varepsilon)/\varepsilon$ so that $(1/\theta) + (1/(2+\varepsilon)) = \frac{1}{2}$. Let

$$\|\pi_h\mathbf{u} - \boldsymbol{\mu}\|_{\mathscr{V}_h} \leqslant \delta \quad \text{and} \quad \|P_h p - \rho\|_{\mathscr{W}_h} \leqslant \delta < 1.$$

Interpret $\mathbf{q}$ and r in (2.10) as

$$\begin{aligned}\mathbf{q} &= \Gamma(P_h p - p) + \alpha(p)(\pi_h\mathbf{u} - \mathbf{u})\\ &\quad + \left[\tilde{\alpha}_{pp}(\rho)\mathbf{u} + \tilde{\boldsymbol{\beta}}_{pp}(\rho)\right](p-\rho)^2 + \tilde{\alpha}_p(\rho)(p-\rho)(\mathbf{u} - \boldsymbol{\mu}),\\ r &= \gamma(P_h p - p) + \tilde{c}_{pp}(\rho)(p-\rho)^2,\end{aligned}$$

and apply Lemma 2.1 to (2.8).

Note that, since $\varepsilon < 1$ implies that $\theta > 4$ and that $2 + \varepsilon < 4/(2 - 2\varepsilon)$, $H^{\varepsilon}(\Omega)^2 \subset L^{2+\varepsilon}(\Omega)^2$, $H^{1+\varepsilon}(\Omega)^2 \subset W^{1,2+\varepsilon}(\Omega)^2$, and $H^{2+\varepsilon}(\Omega)^2 \subset W^{2,2+\varepsilon}(\Omega)^2$. Thus, $H^{1+(\varepsilon/2)}(\Omega)^2 \subset W^{1,2+\varepsilon}(\Omega)^2$ and $\|\lambda\|_{1,2+\varepsilon} \leqslant Q_\varepsilon\|\lambda\|_{1+(\varepsilon/2)}$. Then, (1.5) and (1.8) imply the inequality

$$\begin{aligned}\|P_h p - z\|_{0,\theta} &\leqslant K\left[h^{2/\theta}\|\pi_h\mathbf{u} - \mathbf{y}\|_0 + h^{1+(2/\theta)(1-\delta_{0k})}\|\operatorname{div}(\pi_h\mathbf{u} - \mathbf{y})\|_0 + \|\mathbf{q}\|_0 + \|r\|_0\right]\\
&\leqslant K\Big[h^{2/\theta}\|\pi_h\mathbf{u} - \mathbf{y}\|_0 + h^{1+(2/\theta)(1-\delta_{0k})}\|\operatorname{div}(\pi_h\mathbf{u} - \mathbf{y})\|_0\\
&\qquad + h\|p\|_1 + h\|\mathbf{u}\|_1 + \|p-\rho\|_{0,4}^2 + \|p-\rho\|_{0,\theta}\|\mathbf{u} - \boldsymbol{\mu}\|_{0,2+\varepsilon}\Big]\\
&\leqslant K\Big[h^{2/\theta}\|\pi_h\mathbf{u} - \mathbf{y}\|_0 + h^{1+(2/\theta)(1-\delta_{0k})}\|\operatorname{div}(\pi_h\mathbf{u} - \mathbf{y})\|_0\\
&\qquad + h\|p\|_2 + \|p-\rho\|_{0,\theta}^2 + \left(\|p - P_h p\|_{0,\theta} + \|P_h p - \rho\|_{0,\theta}\right)\\
&\qquad \left(\|\mathbf{u} - \pi_h\mathbf{u}\|_{0,2+\varepsilon} + \|\pi_h\mathbf{u} - \boldsymbol{\mu}\|_{0,2+\varepsilon}\right)\Big]\\
&\leqslant K\Big[h^{2/\theta}\|\pi_h\mathbf{u} - \mathbf{y}\|_0 + h^{1+(2/\theta)(1-\delta_{0k})}\|\operatorname{div}(\pi_h\mathbf{u} - \mathbf{y})\|_0\\
&\qquad + h\|p\|_2 + \delta^2 + \left(h\|p\|_{1,\theta} + \delta\right)\left(h\|\mathbf{u}\|_{1,2+\varepsilon} + \delta\right)\Big]\\
&\leqslant K\left[h^{2/\theta}\|\pi_k\mathbf{u} - \mathbf{y}\|_0 + h^{1+(2/\theta)-\delta_{0k}}\|\operatorname{div}(\pi_h\mathbf{u} - \mathbf{y})\|_0 + (h + \delta^2)\|p\|_{2+\varepsilon}^2\right].\end{aligned} \tag{2.12}$$

If we now take the last term on the left side of each equation in (2.8) over to the right side, the left side becomes exactly the mixed method equations for the operator $-\nabla \cdot (a(p)\nabla)$. It follows from [2] that then

$$(2.13)\qquad \begin{aligned} \|\pi_h \mathbf{u} - \mathbf{y}\|_{\mathbf{V}} &\leqslant K\left[\|P_h p - z\|_0 + \|\mathbf{q}\|_0 + \|r\|_0\right] \\ &\leqslant K\left[\|P_h p - z\|_{0,\theta} + (h + \delta^2)\|p\|_{2+\varepsilon}^2\right]. \end{aligned}$$

If we now substitute (2.13) into (1.12), we see that, for h sufficiently small,

$$(2.14)\qquad \|P_h p - z\|_{0,\theta} \leqslant K_1[h + \delta^2],$$

with K_1 depending on $\|p\|_{2+\varepsilon}^2$ linearly. Putting (2.14) back into (2.13), we obtain (with $K_2 = K(K_1 + \|p\|_{2+\varepsilon}^2)$)

$$(2.15)\qquad \|\operatorname{div}(\pi_h \mathbf{u} - \mathbf{y})\|_0 \leqslant K_2[h + \delta^2],$$

$$(2.16)\qquad \|\pi_h \mathbf{u} - \mathbf{y}\|_0 \leqslant K_2[h + \delta^2].$$

It follows from (2.16), using the quasi-regularity of $\mathcal{T}_h$, that

$$(2.17)\qquad \begin{aligned} \|\pi_h \mathbf{u} - \mathbf{y}\|_{0,2+\varepsilon} &\leqslant K_\varepsilon h^{(2/(2+\varepsilon))-1}\|\pi_h \mathbf{u} - \mathbf{y}\|_0 \\ &\leqslant K_\varepsilon h^{-(\varepsilon/(2+\varepsilon))} K_2(h + \delta^2) \leqslant K_3[h^{2/(2+\varepsilon)} + h^{-(\varepsilon/(2+\varepsilon))}\delta^2]. \end{aligned}$$

We now see that (2.15) and (2.17) imply that

$$(2.18)\qquad \|\pi_h \mathbf{u} - \mathbf{y}\|_{\mathscr{V}_h} \leqslant 2K_3[h^{2/(2+\varepsilon)} + h^{-(\varepsilon/(2+\varepsilon))}\delta^2].$$

Now let $h < (4K_3)^{-(4+2\varepsilon)/(2-\varepsilon)}$ and take $\delta = 4K_3 h^{2/(2+\varepsilon)}$. Observe that in order that $2K_3 h^{2/(2+\varepsilon)} \leqslant \delta/2$ and $2K_3 h^{-(\varepsilon/(2+\varepsilon))}\delta^2 \leqslant \delta/2$, we must have $\delta \in [4K_3 h^{2/(2+\varepsilon)}, (4K_3)^{-1} h^{\varepsilon/(2+\varepsilon)}] \neq \varnothing$, which is satisfied for h and δ as chosen.

The theorem is now proved, since (2.14) and (2.18) imply that $\|P_h p - z\|_{\mathscr{W}_h} \leqslant \delta$ and $\|\pi_h \mathbf{u} - \mathbf{y}\|_{\mathscr{V}_h} \leqslant \delta$; that is, Φ maps the balls of radius $\delta = O(h^{2/(2+\varepsilon)})$, centered at $(\pi_h \mathbf{u}, P_h p)$ into itself. Q.E.D.

3. L^2-Error Estimates. Note that Theorem 2.1 in fact shows that as $h \to 0$ we obtain a sequence $\{(\mathbf{u}_h, p_h)\}_{h \downarrow 0}$ which converges to $(\mathbf{u}, p)$ in $\mathbf{V} \cap L^{2+\varepsilon}(\Omega)^2 \times L^\theta(\Omega)$ and furthermore, that there is a constant $C = 4K_3 + QQ_\varepsilon \|\mathbf{u}\|_{1+\varepsilon}$ such that

$$(3.1)\qquad \max\left\{\|\mathbf{u} - \mathbf{u}_h\|_{0,2+\varepsilon}; \|p - p_h\|_{0,\theta}\right\} \leqslant Ch^{2/(2+\varepsilon)},$$

since

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{0,2+\varepsilon} &\leqslant \|\mathbf{u} - \pi_h \mathbf{u}\|_{0,2+\varepsilon} + \|\pi_h \mathbf{u} - \mathbf{u}_h\|_{0,2+\varepsilon} \\ &\leqslant Qh\|\mathbf{u}\|_{1,2+\varepsilon} + \delta \leqslant QhQ_\varepsilon \|\mathbf{u}\|_{1+\varepsilon} + 4K_3 h^{2/(2+\varepsilon)}, \end{aligned}$$

and

$$\begin{aligned} \|p - p_h\|_{0,\theta} &\leqslant \|p - P_h p\|_{0,\theta} + \|P_h p - p_h\|_{0,\theta} \\ &\leqslant Qh\|p\|_{1,\theta} + \delta \leqslant QhQ_\varepsilon \|p\|_2 + 4K_3 h^{2/(2+\varepsilon)}. \end{aligned}$$

Let us now rewrite (2.4) as

$$(3.2)\qquad \begin{aligned} &\text{(a)}\quad (\alpha(p)[\mathbf{u} - \mathbf{u}_h], \mathbf{v}) - (\operatorname{div} \mathbf{v}, p - p_h) \\ &\qquad\quad + \left(\left[\tilde{\alpha}_p(p_h)\mathbf{u}_h + \tilde{\boldsymbol{\beta}}_p(p_h)\right](p - p_h), \mathbf{v}\right) = 0, \qquad \mathbf{v} \in \mathbf{V}_h, \\ &\text{(b)}\quad (\operatorname{div}[\mathbf{u} - \mathbf{u}_h], w) + \left(\tilde{c}_p(p_h)[p - p_h], w\right) = 0, \qquad w \in W_h, \end{aligned}$$

where $\tilde{\boldsymbol{\beta}}_p(p_h)$ and $\tilde{c}_p(p_h)$ are bounded functions in $\overline{\Omega}$ defined in (2.2) and (2.3).

Observe that (3.2) corresponds to the mixed method for the operator N: $H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ given by

$$Nw = -\nabla \cdot \left(a(p)\nabla w + a(p)\left[\tilde{\alpha}_p(p_h)\mathbf{u}_h + \tilde{\boldsymbol{\beta}}_p(p_h)\right]w\right) + \tilde{c}_p(p_h)w.$$

Its formal adjoint N^*: $H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ is

$$N^*\chi = -\nabla \cdot (a(p)\nabla\chi) + a(p)\left[\tilde{\alpha}_p(p_h)\mathbf{u}_h + \tilde{\boldsymbol{\beta}}_p(p_h)\right] \cdot \nabla\chi + \tilde{c}_p(p_h)\chi. \tag{3.3}$$

Before we turn to the rate of convergence of $(\mathbf{u}_h, p_h)$ to $(\mathbf{u}, p)$, we need the following technical result.

LEMMA 3.1. *There exists an $h_0 > 0$ such that, if $h < h_0$, N^* has a bounded inverse mapping $L^2(\Omega)$ onto $H^2(\Omega) \cap H_0^1(\Omega)$.*

Proof. Since M^{*-1}: $L^2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)$ is bounded and $N^{*-1} = (M^{*-1}N^*)^{-1}M^{*-1}$, it suffices to show that $M^{*-1}N^*$ has a bounded inverse on $H^2(\Omega) \cap H_0^1(\Omega)$. For a linear differential operator D: $X \to Y$, let $|||D|||$ be its norm as a linear functional; e.g.

$$|||M^*||| = \|M^*\|_{\mathscr{L}(H^2(\Omega) \cap H_0^1(\Omega);\, L^2(\Omega))}.$$

Then, all that is needed is to prove that $|||M^{*-1}(M^* - N^*)|||$ is less than one, since this will imply that $I - M^{*-1}(M^* - N^*) = M^{*-1}N^*$ has a bounded inverse. Thus, it is sufficient to show that $|||M^* - N^*|||$ is smaller than $(|||M^{*-1}|||)^{-1}$.

We have, by (2.7) and (3.3),

$$\begin{aligned}(M^* - N^*)\chi &= a(p)\left[\alpha_p(p)\mathbf{u} - \tilde{\alpha}_p(p_h)\mathbf{u}_h + \boldsymbol{\beta}_p(p) - \tilde{\boldsymbol{\beta}}_p(p_h)\right] \cdot \nabla\chi \\ &\quad + \left(c_p(p) - \tilde{c}_p(p_h)\right)\chi \\ &= a(p)\left[\left(\alpha_p(p) - \tilde{\alpha}_p(p_h)\right)\mathbf{u} + \tilde{\alpha}_p(p_h)(\mathbf{u} - \mathbf{u}_h) \right.\\ &\qquad \left. + \left(\beta_p(p) - \tilde{\boldsymbol{\beta}}_p(p_h)\right)\right] \cdot \nabla\chi \\ &\quad + \left(c_p(p) - \tilde{c}(p_h)\right)\chi.\end{aligned} \tag{3.4}$$

Observe that $\xi = p - p_h$,

$$\begin{aligned}\alpha_p(p) - \tilde{\alpha}_p(p_h) &= \int_0^1 \left[\alpha_p(p) - \alpha_p(p_h + t\xi)\right] dt \\ &= \xi \int_0^1 (1-t) \int_0^1 \alpha_{pp}\left(p_h + t\xi + s(1-t)\xi\right) ds\, dt \\ &= \bar{\alpha}_{pp}\xi,\end{aligned} \tag{3.5}$$

where $\bar{\alpha}_{pp}$ is a bounded function. Similarly, we obtain

$$\boldsymbol{\beta}_p(p) - \tilde{\boldsymbol{\beta}}_p(p_h) = \bar{\boldsymbol{\beta}}_{pp}\xi, \qquad c_p(p) - \tilde{c}_p(p_h) = \bar{c}_{pp}\xi, \tag{3.6}$$

where $\bar{\boldsymbol{\beta}}_{pp}$ and $\bar{c}_{pp}$ are bounded functions. Substituting (3.5) and (3.6) into (3.4), we see that

$$\begin{aligned}(M^* - N^*)\chi &= a(p)\left\{\left[\bar{\alpha}_{pp}\mathbf{u} + \bar{\boldsymbol{\beta}}_{pp}\right](p - p_h) + \tilde{\alpha}_p(p_h)(\mathbf{u} - \mathbf{u}_h)\right\} \cdot \nabla\chi \\ &\quad + \bar{c}_{pp}(p - p_h)\chi,\end{aligned}$$

and thus, using (3.1),

$$\begin{aligned}\|(M^* - N^*)\chi\|_0 &\leqslant K_4\big[\|\mathbf{u}\|_{0,\infty}\|p - p_h\|_{0,(4+2\varepsilon)/\varepsilon}\|\nabla\chi\|_{0,2+\varepsilon} \\ &\quad + \|\mathbf{u} - \mathbf{u}_h\|_{0,2+\varepsilon}\|\nabla\chi\|_{0,(4+2\varepsilon)/\varepsilon} + \|p - p_h\|_0\|\chi\|_{0,\infty}\big] \\ &\leqslant K_5\|p\|_{2+\varepsilon}^3\big[h^{2/(2+\varepsilon)}\|\nabla\chi\|_1 + h^{2/(2+\varepsilon)}\|\chi\|_{1+\varepsilon}\big] \\ &\leqslant K_6 h^{2/(2+\varepsilon)}\|\chi\|_2,\end{aligned}$$

since $H^1(\Omega)^2 \subset L^r(\Omega)^2$ for any finite r and $H^{1+\varepsilon}(\Omega) \subset L^\infty(\Omega)$. Now, take h_0 small enough that $K_6 h_0^{2/(2+\varepsilon)} \leqslant (|||M^{*-1}|||)^{-1}$. Q.E.D.

We can now obtain a rate of convergence of $(\mathbf{u}_h, p_h)$ to $(\mathbf{u}, p)$ as $h \to 0$.

THEOREM 3.1. *There is a positive constant C independent of h, depending on $\|p\|_{2+\varepsilon}$ quadratically such that*

(i) $$\|p - p_h\|_0 \leqslant C\begin{cases} h\|p\|_2, & \text{if } k = 0, \\ h^s\|p\|_s, & 2 \leqslant s \leqslant k+1, \text{ if } p \in H^s \text{ and } k > 0,\end{cases}$$

(ii) $$\|\mathbf{u} - \mathbf{u}_h\|_0 \leqslant Ch^s\|p\|_{s+1}, \qquad 1 \leqslant s \leqslant k+1, \text{ if } p \in H^{s+1}(\Omega),$$

(iii) $$\|\operatorname{div}(\mathbf{u} - \mathbf{u}_h)\|_0 \leqslant Ch^s\|p\|_{s+2}, \qquad 0 \leqslant s \leqslant k+1, \text{ if } p \in H^{s+2}(\Omega).$$

Proof. Let $\boldsymbol{\zeta} = \mathbf{u} - \mathbf{u}_h$, $\xi = p - p_h$, $\boldsymbol{\sigma} = \pi_h\mathbf{u} - \mathbf{u}_h$, and $\tau = P_h p - p_h$. Rewrite (3.2) in the form

$$\begin{cases} (\alpha(p)\boldsymbol{\zeta}, v) - (\operatorname{div}\mathbf{v}, \tau) + \big([\tilde{\alpha}_p(p_h)\mathbf{u}_h + \tilde{\boldsymbol{\beta}}_p(p_h)]\tau, \mathbf{v}\big) \\ \qquad = \big([\tilde{\alpha}_p(p_h)\mathbf{u}_h + \tilde{\boldsymbol{\beta}}_p(p_h)][P_h p - p], \mathbf{v}\big), \qquad \mathbf{v} \in \mathbf{V}_h, \\ (\operatorname{div}\boldsymbol{\zeta}, w) + (\tilde{c}_p(p_h)\tau, w) = (\tilde{c}_p(p_h)[P_h p - p], w), \qquad w \in W_h.\end{cases}$$

It follows from Lemma 3.1 of [5] and our Lemma 3.1 that

$$\begin{aligned}\|\tau\|_0 \leqslant K\Big[h\|\boldsymbol{\zeta}\|_0 + h^{2-\delta_{0k}}\|\operatorname{div}\boldsymbol{\zeta}\|_0 + \big\|[\tilde{\alpha}_p(p_h)\mathbf{u}_h + \tilde{\boldsymbol{\beta}}_p(p_h)](P_h p - p)\big\|_0 \\ + \big\|\tilde{c}_p(p_h)(P_h p - p)\big\|_0\Big].\end{aligned} \tag{3.7}$$

If $p \in H^s(\Omega)$, then $p \in H^{s-(2/(2+\varepsilon)),\theta}(\Omega)$ and $\|p\|_{s-(2/(2+\varepsilon)),\theta} \leqslant K\|p\|_s$. Thus, using (1.8) and (3.1), the penultimate term in (3.7) can be bounded by

$$\begin{aligned}&\big\|[\tilde{\alpha}_p(p_h)\mathbf{u}_h + \tilde{\boldsymbol{\beta}}_h(p_h)](P_h p - p)\big\|_0 \\ &\quad\leqslant K\big[\|\mathbf{u}\|_{0,\infty}\|P_h p - p\|_0 + \|\mathbf{u} - \mathbf{u}_h\|_{0,2+\varepsilon}\|P_h p - p\|_{0,(4+2\varepsilon)/\varepsilon}\big] \\ &\quad\leqslant K\|p\|_{2+\varepsilon}^2\big[h^s\|p\|_s + h^{2/(2+\varepsilon)}h^{s-(2/(2+\varepsilon))}\|p\|_{s-(2/(2+\varepsilon)),\theta}\big] \\ &\quad\leqslant K\|p\|_{2+\varepsilon}^2 h^s\|p\|_s.\end{aligned} \tag{3.8}$$

We now derive a preliminary bound for $\|\xi\|_0$. Substituting (1.8) and (3.8) into (3.7) gives the bound

$$\|\tau\|_0 \leqslant K\Big[h\|\boldsymbol{\zeta}\|_0 + h^{2-\delta_{0k}}\|\operatorname{div}\boldsymbol{\zeta}\|_0 + h^s\|p\|_s\|p\|_{2+\varepsilon}^2\Big],$$

which in turn implies, using again (1.8), that

$$\begin{aligned}\|\xi\|_0 = \|p - p_h\|_0 &\leqslant \|p - P_h p\|_0 + \|\tau\|_0 \\ &\leqslant K\Big[\|p\|_{2+\varepsilon}^2 h^s\|p\|_s + h^{2-\delta_{0k}}\|\operatorname{div}\boldsymbol{\zeta}\|_0 + h\|\boldsymbol{\zeta}\|_0\Big].\end{aligned} \tag{3.9}$$

The quasi-regularity of $\mathscr{T}_h$ implies that

$$\|\boldsymbol{\sigma}\|_{0,\infty} \leqslant Kh^{-2/(2+\varepsilon)}\|\boldsymbol{\sigma}\|_{0,2+\varepsilon}. \tag{3.10}$$

The Sobolev imbedding theorem implies that

$$H^{1+\varepsilon}(\Omega)^2 \subset W^{(\varepsilon/2),\infty}(\Omega)^2. \tag{3.11}$$

Using (1.5) with $q = \infty$ and $s = \varepsilon/2$, we obtain from (3.1), (3.10), and (3.11) the inequality

$$\begin{aligned}\|\mathbf{u}_h\|_{0,\infty} &\leqslant \|\mathbf{u}\|_{0,\infty} + \|\pi_h\mathbf{u} - \mathbf{u}\|_{0,\infty} + \|\boldsymbol{\sigma}\|_{0,\infty} \\ &\leqslant K\left[\|p\|_{2+\varepsilon} + h^{\varepsilon/2}\|\mathbf{u}\|_{\varepsilon/2,\infty} + h^{-2/(2+\varepsilon)}\|\boldsymbol{\sigma}\|_{0,2+\varepsilon}\right] \\ &\leqslant K\left[\|p\|_{2+\varepsilon} + h^{-2/(2+\varepsilon)}\left(\|\pi_h\mathbf{u} - \mathbf{u}\|_{0,2+\varepsilon} + \|\mathbf{u} - \mathbf{u}_h\|_{0,2+\varepsilon}\right)\right] \\ &\leqslant K\left[\|p\|_{2+\varepsilon} + h^{-2/(2+\varepsilon)}\left(Qh\|\mathbf{u}\|_{1+\varepsilon} + Ch^{2/(2+\varepsilon)}\right)\right] \leqslant K\left(\|p\|_{2+\varepsilon}^2 + 1\right).\end{aligned} \tag{3.12}$$

If we now rewrite (2.4) as

$$\begin{cases}(\alpha(p)\boldsymbol{\sigma},\mathbf{v}) - (\operatorname{div}\mathbf{v},\tau) = \left(\alpha(p)[\pi_h\mathbf{u} - \mathbf{u}] - \left[\tilde{\alpha}_p(p_h)\mathbf{u}_h + \tilde{\boldsymbol{\beta}}_h(p_h)\right]\xi,\mathbf{v}\right), & \mathbf{v} \in \mathbf{V}_h, \\ (\operatorname{div}\boldsymbol{\sigma}, w) = \left(-\tilde{c}_p(p_h)\xi, w\right), \qquad w \in W_k, & \end{cases}$$

we see using [2] (just as we did to obtain (2.13)), (1.5), and (3.12), that for $\frac{1}{2} < s \leqslant k + 1$ and $p \in H^{s+1}(\Omega)$

$$\|\boldsymbol{\sigma}\|_{\mathbf{V}} \leqslant K\|p\|_{2+\varepsilon}^2\left[\|\pi_h\mathbf{u} - \mathbf{u}\|_0 + \|\xi\|_0\right] \leqslant K\|p\|_{2+\varepsilon}^2\left[\|\xi\|_0 + h^s\|p\|_{s+1}\right]. \tag{3.13}$$

From (3.13) we now obtain by (1.5) and (1.6) the bounds

$$\begin{aligned}\|\boldsymbol{\zeta}\|_0 &\leqslant \|\mathbf{u} - \pi_h\mathbf{u}\|_0 + \|\boldsymbol{\sigma}\|_0 \\ &\leqslant K\left(\|p\|_{2+\varepsilon}^2 + 1\right)\left[\|\xi\|_0 + h^s\|p\|_{s+1}\right], \qquad 1 \leqslant s \leqslant k + 1,\end{aligned} \tag{3.14}$$

$$\begin{aligned}\|\operatorname{div}\boldsymbol{\zeta}\|_0 &\leqslant \|\operatorname{div}(\mathbf{u} - \pi_h\mathbf{u})\|_0 + \|\operatorname{div}\boldsymbol{\sigma}\|_0 \\ &\leqslant K\left(\|p\|_{2+\varepsilon}^2 + 1\right)\left[\|\xi\|_0 + h^s\|p\|_{s+2}\right], \qquad 0 \leqslant s \leqslant k + 1\end{aligned} \tag{3.15}$$

which, when substituted into (3.9), yield the estimate

$$\begin{aligned}\|\xi\|_0 &\leqslant K\|p\|_{2+\varepsilon}^2\left[h\|\xi\|_0 + h^{s-\delta_{0k}}\|p\|_s + h^s\|p\|_s\right] \\ &\leqslant K\left(\|p\|_{2+\varepsilon}^2 + 1\right)\left[h\|\xi\|_0 + h^{s-\delta_{0k}}\|p\|_s\right], \qquad 2 \leqslant s \leqslant k + 1 + \delta_{0k}.\end{aligned} \tag{3.16}$$

But (3.16) now implies (i) holds if h is small enough. Applying (i) to (3.14) and (3.15) shows that (ii) and (iii) also hold. Q.E.D.

Observe that Theorem 3.1 shows that $\{(\mathbf{u}_h, p_h)\}_{h\downarrow 0}$ converges in $\mathbf{V} \times W$ to $(\mathbf{u}, p)$ both at an optimal rate (for any h) and with minimal smoothness requirements on the solution of (1.1) (if $k \geqslant 1$).

4. A Uniqueness Result for the Discrete Problem. We shall now prove a uniqueness result.

THEOREM 4.1. *If* $f \in H^{2+\varepsilon+(1-\varepsilon)\delta_{0k}}(\Omega)$ *and h is sufficiently small, then for any* $K > 0$ *there is a unique solution of* (1.11) *in the intersection of the balls*

$$\left\{ \|\mathbf{u}-\mathbf{u}_h\|_{\mathscr{V}_h} + \|p-p_h\|_{\mathscr{W}_h} \leqslant \left(K_4 \|\mathbf{u}\|_{0,\infty} \|M^{*-1}\|_{\mathscr{L}(L^2(\Omega);H^2(\Omega)\cap H_0^1(\Omega))} \right)^{-1} \right\}$$
$$\cap \left\{ \|\mathbf{u}-\mathbf{u}_h\|_{0,\infty} + \|p-p_h\|_{0,\infty} \leqslant K \right\} = B,$$

where K_4 *is the constant of Lemma* 3.1 *appearing after* (3.6).

Proof. First note that Theorem 3.1 in fact shows that any solution $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ of (1.11) lying in B will verify the bounds (i), (ii), and (iii) of Theorem 3.1. Assume now that, for $i = 1$ and 2, $(\mathbf{u}_h^{(i)}, p_h^{(i)}) \in \mathbf{V}_h \times W_h$ is a solution which satisfies the above hypotheses. Let $\boldsymbol{\zeta}_i = \mathbf{u} - \mathbf{u}_h^{(i)}$, $\xi_i = p - p_h^{(i)}$, $\mathbf{U} = \mathbf{u}_h^{(1)} - \mathbf{u}_h^{(2)}$, $P = p_h^{(1)} - p_h^{(2)}$, and $\boldsymbol{\sigma}_i = \pi_h \mathbf{u} - \mathbf{u}_h^{(i)}$. Theorem 5.1(b) below will then imply that

$$\|\xi_i\|_{0,\infty} \leqslant Q h^{\varepsilon/2} \|p\|_{2+\varepsilon+(1-\varepsilon)\delta_{0k}}, \qquad i = 1, 2. \tag{4.1}$$

Also, Theorem 3.1(b) implies

$$\|\boldsymbol{\zeta}_i\|_0 \leqslant Q h \|p\|_2, \qquad i = 1, 2. \tag{4.2}$$

It follows from (1.5), (3.10), and (4.2) that

$$\begin{aligned} \|\boldsymbol{\zeta}_i\|_{0,\infty} &\leqslant \|\mathbf{u} - \pi_h \mathbf{u}\|_{0,\infty} + \|\boldsymbol{\sigma}_i\|_{0,\infty} \leqslant K\left[h^{\varepsilon/2} \|\mathbf{u}\|_{\varepsilon/2,\infty} + h^{-1} \|\boldsymbol{\sigma}_i\|_0 \right] \\ &\leqslant K\left[h^{\varepsilon/2} \|\mathbf{u}\|_{1+\varepsilon} + h^{-1}\left(\|\pi_h \mathbf{u} - \mathbf{u}\|_0 + \|\boldsymbol{\zeta}_i\|_0 \right) \right] \\ &\leqslant K\left[h^{\varepsilon/2} \|p\|_{2+\varepsilon} + h^{-1} h \|p\|_2 \right] \leqslant K(p), \qquad i = 1, 2, \end{aligned} \tag{4.3}$$

where $K(p)$ depends on $\|p\|_{2+\varepsilon}$ quadratically.

It follows from (1.3) and (1.11) that

$$\begin{aligned} (\alpha(p)\mathbf{U}, \mathbf{v}) - (\operatorname{div} \mathbf{v}, P) &= \left(\left[\alpha(p_h^{(2)}) - \alpha(p_h^{(1)}) \right] \mathbf{u}_h^{(2)}, \mathbf{v} \right) \\ &\quad + \left(\left[\alpha(p) - \alpha(p_h^{(1)}) \right] \mathbf{U} + \boldsymbol{\beta}(p_h^{(2)}) - \boldsymbol{\beta}(p_h^{(1)}), \mathbf{v} \right), \\ &\qquad\qquad \mathbf{v} \in \mathbf{V}_h, \\ (\operatorname{div} \mathbf{U}, w) &= \left(c(p_h^{(2)}) - c(p_h^{(1)}), w \right), \qquad w \in W_h. \end{aligned}$$

Let

$$\alpha(p_h^{(1)}) - \alpha(p_h^{(2)}) = \bar{\alpha}_p(\bar{P})P, \qquad \alpha(p) - \alpha(p_h^{(1)}) = \bar{\alpha}_p(\xi_1)\xi_1,$$
$$\boldsymbol{\beta}(p_h^{(1)}) - \boldsymbol{\beta}(p_h^{(2)}) = \bar{\boldsymbol{\beta}}_p(\bar{P})P, \qquad c(p_h^{(1)}) - c(p_h^{(2)}) = \bar{c}_p(\bar{P})P,$$

where $\bar{\alpha}_p(\bar{P})$, $\bar{\alpha}_p(\xi_1)$, $\bar{\boldsymbol{\beta}}_p(\bar{P})$, and $\bar{c}_p(\bar{P})$ are bounded functions in $\bar{\Omega}$, where $\bar{P}$ is some convex combination of $p_h^{(1)}$ and $p_h^{(2)}$. Then

$$\left\{ \begin{aligned} (\alpha(p)\mathbf{U}, \mathbf{v}) - (\operatorname{div} \mathbf{v}, P) &= -\left(\left[\bar{\alpha}_p(\bar{P})\mathbf{u}_h^{(2)} + \bar{\boldsymbol{\beta}}_p(\bar{P}) \right] P, \mathbf{v} \right) \\ &\quad + \left(\bar{\alpha}_p(\xi_1)\xi_1 \mathbf{U}, \mathbf{v} \right), \qquad \mathbf{v} \in \mathbf{V}_h, \\ (\operatorname{div} \mathbf{U}, w) &= -\left(\bar{c}_p(\bar{P})P, w \right), \qquad w \in W_h. \end{aligned} \right. \tag{4.4}$$

It follows from [2], (4.1), and (4.3) that

$$\|\mathbf{U}\|_0 \leqslant K(p)\left[\|P\|_0 + h\|\mathbf{U}\|_0 \right], \qquad \|\operatorname{div} \mathbf{U}\|_0 \leqslant K(p)\left[\|P\|_0 + h\|\mathbf{U}\|_0 \right]. \tag{4.5}$$

For h sufficiently small, (4.5) implies

$$\|\mathbf{U}\|_0 \leqslant K(p)\|P\|_0, \tag{4.6}$$

(4.7) $$\|\operatorname{div} \mathbf{U}\|_0 \leqslant K(p)\|P\|_0.$$

Rewrite (4.4) in the form

$$(\alpha(p)\mathbf{U}, \mathbf{v}) - (\operatorname{div} \mathbf{v}, P) + \left(\left[\bar{\alpha}_p(\bar{P})\mathbf{u}_h^{(2)} + \bar{\boldsymbol{\beta}}_p(\bar{P})\right]P, \mathbf{v}\right) = \left(\bar{\alpha}_p(\xi_1)\xi_1\mathbf{U}, \mathbf{v}\right), \quad \mathbf{v} \in \mathbf{V}_h,$$
$$(\operatorname{div} \mathbf{U}, w) + \left(\bar{c}_p(\bar{P})P, w\right) = 0, \qquad w \in W_h.$$

Then, it follows from [4], an obvious variation of Lemma 3.1, and (4.1) that

(4.8) $$\|P\|_0 \leqslant K(p)\left[h\|\mathbf{U}\|_0 + h\|\operatorname{div} \mathbf{U}\|_0\right].$$

If we now substitute (4.6) and (4.7) into (4.8), we have $\|P\|_0 \leqslant Kh\|P\|_0$, which implies that $P = 0$ for h sufficiently small. Then (4.6) implies that $U = 0$. Q.E.D.

5. L^q-Error Estimates ($2 \leqslant q \leqslant \infty$). We shall first obtain a negative norm estimate for τ.

LEMMA 5.1. *There exists a positive constant $C \leqslant K(\|p\|_{2+\varepsilon}^4 + 1)$, independent of h, such that, if $\partial\Omega$ and the coefficients a, $\mathbf{b}$, and c of* (1.1a) *are sufficiently smooth, then for* $0 \leqslant s \leqslant k$,

$$\|\tau\|_{-s} \leqslant C\left[h^{r_1+s+1}\|p\|_{r_1+1+(s-k+1)^+} + h^{r_2+l-\varepsilon}\|p\|_{r_2+1}\|p\|_{l+\delta_{0k}}\right],$$

$1 \leqslant r_1, r_2 \leqslant k+1$, $2 - \delta_{0k} \leqslant l \leqslant k+1$, *where p is the solution of* (1.1).

Proof. Since

$$\|\tau\|_{-s} = \sup_{\substack{\psi \in H^s(\Omega) \\ \psi \neq 0}} \frac{(\tau, \psi)}{\|\psi\|_s},$$

we wish to bound (τ, ψ) for $\psi \in H^s(\Omega)$. Let $\phi \in H^{s+2}(\Omega)$ be the (unique) solution of $M^*\phi = \psi$ in Ω, $\phi = 0$ on $\partial\Omega$, the existence of which we shall assume. Assume also that $\|\phi\|_{s+2} \leqslant K\|\psi\|_s$. Note that (1.5) and the Sobolev imbedding theorem give the bound

(5.1) $$\|a(p)\nabla\phi - \pi_h a(p)\nabla\phi\|_{0,(2+\varepsilon)/\varepsilon} \leqslant Kh^{\varepsilon/2}\|a(p)\nabla\phi\|_{\varepsilon/2,(2+\varepsilon)/\varepsilon} \leqslant Kh^{\varepsilon/2}\|\nabla\phi\|_1 \leqslant Kh^{\varepsilon/2}\|\phi\|_{s+2}.$$

Also, the Sobolev imbedding theorem implies that $a(p)\nabla\phi \in L^t(\Omega)^2$ for any finite t, with $\|a(p)\nabla\phi\|_{0,t} \leqslant K\|\phi\|_2$; the quasi-regularity of $\mathcal{T}_h$ implies that if $\chi \in \mathbf{V}_h$ and $\pi \in W_h$,

(5.2) $$\|\chi\|_{0,2+\varepsilon} \leqslant Kh^{-\varepsilon/(2+\varepsilon)}\|\chi\|_0, \qquad \|\pi\|_{0,2+\varepsilon} \leqslant Kh^{-\varepsilon/(2+\varepsilon)}\|\pi\|_0.$$

Let $\boldsymbol{\kappa} = \tilde{\alpha}_{pp}(p_h)\mathbf{u} + \tilde{\boldsymbol{\beta}}_{pp}(p_h)$, $\lambda = \tilde{\alpha}_p(p_h)$, and $\rho = \tilde{c}_{pp}(p_h)$. We have $\lambda, \rho \in L^\infty(\Omega)$ and $\boldsymbol{\kappa} \in L^\infty(\Omega)^2$. Rewrite (2.5) as

(5.3)
$$\text{(a)} \quad (\alpha(p)\boldsymbol{\zeta}, \mathbf{v}) - (\operatorname{div} \mathbf{v}, \tau) + (\Gamma\tau, \mathbf{v}) = ([\boldsymbol{\kappa}\xi + \lambda\boldsymbol{\zeta}]\xi + \Gamma[P_h p - p], \mathbf{v}), \qquad \mathbf{v} \in \mathbf{V}_h,$$
$$\text{(b)} \quad (\operatorname{div} \boldsymbol{\zeta}, w) + (\gamma\tau, w) = \left(\rho\xi^2 + \gamma[P_h p - p], w\right), \qquad w \in W_h,$$

where ξ, $\boldsymbol{\zeta}$, and τ are the same as in the proof of Theorem 3.1. It follows from (1.4), (1.10), (5.3), and integration by parts (exactly as in Lemma 2.1) that

(5.4)
$$\begin{aligned}(\tau, \psi) = {} & ([\boldsymbol{\kappa}\xi + \lambda\boldsymbol{\zeta}]\xi + \Gamma[P_h p - p], a(p)\nabla\phi) \\ & + ([\boldsymbol{\kappa}\xi + \lambda\boldsymbol{\zeta}]\xi + \Gamma[P_h p - p], \pi_h a(p)\nabla\phi - a(p)\nabla\phi) \\ & + (\alpha(p)\boldsymbol{\zeta} + \Gamma\tau, a(p)\nabla\phi - \pi_h a(p)\nabla\phi) + (\operatorname{div} \boldsymbol{\zeta} + \gamma\tau, \phi - P_h\phi) \\ & + \left(\rho\xi^2 + \gamma[P_h p - p], \phi\right) + \left(\rho\xi^2 + \gamma[P_h p - p], P_h\phi - \phi\right).\end{aligned}$$

First, note that (1.5), (1.8), (5.1), (5.2), and Theorem 3.1 imply

$$
\begin{aligned}
([\kappa\xi + \lambda\zeta]\xi, a(p)\nabla\phi) &\leqslant K(\|\xi\|_{0,2+\varepsilon} + \|\zeta\|_{0,2+\varepsilon})\|\xi\|_{0,2+\varepsilon}\|\nabla\phi\|_{0,(2+\varepsilon)/\varepsilon} \\
&\leqslant K(\|p - P_h p\|_{0,2+\varepsilon} + \|\tau\|_{0,2+\varepsilon} + \|\mathbf{u} - \pi_h \mathbf{u}\|_{0,2+\varepsilon} + \|\sigma\|_{0,2+\varepsilon}) \\
&\quad \times(\|p - P_h p\|_{0,2+\varepsilon} + \|\tau\|_{0,2+\varepsilon})\|\nabla\phi\|_1 \\
&\leqslant K\Big(h^{r_2-\varepsilon/(2+\varepsilon)}\|p\|_{r_2-\varepsilon/(2+\varepsilon),2+\varepsilon} + h^{-\varepsilon/(2+\varepsilon)}\|\tau\|_0 \\
&\quad + h^{r_2-\varepsilon/(2+\varepsilon)}\|\mathbf{u}\|_{r_2-\varepsilon/(2+\varepsilon),2+\varepsilon} + h^{-\varepsilon/(2+\varepsilon)}\|\sigma\|_0\Big) \\
&\quad \times\left(h^{l-\varepsilon/(2+\varepsilon)}\|p\|_{l-\varepsilon/(2+\varepsilon),2+\varepsilon} + h^{-\varepsilon/(2+\varepsilon)}\|\tau\|_0\right)\|\phi\|_2 \\
&\leqslant Kh^{-2\varepsilon/(2+\varepsilon)}\left(h^{r_2}\|p\|_{r_2+1} + \|p_h p - p\|_0 + \|\xi\|_0 + \|\pi_h\mathbf{u} - \mathbf{u}\|_0 + \|\zeta\|_0\right) \\
&\quad \times\left(h^l\|p\|_l + \|P_h p - p\|_0 + \|\xi\|_0\right)\|\phi\|_{s+2} \\
&\leqslant Kh^{-2\varepsilon/(2+\varepsilon)}h^{r_2}\|p\|_{r_2+1}h^l\|p\|_{l+\delta_{0k}}\|\phi\|_{s+2} \\
&= Kh^{l+r_2-2\varepsilon/(2+\varepsilon)}\|p\|_{l+\delta_{0k}}\|p\|_{r_2+1}\|\phi\|_{s+2}, \\
&\qquad 2-\delta_{0k} \leqslant l \leqslant k+1,\ 1 \leqslant r_2 \leqslant k+1,
\end{aligned} \tag{5.5}
$$

$$
\begin{aligned}
([\kappa\xi + \lambda\zeta]\xi, &\ \pi_h a(p)\nabla\phi - a(p)\nabla\phi) \\
&\leqslant Kh^{-2\varepsilon/(2+\varepsilon)}h^{r_2}\|p\|_{r_2+1}h^l\|p\|_{l+\delta_{0k}}\|\pi_h a(p)\nabla\phi - a(p)\nabla\phi\|_{0,(2+\varepsilon)/\varepsilon} \\
&\leqslant Kh^{l+r_2-2\varepsilon/(2+\varepsilon)}\|p\|_{l+\delta_{0k}}\|p\|_{r_2+1}h^{\varepsilon/2}\|\phi\|_{s+2}, \\
&\qquad 2-\delta_{0k} \leqslant l \leqslant k+1,\ 1 \leqslant r_2 \leqslant k+1,
\end{aligned} \tag{5.6}
$$

$$
\begin{aligned}
(\rho\xi^2, \phi) &\leqslant K\|\xi\|_{0,2+\varepsilon}^2\|\phi\|_{0,(2+\varepsilon)/\varepsilon} \\
&\leqslant Kh^{-2\varepsilon/(2+\varepsilon)}h^l\|p\|_{l+\delta_{0k}}h^{r_2}\|p\|_{r_2+\delta_{0k}}\|\phi\|_2 \\
&\leqslant Kh^{l+r_2-2\varepsilon/(2+\varepsilon)}\|p\|_{l+\delta_{0k}}\|p\|_{r_2+1}\|\phi\|_{s+2}, \\
&2-\delta_{0k} \leqslant l \leqslant k+1,\ 1 \leqslant r_2 \leqslant k+1,
\end{aligned} \tag{5.7}
$$

$$
\begin{aligned}
(\rho\xi^2, P_h\phi - \phi) &\leqslant K\|\xi\|_{0,2+\varepsilon}^2\|P_h\phi - \phi\|_{0,(2+\varepsilon)/\varepsilon} \\
&\leqslant Kh^{l+r_2-2\varepsilon/(2+\varepsilon)}\|p\|_{l+\delta_{0k}}\|p\|_{r_2+1}h^{\varepsilon/2}\|\phi\|_1 \\
&\leqslant Kh^{l+r_2-2\varepsilon/(2+\varepsilon)}\|p\|_{l+\delta_{0k}}\|p\|_{r_2+1}\|\phi\|_{s+2}, \\
&2-\delta_{0k} \leqslant l \leqslant k+1,\ 1 \leqslant r_2 \leqslant k+1.
\end{aligned} \tag{5.8}
$$

Next, we obtain from (1.9) the bound

$$
\begin{aligned}
(\Gamma[P_h p - p], a(p)\nabla\phi) &+ (\gamma[P_h p - p], \phi) \\
&\leqslant K\|P_h p - p\|_{-s-1}(\|\nabla\phi\|_{s+1} + \|\phi\|_{s+1}) \\
&\leqslant Kh^{r_1+s+1}\|p\|_{r_1}\|\phi\|_{s+2}, \qquad 0 \leqslant r_1 \leqslant k+1.
\end{aligned} \tag{5.9}
$$

Also, (1.5) and (1.8) give the estimates

$$
\begin{aligned}
(\Gamma[P_h p - p], \pi_h a(p)\nabla\phi - a(p)\nabla\phi) &+ (\gamma[P_h p - p], P_h\phi - \phi) \\
&\leqslant K\|P_h p - p\|_0(\|\pi_h a(p)\nabla\phi - a(p)\nabla\phi\|_0 + \|P_h\phi - \phi\|_0) \\
&\leqslant Kh^{r_1+s+1}\|p\|_{r_1}\|\phi\|_{s+2}, \qquad 0 \leqslant r_1 \leqslant k+1,
\end{aligned} \tag{5.10}
$$

$$(5.11)\quad \begin{aligned}&(\Gamma\tau, a(p)\nabla\phi - \pi_h a(p)\nabla\phi) + (\gamma\tau, \phi - P_h\phi)\\ &\qquad \leqslant K\|\tau\|_0\big(\|a(p)\nabla\phi - \pi_h a(p)\nabla\phi\|_0 + \|\phi - P_h\phi\|_0\big)\\ &\qquad \leqslant Kh^{s+1}\|\tau\|_0\|\phi\|_{s+2}.\end{aligned}$$

Finally, (1.5), (1.8), and Theorem 3.1 imply that

$$(5.12)\quad \begin{aligned}&(\alpha(p)\zeta, a(p)\nabla\phi - \pi_h a(p)\nabla\phi)\\ &\qquad \leqslant K\|\zeta\|_0\|a(p)\nabla\phi - \pi_h a(p)\nabla\phi\|_0 \leqslant Kh^{r_1+s+1}\|p\|_{r_1+1}\|\phi\|_{s+2},\\ &\qquad\qquad 1 \leqslant r_1 \leqslant k+1,\end{aligned}$$

$$(5.13)\quad \begin{aligned}&(\operatorname{div}\zeta, \phi - P_h\phi) \leqslant K\|\operatorname{div}\zeta\|_0\|\phi - P_h\phi\|_0\\ &\qquad \leqslant Kh^{r_1+s+1}\begin{cases}\|p\|_{r_1+1}\|\phi\|_{s+2}, & 1 \leqslant r_1 \leqslant k+2, \text{ if } s \leqslant k-1,\\ \|p\|_{r_1+2+s-k}\|\phi\|_{s+1+k-s}, & 0 \leqslant r_1 \leqslant k+1, \text{ if } s > k-1.\end{cases}\end{aligned}$$

If we combine now (5.4)–(5.13), we see that

$$(5.14)\quad \|\tau\|_{-s} \leqslant K\Big[h^{s+1}\|\tau\|_0 + h^{l+r_2-\varepsilon}\|p\|_{l+\delta_{0k}}\|p\|_{r_2+1} + h^{r_1+s+1}\|p\|_{r_1+1+(s-k+1)^+}\Big]$$

for $2 - \delta_{0k} \leqslant l \leqslant k+1$, $1 \leqslant r_1, r_2 \leqslant k+1$, where K contains a multiple of $\|p\|_{2+\varepsilon}^4$. If we now take $s = 0$ in (5.14), we obtain the estimate

$$\|\tau\|_0 \leqslant K\Big[h\|\tau\|_0 + h^{r_2+l-\varepsilon}\|p\|_{l+\delta_{0k}}\|p\|_{r_2+1} + h^{r_1+1}\|p\|_{r_1+1+\delta_{0k}}\Big],$$

which, for sufficiently small h, implies that

$$(5.15)\quad \|\tau\|_0 \leqslant K\Big[h^{l+r_2-\varepsilon}\|p\|_{l+\delta_{0k}}\|p\|_{r_2+1} + h^{r_1+1}\|p\|_{r_1+1+\delta_{0k}}\Big].$$

Substituting (5.15) into (5.14) completes the proof of the lemma. Q.E.D.

We can now demonstrate the convergence of p_h to p to be at an optimal rate in $L^q(\Omega)$, $2 \leqslant q \leqslant \infty$ (for $k > 0$ if $q > 2/\varepsilon$).

THEOREM 5.1. *There are positive constants C_q and C (both containing a multiple of $\|p\|_{2+\varepsilon}^4$), independent of h, such that, if $p \in W^{r,q}(\Omega) \cap H^{r+1+\delta_{0k}+t(q,r)}(\Omega)$, $1 \leqslant r \leqslant k + 1 - (\varepsilon - 2/q)^+\delta_{0k}$, where $t(q, r) = -(2/q) + (1 + (2/q) - r)^+$, then*

$$\text{(a)}\quad \|p - p_h\|_{0,q} \leqslant C_q h^r\|p\|_{r+1+\delta_{0k}+t(q,r)}, \qquad q < \infty,$$

$$\text{(b)}\quad \|p - p_h\|_{0,\infty} \leqslant Ch^r\big[\|p\|_{r,\infty} + \|p\|_{r+1+\delta_{0k}}\big].$$

Proof. Using Lemma 5.1 (with $s = 0$, $l = 1 + \varepsilon(1 - \delta_{0k})$, $r_1 + 2/q = r_2 + (2/q) - \varepsilon\delta_{0k} = r$), (1.8), and the quasi-regularity of $\mathscr{T}_h$, we obtain the bounds

$$\begin{aligned}\text{(a)}\quad \|p - p_h\|_{0,q} &\leqslant \|p - P_h p\|_{0,q} + \|\tau\|_{0,q}\\ &\leqslant K_q\big[h^r\|p\|_{r,q} + h^{-(q-2)/q}\|\tau\|_0\big]\\ &\leqslant C_q\big[h^r\|p\|_{r,q} + h^r\|p\|_{r+1+\delta_{0k}+t(q,r)^+}\big],\end{aligned}$$

$$\begin{aligned}\text{(b)}\quad \|p - p_h\|_{0,\infty} &\leqslant \|p - P_h p\|_{0,\infty} + \|\tau\|_{0,\infty}\\ &\leqslant K\big[h^r\|p\|_{r,\infty} + h^{-1}\|\tau\|_0\big]\\ &\leqslant K\big[h^r\|p\|_{r,\infty} + h^{-1+r+1}\|p\|_{r+1+\delta_{0k}}\big]. \quad \text{Q.E.D.}\end{aligned}$$

6. Error Estimates in $H^s(\Omega)'$ and $(H^s(\Omega)^2)'$. We can also derive negative norm error estimates from Lemma 5.1.

THEOREM 6.1. *There exists a constant $C > 0$ (containing a multiple of $\|p\|_{2+\varepsilon}^4$), independent of h, such that, if p is sufficiently smooth, then for $0 \leqslant s \leqslant k+1$ we have the following estimates*:

$$\text{(i)}\quad \|p - p_h\|_{-s} \leqslant C\left[h^{r_1+s}\|p\|_{r_1+(s-k+1)^+} + h^{r_2+l-\varepsilon}\|p\|_{l+\delta_{0k}}\|p\|_{r_2+1}\right],$$
$$2-\delta_{0k} \leqslant l \leqslant k+1,\ 1 \leqslant r_1, r_2 \leqslant k+1,$$

$$\text{(ii)}\quad \|\mathbf{u} - \mathbf{u}_h\|_{-s} \leqslant C\left[h^{r_1+s}\|p\|_{r_1+1+(s-k)^+} + h^{r_2+l-\varepsilon}\|p\|_{l+\varepsilon+\delta_{0k}}\|p\|_{r_2+1+\varepsilon\delta_{0k}}\right],$$
$$2-\delta_{0k} \leqslant l \leqslant k+1,\ 1 \leqslant r_1, r_2 \leqslant k+1,$$

$$\text{(iii)}\quad \|\operatorname{div}(\mathbf{u} - \mathbf{u}_h)\|_{-s} \leqslant C\left[h^{r_1+s}\|p\|_{r_1+2} + h^{r_2+l-\varepsilon}\|p\|_{l+\varepsilon+\delta_{0k}}\|p\|_{r_2+1+\varepsilon\delta_{0k}}\right],$$
$$2-\delta_{0k} \leqslant l \leqslant k+1,\ 0 \leqslant r_1, r_2 \leqslant k+1.$$

Proof. (i) follows directly from Lemma 5.1 and (1.9), since

$$\xi = \tau + (P_h p - p), \quad \left(\|\xi\|_{-k-1} \leqslant \|\tau\|_{-k} + \|P_h p - p\|_{-k-1}\right).$$

Rewrite (5.3) as

$$(6.1)\quad \begin{cases} (\alpha(p)\zeta, \mathbf{v}) = (\operatorname{div}\mathbf{v}, \tau) - (\Gamma\xi, \mathbf{v}) + ([\kappa\xi + \lambda\zeta]\xi, \mathbf{v}), & \mathbf{v} \in \mathbf{V}_h, \\ (\operatorname{div}\zeta, w) = (-\gamma\xi, w) + (\rho\xi^2, w), & w \in W_h. \end{cases}$$

Note that (iii) for $s = 0$ is just a consequence of (iii) of Theorem 3.1. Let then $\psi \in H^s(\Omega)$, $1 \leqslant s \leqslant k+1$. Then, $\psi \in L^{\theta/2}(\Omega)$ with $\|\psi\|_{\theta/2} \leqslant K\|\psi\|_s$. The second equation of (6.1) and (1.8) imply that

$$\begin{aligned}
(\operatorname{div}\zeta, \psi) &= (\operatorname{div}\zeta, P_h\psi) + (\operatorname{div}\zeta, \psi - P_h\psi) \\
&= -(\gamma\xi, P_h\psi) + (\rho\xi^2, P_h\psi) + (\operatorname{div}\zeta, \psi - P_h\psi) \\
&= -(\gamma\xi, \psi) + (\gamma\xi, \psi - P_h\psi) + (\rho\xi^2, \psi) + (\rho\xi^2, P_h\psi - \psi) \\
&\quad + (\operatorname{div}\zeta, \psi - P_h\psi) \\
&\leqslant K\|\psi\|_s\left[\|\xi\|_{-s} + h^s\|\xi\|_0 + \|\xi\|_{0,2+\varepsilon}^2 + h^s\|\xi\|_{0,4}^2 + h^s\|\operatorname{div}\zeta\|_0\right]
\end{aligned}$$

since $[1/(2+\varepsilon)] + [1/(2+\varepsilon)] + [1/(\theta/2)] = 1$. Therefore, (iii) follows from (i), (a) of Theorem 5.1, (iii) of Theorem 3.1, and interpolation [10] for $0 < s < 1$.

Finally, assume that the coefficient a is smooth enough that for $\chi \in H^{s-1}(\Omega)$ there is a unique $\phi \in H^{s+1}(\Omega)$ such that $\nabla \cdot (a(p)\nabla\phi) = \chi$ in Ω, $\phi = 0$ on $\partial\Omega$, with $\|\phi\|_{s+1} \leqslant K\|\chi\|_{s-1}$, $0 \leqslant s \leqslant k+1$. Let $\psi \in H^s(\Omega)^2$; let $\phi \in H^{s+1}(\Omega)$ be the solution of $\nabla(a(p)\nabla\phi) = \operatorname{div}\psi$ in Ω, $\phi = 0$ on $\partial\Omega$, and let $\chi = \psi - a(p)\nabla\phi$. Then $\operatorname{div}\chi = 0$, $\|\chi\|_s \leqslant K\|\psi\|_s$, and $\|\phi\|_{s+1} \leqslant K\|\psi\|_s$. It follows from (1.4), (1.5), (1.3), (1.8), (1.10), (5.1), integration by parts, Theorem 3.1 and Theorem 5.1 that

$$\begin{aligned}
(\alpha(p)\zeta, \psi) &= (\alpha(p)\zeta, \chi) + (\zeta, \nabla\phi) \\
&= -(\operatorname{div}\zeta, \phi) + (\alpha(p)\zeta, \pi_h\chi) + (\alpha(p)\zeta, \chi - \pi_h\chi) \\
&= -(\operatorname{div}\zeta, P_h\phi) + (\operatorname{div}\zeta, P_h\phi - \phi) + (\alpha(p)\zeta, \chi - \pi_h\chi) + (\alpha(p)\zeta, \pi_h\chi) \\
&= (\gamma\xi, P_h\phi) - (\rho\xi^2, P_h\phi) + (\operatorname{div}\zeta, P_h\phi - \phi) + (\alpha(p)\zeta, \chi - \pi_h\chi) \\
&\quad -(\Gamma\xi, \pi_h\chi) + ([\kappa\xi + \lambda\zeta]\xi, \pi_h\chi) \\
&= (\gamma\xi, \phi) + (\gamma\xi, P_h\phi - \phi) - (\rho\xi^2, \phi) - (\rho\xi^2, P_h\phi - \phi) \\
&\quad + (\operatorname{div}\zeta, P_h\phi - \phi) + (\alpha(p)\zeta, \chi - \pi_h\chi) - (\Gamma\xi, \chi) \\
&\quad -(\Gamma\xi, \pi_h\chi - \chi) + ([\kappa\xi + \lambda\zeta]\xi, \chi) + ([\kappa\xi + \lambda\zeta]\xi, \pi_h\chi - \chi) \\
&\leqslant K\|\psi\|_s\Big[\|\xi\|_{-s-1} + h^{s+1}\|\xi\|_0 + \|\xi\|_{0,4}^2 + h^{s+1}\|\xi\|_{0,4}^2 \\
&\quad + h^{s+1}\|\operatorname{div}\zeta\|_0 + h^s\|\zeta\|_0 + \|\xi\|_{-s} + h^s\|\xi\|_0 \\
&\quad + \left(\|\zeta\|_{0,2+\varepsilon} + \|\xi\|_{0,2+\varepsilon}\right)\|\xi\|_{0,(2+\varepsilon)/\varepsilon} + h^s\left(\|\zeta\|_0 + \|\xi\|_0\right)\|\xi\|_{0,\infty}\Big] \\
&\leqslant K\|\psi\|_s\left[h^{r_2+l-\varepsilon}\|p\|_{r_2+1+\varepsilon\delta_{0k}}\|p\|_{l+\varepsilon+\delta_{0k}} + h^{r_1+s}\|p\|_{r_1+1+(s-k)^+}\right],
\end{aligned}$$

from which (ii) follows immediately. Q.E.D.

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