

On Polynomial Approximation in the Complex Plane with Application to Conformal Mapping

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Abstract. We consider the selection of polynomial bases for polynomial approximation of analytic functions on bounded simply connected regions in the complex plane. While a monomial basis may be very ill-conditioned, we show that a basis of Lagrange polynomials with Fejér points as nodes is well-conditioned. Numerical examples, where we compute polynomial approximations of conformal mappings, conclude the paper.

1. Introduction. Important for the success of any numerical method for polynomial approximation is the selection of a well-conditioned polynomial basis. For polynomial approximation on the real axis the numerical condition of various bases has been investigated by Gautschi [7], [8], [9]. We consider the numerical condition of polynomial bases for approximation on bounded simply connected regions Ω of the complex plane \mathbf{C} . The polynomials of a well-conditioned basis depend on the shape of Ω . For disks with radius ρ and center $z = 0$, the monomials $p_k(z) := (z/\rho)^k$, $k = 0, 1, 2, \dots$ are well-conditioned, see Example 2.1 below, but on flat ellipses this basis is severely ill-conditioned, see Example 2.2. We will show that a basis of Lagrange polynomials

$$(1.1) \quad l_k(z) := \prod_{\substack{j=0 \\ j \neq k}}^n \frac{z - z_j}{z_k - z_j}, \quad k = 0(1)n,$$

where the z_k are Fejér points on the boundary of the region, is well-conditioned and simple to compute. Fejér points are defined as follows, see also Gaier [6]. Denote by $\partial\Omega$ the boundary of Ω and assume that $\partial\Omega$ is a Jordan curve. Let $\bar{\Omega} := \Omega \cup \partial\Omega$ and $\Omega_c := \mathbf{C} \setminus \bar{\Omega}$. Let $w = \psi(z)$ be a conformal map from Ω_c to $|w| > 1$ with $\psi(\infty) = \infty$. ψ can be continued continuously to a bijective map from $\Omega_c \cup \partial\Omega$ to $|w| \geq 1$. ψ is uniquely determined by the requirement $\psi(z_1) = 1$. The points $\{z_k\}_{k=1}^n$ are called Fejér points if they satisfy

$$(1.2) \quad \psi(z_k) = \exp(2\pi i(k-1)/n), \quad k = 1(1)n.$$

Note that z_1 is an arbitrary point on $\partial\Omega$. To determine a set of Fejér points, we need the restriction of ψ to $\partial\Omega$. This restriction we compute by solving a modified Symm's

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integral equation. The modification makes Symm's integral equation uniquely solvable.

We next turn to our application, the polynomial approximation of conformal mappings from Ω to the unit disk. Besides illustrating the good numerical condition of the Lagrange basis (1.1), polynomial approximants of conformal mappings are important in themselves. They provide an approximation of particularly simple form, easy to evaluate at interior as well as at boundary points. Assume from now on that Ω is open and contains the origin. Let $w = \phi(z)$ denote a conformal mapping from Ω to $|w| < 1$, such that $\phi(0) = 0$. ϕ can be continued continuously to a bijective map from $\bar{\Omega}$ to $|w| \leq 1$. The restriction of ϕ to $\partial\Omega$ we compute by a modified Symm's integral equation and determine a polynomial approximant of ϕ by interpolation in a set of Fejér points. The norm of this interpolation projection grows only logarithmically with the degree of the polynomial, see Theorem 2.1. The numerical scheme we present to compute the polynomial approximant requires only little computation once a set of Fejér points has been determined.

The paper is organized as follows. Section 2 contains results on the numerical condition of some polynomial bases. In Section 3, we introduce the modified Symm's integral equation, and in Section 4, we present the numerical scheme for computing polynomial approximants to conformal mappings. This section also contains computed examples.

2. Condition of Some Polynomial Bases. We consider the numerical condition of the Lagrange basis with Fejér nodes, the monomial basis and a basis of Newton polynomials with Fejér nodes. Let $q_j(z)$, $j = 0(1)n$ be polynomials such that $\text{span}\{q_j\}_{j=0}^n = \text{span}\{z^j\}_{j=0}^n$. We wish to investigate the sensitivity of

$$(2.1) \quad Q_n(z) := \sum_{j=0}^n a_j q_j(z), \quad z \in \bar{\Omega},$$

with respect to perturbations in the coefficients a_j . Following Gautschi [9], we introduce the map $M_n: \mathbb{C}^{n+1} \rightarrow \Pi_n$, mapping the coefficient space on the space of polynomials of degree $\leq n$. With $\mathbf{a} := (a_0, a_1, \dots, a_n) \in \mathbb{C}^{n+1}$, we define

$$(2.2) \quad (M_n \mathbf{a})(z) := \sum_{k=0}^n a_k q_k(z), \quad z \in \bar{\Omega}.$$

For the inverse of M_n , we have $M_n^{-1} Q_n = \mathbf{a}$. In \mathbb{C}^{n+1} we introduce the maximum norm $\|\mathbf{a}\|_\infty := \max_{0 \leq k \leq n} |a_k|$. Also Π_n we equip with the maximum norm $\|Q_n\|_{\partial\Omega} := \sup_{z \in \partial\Omega} |Q_n(z)|$. Let $\|M_n\|$ and $\|M_n^{-1}\|$ be the induced operator norms. Our interest of this section is how the condition of the map M_n , defined by

$$(2.3) \quad \text{cond } M_n := \|M_n\| \|M_n^{-1}\|,$$

grows with n for various choices of q_j .

THEOREM 2.1. *Let $q_k(z)$ be the Lagrange polynomials (1.1) with the z_k being Fejér points, and assume that $\partial\Omega$ is analytic. Then, for all $n \geq 1$,*

$$\text{cond } M_n \leq \frac{2}{\pi} \ln n + \alpha$$

where α is a constant depending on the shape of Ω .

Proof. The theorem may be regarded as a corollary to Theorem B of Curtiss [2]. We first note that $\|M_n^{-1}\| = 1$ and therefore

$$\text{cond } M_n = \|M_n\| = \left\| \sum_{k=0}^n |l_k(z)| \right\|_{\partial\Omega}.$$

Curtiss [2] shows that when f is a function analytic in Ω , continuous on $\bar{\Omega}$ and of bounded variation on $\partial\Omega$, then the interpolation projection

$$(2.4) \quad (L_n f)(z) := \sum_{k=0}^n f(z_k) l_k(z)$$

converges to $f(z)$ in norm $\|\cdot\|_{\partial\Omega}$. Combining (3.3), (3.6), (4.2) and (4.6) of Curtiss [2], one obtains after some simplifications

$$(2.5) \quad \|L_n\| \leq \|L_n^d\| + \beta + O(r^n), \quad n \rightarrow \infty,$$

where $0 < r < 1$, $\beta \geq 0$ are constants depending on the location of the point set in Ω on which no analytic continuation of the conformal map ψ can be defined. L_n^d denotes the interpolation projection for the case when Ω is the unit disk. The theorem now follows from $\|L_n^d\| \leq (2/\pi) \ln n + 0.08 + o(1)$; see Geddes and Mason [11]. \square

Remark 2.1. $\|L_n\|$ has been computed numerically for ellipses (Geddes [10]), for polygons (Afolabi and Geddes [1]), and for a semidisk (Elliott [4]). Also for these latter contours with corners $\|L_n\|$ was found to grow like $\ln n$, $n \rightarrow \infty$. A survey of these and related results has recently been given by Mason [14]. \square

We next turn to the monomial basis

$$(2.6) \quad q_j(z) := c_j z^j, \quad j = 0, 1, 2, \dots,$$

where we choose real positive coefficients c_j so that $\|q_j\|_{\partial\Omega} = 1$.

Example 2.1. Let Ω be the unit disk. Then

$$(2.7) \quad \|M_n\| = \max_{\|a\|_\infty=1} \left\| \sum_{j=0}^n a_j z^j \right\|_{\partial\Omega} = n + 1,$$

$$\|M_n^{-1}\| = \left(\min_{\|a\|_\infty=1} \left\| \sum_{j=0}^n a_j z^j \right\|_{\partial\Omega} \right)^{-1} \geq 1.$$

From

$$\|Q_n\|_\infty \geq \left(\frac{1}{2\pi} \int_0^{2\pi} |Q_n(e^{i\theta})|^2 d\theta \right)^{1/2} = \left(\sum_{j=0}^n |a_j|^2 \right)^{1/2},$$

we obtain

$$\|M_n^{-1}\| \leq \left(\min_{\|a\|_\infty=1} \left(\sum_{j=0}^n |a_j|^2 \right)^{1/2} \right)^{-1} = 1.$$

Hence, $\text{cond } M = n + 1$, showing that the basis is fairly well-conditioned. \square

Example 2.2. Let $\partial\Omega$ be the ellipse $E(c, d) := \{z = x + iy, (x/c)^2 + (y/d)^2 = 1\}$, where c, d are real constants, $c \geq d$. The scaling factors are $c_k := c^{-k}$ which is the reason for introducing the new variable $w := z/c$. Then $w \in E(1, d/c)$, an ellipse

with foci $\pm f$, where $f = \sqrt{1 - (d/c)^2}$. The Chebyshev polynomials for $E(1, d/c)$ with highest-order coefficient 1 are

$$(2.8) \quad T_n(w, f) := \frac{f^n}{2^{n-1}} T_n\left(\frac{w}{f}\right) = \frac{1}{2^n} \left((w + \sqrt{w^2 - f^2})^n + (w - \sqrt{w^2 - f^2})^n \right),$$

$n = 0, 1, 2, \dots,$

where $T_n(\cdot)$ is the Chebyshev polynomial of the first kind. Write the points $w \in E(1, d/c)$ as

$$w = \frac{1}{2} \left(\left(1 + \frac{d}{c}\right) e^{i\theta} + \left(1 - \frac{d}{c}\right) e^{-i\theta} \right), \quad 0 \leq \theta < 2\pi,$$

and substitute into (2.8). This yields

$$T_n(w, f) = \left(\frac{1}{2} \left(1 + \frac{d}{c}\right) \right)^n e^{in\theta} + \left(\frac{1}{2} \left(1 - \frac{d}{c}\right) \right)^n e^{-in\theta} \sim \left(\frac{1}{2} \left(1 + \frac{d}{c}\right) \right)^n e^{in\theta},$$

$n \rightarrow \infty,$

and therefore,

$$(2.9) \quad \|T_n(w, f)\|_{E(1, d/c)} \sim \left(\frac{1}{2} \left(1 + \frac{d}{c}\right) \right)^n, \quad n \rightarrow \infty.$$

Let $\tau^{(n)} := (\tau_0^{(n)}, \tau_1^{(n)}, \dots, \tau_{[n/2]}^{(n)})$ denote the coefficient vector of $T_n(w, f)$,

$$T_n(w, f) = \sum_{k=0}^{[n/2]} \tau_k^{(n)} w^{n-2k}, \quad \text{where } [s] \text{ denotes the largest integer } \leq s.$$

For $f = 0$, one has $\tau_0^{(n)} = 1$, and $\tau_k^{(n)} = 0, k > 0$. This corresponds to the ellipse being a circle, a case that has been treated in the previous example. Assume $f > 0$. A lower bound for $\|M_n^{-1}\|$ is obtained from

$$(2.10) \quad \|M_n^{-1}\| = \left(\min_{\|a\|_\infty=1} \left\| \sum a_k w^k \right\|_{E(1, d/c)} \right)^{-1} \geq \frac{\|\tau^{(n)}\|_\infty}{\|T_n\|_{E(1, d/c)}}.$$

Previously Gautschi [9, Eq. (2.2)] has used this bound to obtain estimates for the numerical condition of the monomial basis on intervals $[-c, c]$. We use the estimate for $\|\tau^{(n)}\|_\infty$ derived in [9],

$$(2.11) \quad \|\tau^{(n)}\| \sim \sqrt{\frac{2}{\pi}} \cdot \frac{(1 + f^2)^{3/4}}{f} n^{-1/2} \left(\frac{1 + \sqrt{1 + f^2}}{2} \right)^n, \quad n \rightarrow \infty.$$

Combining (2.9–11) and noting that $\|M_n\| = n + 1$ yields

$$\|M_n^{-1}\| \geq \sqrt{\frac{2}{\pi}} \frac{(1 + f^2)^{3/4}}{f} n^{-1/2} \left(\frac{1 + \sqrt{1 + f^2}}{1 + d/c} \right)^n, \quad n \rightarrow \infty,$$

and

$$\text{cond } M_n \geq \sqrt{\frac{2}{\pi}} \frac{(1 + f^2)^{3/4}}{f} \sqrt{n} \left(\frac{1 + \sqrt{1 + f^2}}{1 + d/c} \right)^n, \quad n \rightarrow \infty,$$

where \geq stands for asymptotically larger than or equal to. Introduce $\gamma := d/c$ and $F(\gamma)$,

$$\frac{1 + \sqrt{1 + f^2}}{1 + d/c} = \frac{1 + \sqrt{2 - \gamma^2}}{1 + \gamma} =: F(\gamma).$$

We find that

$$\begin{aligned}
 F(0) &= 1 + \sqrt{2}, & \text{this corresponds to } \partial\Omega \text{ being the interval } [-c, c]. \\
 F(1) &= 1 \\
 F'(\gamma) &< 0, & 0 \leq \gamma \leq 1.
 \end{aligned}$$

Hence, if the ellipse has not degenerated into a circle, the condition number grows exponentially with n , and the growth rate increases with the ratio c/d . \square

Examples 2.1 and 2.2 demonstrate that when using the monomial basis for approximation on regions not very disk-like, the ill-conditioning of the basis may cause numerical difficulties.

3. A Modified Symm's Equation. Symm's integral equation and our modification of it differ in that

- (1) the modified equation is uniquely solvable for all scalings of the contour;
- (2) the modified integral equations for $\psi|_{\partial\Omega}$ (= the restriction of ψ to $\partial\Omega$) and for $\phi|_{\partial\Omega}$ differ only in their right-hand sides. Therefore both restrictions can be computed to practically the cost of one. We need $\psi|_{\partial\Omega}$ in order to allocate a set of Fejér points. $\phi|_{\partial\Omega}$ is needed when determining polynomial approximants to ϕ .

THEOREM 3.1. *Let $\theta(z) := \arg(\psi(z))$ for $z \in \partial\Omega$, and let γ denote the capacity of $\partial\Omega$. Then the unique solution $\{q^*, \sigma^*\}$, where $\sigma^* = \sigma^*(z) \in L^2(\partial\Omega)$ and q^* is a constant, of the modified Symm's equation*

$$(3.1) \quad \begin{cases} q + \int_{\partial\Omega} \ln|z - \zeta| \sigma(\zeta) |d\zeta| = 0, & z \in \partial\Omega, \\ \int_{\partial\Omega} \sigma(\zeta) |d\zeta| = 1, \end{cases}$$

satisfies

$$(3.2) \quad \theta(z) = 2\pi \int_{z_1}^z \sigma^*(\zeta) |d\zeta|, \quad q^* = -\ln \gamma.$$

Integration in (3.2) is along $\partial\Omega$ in the positive direction.

Proof. Introduce the parametric representation $s \rightarrow \zeta(s)$ of $\partial\Omega$, $0 \leq s < l$, $\zeta(0) = z_1$, where s is the arclength of $\partial\Omega$. For $\zeta \in \partial\Omega$ we may regard $\theta(\zeta)$ also as a function of s , $\tilde{\theta}(s) := \theta(\zeta(s))$. Gaier [5] shows that for rectifiable contours $\partial\Omega$, and $\gamma \neq 1$, the integral equation

$$(3.3) \quad \int_{\partial\Omega} \ln|z - \zeta| v(\zeta) |d\zeta| = 1 \quad \text{a.e.}$$

has the unique, integrable solution

$$v(\zeta) = \frac{1}{2\pi} \cdot \frac{1}{\ln \gamma} \tilde{\theta}'(s), \quad \zeta = \zeta(s),$$

where differentiation of $\tilde{\theta}$ is with respect to s . Therefore (3.1) has the unique solution

$$(3.4) \quad \begin{cases} \sigma^*(\zeta) = \frac{1}{2\pi} \tilde{\theta}'(s), & \zeta = \zeta(s), \\ q^* = -\ln \gamma. \end{cases}$$

In [15] we showed that (3.1) has a unique solution for any scaling of $\partial\Omega$, and that σ^* is invariant under scaling and q^* varies continuously with the scaling. Therefore (3.4) is the unique solution also for $\gamma = 1$. \square

THEOREM 3.2. *Let $\theta_i(z) := \arg(\phi(z))$ for $z \in \partial\Omega$ with $\theta_i(z_1) = 0$. Then the unique solution $\{q_i^*, \sigma_i^*\}$, where q_i^* is a constant and $\sigma_i^* = \sigma_i^*(z) \in L^2(\partial\Omega)$, of the system of integral equations*

$$(3.5) \quad \begin{cases} q_i + \int_{\partial\Omega} \ln|z - \zeta| |\sigma_i(\zeta)| d\zeta = \ln|z|, & z \in \partial\Omega, \\ \int_{\partial\Omega} \sigma_i(\zeta) |d\zeta| = 1, \end{cases}$$

satisfies

$$(3.6) \quad \begin{cases} \theta_i(z) = 2\pi \int_{z_1}^z \sigma_i^*(\zeta) |d\zeta|, \\ q_i^* = 0. \end{cases}$$

Integration is along $\partial\Omega$ in the positive direction.

Proof. As in the proof of Theorem 3.1, we use results of Gaier [5]. Let γ be the capacity of $\partial\Omega$ and assume that $\gamma \neq 1$. By Gaier [5],

$$(3.7) \quad \int_{\partial\Omega} \ln|z - \zeta| |\rho(\zeta)| d\zeta = \ln|z|, \quad z \in \Gamma,$$

has a unique solution $\rho^*(\zeta)$ which, moreover, satisfies

$$\int_{\partial\Omega} \rho^*(\zeta) |d\zeta| = 1.$$

Therefore $\sigma_i^* = \rho^*$ and $q_i^* = 0$ is a solution of (3.5). By [15], (3.5) has a unique solution, which is invariant under scaling. Therefore (3.6) also holds for curves with $\gamma = 1$. \square

In [16] we describe a fast Fourier-Galerkin method for the numerical solution of (3.1) and (3.5).

4. Polynomial Approximation of the Mapping ϕ . The numerical scheme is as follows:

1. Compute $\theta(z)$ and $\theta_i(z)$ defined in Theorems 3.1 and 3.2.
2. Determine $n + 1$ Fejér points z_k by solving the equations

$$\theta(z_k) = 2\pi \frac{k}{n+1}, \quad k = 0(1)n.$$

3. Compute the images w_k of z_k under ϕ by evaluating

$$w_k := \exp(i\theta_i(z_k)), \quad k = 0(1)n.$$

Determine a polynomial approximant $p_n(z)$ of $\phi(z)/z$ of degree $\leq n$ by interpolating $\phi(z)/z$ at the Fejér points z_k , $k = 0(1)n$. An approximation of $\phi(z)$ is given by

$$(4.1) \quad \phi_{n+1}(z) := zp_n(z).$$

The accuracy of ϕ_{n+1} depends on the accuracy of $\theta_i(z)$, $\theta(z)$, and on the interpolation error. In the examples, the approximation error in ϕ_{n+1} is generally determined by how accurately we have computed $\theta_i(z)$. The basis of Lagrange polynomials has not caused any problems due to ill-conditioning. We had no difficulty with polynomials of degree 80–100, despite the fact that all computations were carried out on a DEC-10 computer in single precision, i.e., with only 8 significant digits. For polynomials of such high degree the interpolation error was much smaller than the error in the computed $\theta_i(z)$. In the examples, we have computed $\theta_i(z)$ and chosen the degree of the polynomial so that the interpolation error and the error in θ_i are of the same order of magnitude.

When using a monomial basis, ill-conditioning caused difficulties. The monomial basis used is $\{(z/c)^k\}_{k=0}^n$, where $c := \max_{\partial\Omega} |z|$. The condition number of the interpolation matrix

$$(4.2) \quad A_n := [a_{jk}], \quad a_{jk} = (z_k/c)^j, \quad 0 \leq j, k \leq n,$$

is measured in the maximum norm.

Example 4.1. Let $\partial\Omega = \{z = 2 \cos(t) + i(\sin(t) + 2 \cos^3(t)), 0 \leq t < 2\pi\}$.

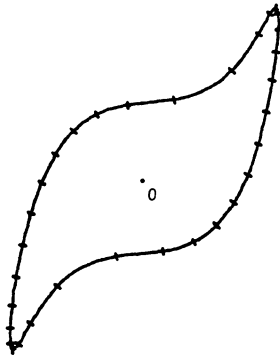


FIGURE 4.1

Figure 4.1 shows $\partial\Omega$ and 32 Fejér points marked with dashes.

n	Basis*	$\ \phi_n(z) - 1 \ _{\partial\Omega}$
16	M	$3 \cdot 10^{-2}$
32	M	$3 \cdot 10^{-3}$
64	L	$9 \cdot 10^{-6}$

*M = monomial basis, L = Lagrange basis.

The condition number of A_{32} was $5 \cdot 10^9$. To obtain a significantly smaller error, the use of a better-conditioned basis is necessary.

Approximations of $\phi(z)$ using polynomials have previously been computed by Ellacott [3], who determined polynomial approximants to $\ln(\phi(z)/z)$ by minimizing $\| \ln z - \text{Re}(\sum_0^n a_k z^k) \|_{\partial\Omega}$ w.r.t. the a_k using the Lawson algorithm. The computations were carried out for different n , and the smallest error in the approximate conformal maps obtained was $3 \cdot 10^{-2}$. \square

Example 4.2. Let $\partial\Omega = \{z = \cos(t) + ib \sin(t), 0 \leq t < 2\pi\}$, b is a constant, $0 < b < 1$. A conformal mapping ϕ from Ω to the unit disk, $\phi(0) = 0$, is given by an elliptic sine, see Kober [12, p. 177], which shows that the singularities of $\phi(z)$ closest to $\partial\Omega$ are poles at $\zeta_{1,2} = \pm i2b/\sqrt{1-b^2}$. As is shown in this and in the next example, it is sometimes possible to obtain good approximations of the singularities by locally approximating ϕ by a Möbius transformation ϕ_M . In the present example, we consider the Möbius transformation that maps $0 \rightarrow 0$, and the circle of curvature through ib onto the unit circle. It has a pole at

$$\zeta_1^* := i2b \frac{1 - \frac{1}{2}b^2}{1 - b^2}.$$

Note $\zeta_1^* = \zeta_1 + O(b^4)$ as $b \rightarrow 0$. We approximate $\phi(z)(z - \zeta_1^*)(z - \bar{\zeta}_1^*)$ by polynomials. This reduces the error between a factor 5 and 10 in the following examples, compared to approximating $\phi(z)$ by polynomials.

b	n	basis	$\ \phi_n(z) - 1 \ _{\partial\Omega}$
1/4	8	M	$1 \cdot 10^{-4}$
1/4	32	L	$1 \cdot 10^{-6}$
1/8	16	M	$8 \cdot 10^{-5}$
1/8	32	L	$5 \cdot 10^{-6}$

When computations were done with a Lagrange basis, a monomial basis could not be used due to the ill-conditioning of the matrix (4.2).

The introduction of “approximate singularities” appears quite useful in the approximation of analytic function. In a different conformal mapping method, which also uses polynomials to approximate ϕ , Levin et al. [13] reduced their approximation error by including extra basis functions with singularities at $\pm \zeta_1$ or at $\pm \zeta_1^*$.

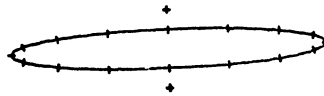


FIGURE 4.2

Figure 4.2 shows $\partial\Omega$ for $b = 1/8$. The crosses mark $\pm \zeta_1^*$. □

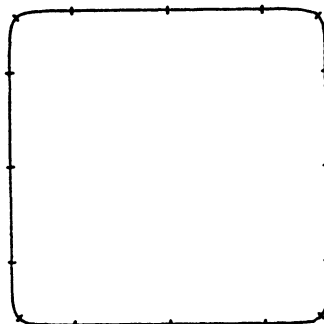


FIGURE 4.3

Example 4.3. Let $\partial\Omega = \{z = x + iy, x^4 + y^4 = 1\}$. Figure 4.3 shows $\partial\Omega$ and 16 Fejér points. $\|\phi_{16}(z) - 1\|_{\partial\Omega} = 6 \cdot 10^{-4}$. $\partial\Omega$ is circle-like enough to allow the power basis to be used even for ϕ_{32} and ϕ_{64} . \square

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