MATHEMATICS OF COMPUTATION
VOLUME 44, NUMBER 170
APRIL 1985, PAGES 435-441

On the Differential-Difference Properties of the Extended Jacobi Polynomials

By S. Lewanowicz

Abstract. We discuss differential-difference properties of the extended Jacobi polynomials

$$P_n(x) = {}_{p+2}F_q(-n, n+\lambda, a_p; b_q; x) \qquad (n = 0, 1, \ldots).$$

The point of departure is a corrected and reformulated version of a differential-difference equation satisfied by the polynomials $P_n(x)$, which was derived by Wimp (*Math. Comp.*, v. 29, 1975, pp. 577-581).

1. Introduction. Here we are concerned with the properties of the extended Jacobi polynomials [3, Vol. 1, Section 7.4],

$$(1)\qquad \begin{aligned} P_n(x) &= {}_{p+2}F_q\left(\begin{matrix} -n, n+\lambda, a_p \\ b_q \end{matrix} \middle| \, x \right) \\ &= \sum_{k=0}^{n} \frac{(-n)_k (n+\lambda)_k (a_p)_k}{k!(b_q)_k} x^k \qquad (n = 0, 1, \ldots), \end{aligned}$$

where

$$(\alpha)_k = \Gamma(\alpha + k)/\Gamma(\alpha)$$

is Pochhammer's symbol and the above contracted notation will be used throughout the paper:

$$f(a_p) = \prod_{i=1}^{p} f(a_i), \qquad f(b_q) = \prod_{j=1}^{q} f(b_j),$$

f being a given function. We assume that no a_i equals any b_j ($i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, q$) and set

$$(2)\qquad b_j = 1 \quad \text{for } j = q + 1.$$

Let

$$(3)\qquad \sigma = \max\{p + 1, q\}.$$

Wimp ([5]; see also [3, Vol. 2, Section 12.2]) has proved the following.

Received January 31, 1984.
1980 *Mathematics Subject Classification*. Primary 33A30.

THEOREM 1.1. *Let* λ, a_i, b_j $(i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, q + 1)$ *be complex constants such that none of the quantities* λ, $\lambda + 1 - b_j$ $(j = 1, 2, \ldots, q)$ *are negative integers or zero. Then the polynomials* $P_n(x)$, *see* (1), *satisfy the difference equation*

$$\sum_{m=0}^{\sigma+1} [C_m(n; \sigma + 1) + xD_m(n; \sigma + 1)] P_{n-m}(x) = 0 \qquad (n \geqslant \sigma + 1), \tag{4}$$

where

$$\begin{aligned} C_m(n; t) &= \frac{(n - m + 1)_m (2n + \lambda - 2m)_{2m} (n - m - 1 + b_{q+1})}{m!(n + \lambda - m)_m (2n + \lambda - m - t)_m (n - 1 + b_{q+1})} \\ &\quad \times {}_{q+3}F_{q+2}\left(\left. \begin{matrix} -m, 2n + \lambda - m - t, n - m + b_{q+1} \\ 2n + \lambda - 2m + 1, n - m - 1 + b_{q+1} \end{matrix} \right| 1 \right), \end{aligned} \tag{5}$$

$$\begin{aligned} D_m(n; t) &= \frac{(n - m + 1)_m (2n + \lambda - 2m)_{2m} (n - m + a_p)}{\Gamma(m)(n + \lambda - m)_m (2n + \lambda - m - t + 1)_{m-1} (n - 1 + b_{q+1})} \\ &\quad \times {}_{p+2}F_{p+1}\left(\left. \begin{matrix} 1 - m, 2n + \lambda - m - t + 1, n - m + 1 + a_p \\ 2n + \lambda - 2m + 1, n - m + a_p \end{matrix} \right| 1 \right). \end{aligned} \tag{6}$$

Moreover, these polynomials do not satisfy a nontrivial equation of the form (4) *of lower order than* $\sigma + 1$.

The next theorem supplements Theorem 1.1 in a certain particular case.

THEOREM 1.2 [2]. *If* $p + 1 = q$ *and* $x = 1$ *in* (1) *then the sequence* $P_n(1)$ *satisfies the recurrence relation*

$$\sum_{m=0}^{\sigma} [C_m(n; \sigma) + D_m(n; \sigma)] P_{n-m}(1) = 0 \qquad (n \geqslant \sigma). \tag{7}$$

The above result contains as a special case the recurrence relation given by Bailey [1] for

$${}_3F_2\left(\left. \begin{matrix} -n, n + \lambda, a_1 \\ b_1, b_2 \end{matrix} \right| 1 \right). \tag{8}$$

In this paper, our attention is focussed on another result of Wimp which is contained in the following theorem.

THEOREM 1.3 ([6]; see also [4, Section 5.13]). *Let* $\lambda \neq 1, 2, \ldots$ *and let the assumptions of Theorem* 1.1 *be satisfied. Then the polynomials* $P_n(x)$ *satisfy the differential-difference equation*

$$x(\delta x - \varepsilon) \frac{d}{dx} P_n(x) = \sum_{m=0}^{\sigma} [A_m(n) + xB_m(n)] P_{n-m}(x), \tag{9}$$

where

$$\delta = \begin{cases} 1 & (p+1 \geqslant q), \\ 0 & (p+1 < q), \end{cases} \qquad \varepsilon = \begin{cases} 0 & (p+1 > q), \\ 1 & (p+1 \leqslant q), \end{cases} \tag{10}$$

and

$$A_m(n) = \begin{cases} \alpha_m(n)\left[\dfrac{1}{m!}\displaystyle\sum_{k=0}^{m}\frac{(-m)_k(n-k-1+b_{q+1})}{k!(2n+\lambda-k-\sigma)_{\sigma+1-m}} - \varepsilon\right] & (m>0), \\ \omega(n) - \varepsilon\alpha_0(n) & (m=0), \end{cases} \tag{11}$$

$$B_m(n) = \begin{cases} \alpha_m(n)\left[\delta - \dfrac{1}{\Gamma(m)}\displaystyle\sum_{k=0}^{m-1}\frac{(1-m)_k(n-k-1+a_p)}{k!(2n+\lambda-k-\sigma)_{\sigma-m}}\right] & (m>0), \\ \delta\alpha_0(n) & (m=0), \end{cases} \tag{12}$$

and

$$\alpha_m(n) = \begin{cases} (-1)^m\dfrac{(n-m+1)_m(2n+\lambda-2m)}{(n+\lambda-m)_m} & (m>0), \\ n & (m=0), \end{cases} \tag{13}$$

$$\omega(n) = \frac{n-1+b_{q+1}}{(2n+\lambda-\sigma)_\sigma}. \tag{14}$$

Moreover, no equation of the type (9) *of lower order* $\sigma' < \sigma$ *exists.*

Note that the formulae (5) and (6) of the paper [6], defining the coefficients A_m and B_m, respectively, are not correct as can be seen by considering the case of $p+1=q=2$ and $x=1$ and observing that the resulting second-order (pure) difference equation for quantities (8) disagrees with that given by Bailey [1]. An inspection of Wimp's proof of the equation (9) (see [6, esp. the last paragraph of Section II]) reveals how the formulae should be corrected.

Wimp's theorem was reproduced in the book [4] (see Theorem 2 in Section 5.13.2). Unfortunately, in the Russian edition of that book (Mir Publs., Moscow, 1980) the list of errors was increased by six other misprints.

2. Alternative Forms for A_m and B_m. In Theorem 2.1, below, we give some alternative forms for the coefficients A_m and B_m, see (1.11) and (1.12), respectively. We need the following lemma.

LEMMA 2.1. *Let m, r, s be integers $\geqslant 0$. Let none of the complex constants γ, c_i ($i = 1, 2, \ldots, r$) be integers. Then the identity*

$$\sum_{k=0}^{m}\frac{(-m)_k(c_r-k)}{k!(\gamma-s-k)_{s+1-m}} = (-1)^m\frac{(\gamma-2m+1)_{2m-1}(c_r-m)}{(\gamma-s-m)_{s+m}} \times {}_{r+2}F_{r+1}\left(\begin{matrix} -m, \gamma-s-m, 1-m+c_r \\ \gamma+1-2m, c_r-m \end{matrix}\,\middle|\, 1\right) \tag{1}$$

holds.

Proof. Making use of some properties of Pochhammer's symbol [3, Vol. 1, Section 2.1], we obtain

$$
\begin{aligned}
(-1)^m & \frac{(\gamma - s - m)_{s+m}}{(\gamma - 2m + 1)_{2m-1}(c_r - m)} \sum_{k=0}^{m} \frac{(-m)_k(c_r - k)}{k!(\gamma - s - k)_{s+1-m}} \\
&= (-1)^m \sum_{k=0}^{m} \frac{(-m)_k(\gamma - s - m)_{m-k}(c_r - k)}{k!(\gamma - 2m + 1)_{m-k}(c_r - m)} \\
&= (-1)^m \sum_{k=0}^{m} \frac{(-m)_{m-k}(\gamma - s - m)_k(c_r - m + k)}{(m - k)!(\gamma - 2m + 1)_k(c_r - m)} \\
&= \sum_{k=0}^{m} \frac{(-m)_k(\gamma - s - m)_k(c_r - m + 1)_k}{k!(\gamma - 2m + 1)_k(c_r - m)_k} \\
&= {}_{r+2}F_{r+1}\left(\begin{matrix} -m, \gamma - s - m, c_r - m + 1 \\ \\ \gamma - 2m + 1, c_r - m \end{matrix} \middle| \, 1 \right). \quad \square
\end{aligned}
$$

THEOREM 2.1. *The equations* (1.11) *and* (1.12) *can be rewritten in the form*

$$
\begin{aligned}
&(2) \qquad A_m(n) = \omega(n)C_m(n; \sigma) - \varepsilon\alpha_m(n), \\
&(3) \qquad B_m(n) = \omega(n)D_m(n; \sigma) + \delta\alpha_m(n)
\end{aligned}
\qquad (m = 0, 1, \ldots, \sigma),
$$

respectively. Here the notation is that of (1.3), (1.5), (1.6), (1.10), (1.13) *and* (1.14).

Proof. Putting $r = q + 1$, $\gamma = 2n + \lambda$, $s = \sigma$, $c_i = n - 1 + b_i$ $(i = 1, 2, \ldots, q + 1)$ in (1), we obtain

$$
\begin{aligned}
\sum_{k=0}^{m} & \frac{(-m)_k(n - k - 1 - b_{q+1})}{k!(2n + \lambda - \sigma - k)_{\sigma+1-m}} \\
&= (-1)^m \frac{(2n + \lambda - 2m + 1)_{2m-1}(n - m - 1 + b_{q+1})}{(2n + \lambda - \sigma - m)_{\sigma+m}} \\
&\quad \times {}_{q+2}F_{q+1}\left(\begin{matrix} -m, 2n + \lambda - m - \sigma, n - m + b_{q+1} \\ \\ 2n + \lambda - 2m + 1, n - m - 1 + b_{q+1} \end{matrix} \middle| \, 1 \right) \\
&= \frac{m!\omega(n)}{\alpha_n(n)} C_m(n; \sigma).
\end{aligned}
$$

Now, it readily follows that the first part of (1.11) may be written in the form (2). Obviously, the second part of (1.11) can be written as

$$A_0(n) = \omega(n)C_0(n; \sigma) - \varepsilon\alpha_0(n)$$

because $C_0(n; \sigma) = 1$. Proceeding in a similar fashion, one arrives at (3). $\square$

Note that if $p + 1 = q$ and $x = 1$ then the equation (1.9) takes the form (1.7) as, in view of (2) and (3), we have

$$A_m(n) + B_m(n) = \omega(n)[C_m(n; \sigma) + D_m(n; \sigma)].$$

3. Further Differential-Difference Equations. With the aid of Theorems 1.3 and 2.1 we derive further differential-difference equations satisfied by the polynomials $P_n(x)$. We require one more lemma.

LEMMA 3.1. *We have*

$$(1)\quad C_m(n;\sigma)+\theta(n)C_{m-1}(n-1;\sigma)=C_m(n;\sigma+1)\qquad (m=1,2,\ldots,\sigma+1),$$

$$(2)\quad C_{\sigma+1}(n;\sigma)=\varepsilon\alpha_{\sigma+1}(n)/\omega(n),$$

$$(3)\quad D_m(n;\sigma)+\theta(n)D_{m-1}(n-1;\sigma)=D_m(n;\sigma+1)\qquad (m=1,2,\ldots,\sigma+1),$$

$$(4)\quad D_{\sigma+1}(n;\sigma)=-\delta\alpha_{\sigma+1}(n)/\omega(n),$$

where

$$(5)\quad \theta(n)=\frac{n\omega(n-1)}{(n+\lambda-1)\omega(n)},$$

and the notation used is that of (1.3), (1.5), (1.6), (1.10), (1.13) *and* (1.14).

Proof. Equations (1) and (3) can be checked by a straightforward calculation, using the definitions (1.5) and (1.6), respectively, and (5).

We prove the formula (2). We have

$$C_{\sigma+1}(n;\sigma)=\frac{(n-\sigma)_{\sigma+1}(2n+\lambda-2\sigma-2)_{2\sigma+2}(n-\sigma-2+b_{q+1})}{(\sigma+1)!(n+\lambda-\sigma-1)_{\sigma+1}(2n+\lambda-2\sigma-1)_{\sigma+1}(n-1+b_{q+1})}f,$$

where

$$f={}_{q+2}F_{q+1}\left(\begin{matrix}-\sigma-1,\, n-\sigma-1+b_{q+1}\\ n-\sigma-2+b_{q+1}\end{matrix}\,\middle|\, 1\right)$$

(see (1.5)). Now, observe that

$$\begin{aligned} f&=\frac{(-1)^{\sigma+1}(\sigma+1)!}{(n-\sigma-2+b_{q+1})} && (\sigma=q\geqslant p+1),\\ &=0 && (\sigma=p+1>q)\end{aligned}$$

(cf. [3, Vol. 1, Section 2.9]). Thus, for arbitrary p and q,

$$\begin{aligned} C_{\sigma+1}(n;\sigma)&=\varepsilon(-1)^{\sigma+1}\frac{(n-\sigma)_{\sigma+1}(2n+\lambda-\sigma)_{\sigma}(2n+\lambda-2-\sigma)}{(n+\lambda-\sigma-1)_{\sigma+1}(n-1+b_{q+1})}\\ &=\varepsilon\alpha_{\sigma+1}(n)/\omega(n).\end{aligned}$$

Equation (4) can be proved in a similar way. □

THEOREM 3.1. *Under the assumptions of Theorem* 1.3, *the polynomials* $P_n(x)$ *satisfy the equations*

$$(6)\quad x\left[\frac{dP_n(x)}{dx}+\frac{n}{n+\lambda-1}\frac{dP_{n-1}(x)}{dx}\right]=n\left[P_n(x)-P_{n-1}(x)\right],$$

$$(7)\quad x\frac{dP_n(x)}{dx}=\sum_{k=1}^{n}(-1)^k\frac{(-n)_k(2n+\lambda-2k)}{(1-\lambda-n)_k}P_{n-k}(x)+nP_n(x).$$

Proof. First observe that Eq. (1.9) can, by virtue of Theorem 2.1, be written in the form

$$x(\delta x-\varepsilon)\frac{dP_n(x)}{dx}=(\delta x-\varepsilon)\sum_{m=0}^{\sigma}\alpha_m(n)P_{n-m}(x)+\omega(n)\sum_{m=0}^{\sigma}\left[C_m(n;\sigma)+xD_m(n;\sigma)\right]P_{n-m}(x). \tag{8}$$

Now, replace in the above equation n by $n-1$, multiply the resulting equation by $n/(n+\lambda-1)$ and add to (8). The result is

$$\begin{aligned} x(\delta x-\varepsilon)&\left[\frac{dP_n(x)}{dx}+\frac{n}{n+\lambda-1}\frac{dP_{n-1}(x)}{dx}\right]\\ &=(\delta x-\varepsilon)\left[\sum_{m=0}^{\sigma}\alpha_m(n)P_{n-m}(x)+\frac{n}{n+\lambda-1}\sum_{m=1}^{\sigma+1}\alpha_{m-1}(n-1)P_{n-m}(x)\right]\\ &\quad+\omega(n)\Bigg\{\sum_{m=0}^{\sigma}\left[C_m(n;\sigma)+xD_m(n;\sigma)\right]P_{n-m}(x)\\ &\qquad+\theta(n)\sum_{m=1}^{\sigma+1}\left[C_{m-1}(n-1;\sigma)+xD_{m-1}(n-1;\sigma)\right]P_{n-m}(x)\Bigg\}. \end{aligned} \tag{9}$$

Using Lemma 3.1 and considering that $C_0(n;t)=1$, $D_0(n;t)=0$, we write the right-hand side of (9) in the form

$$(\delta x-\varepsilon)\sum_{m=0}^{\sigma+1}\beta_m(n)P_{n-m}(x)+\omega(n)\sum_{m=0}^{\sigma+1}\left[C_m(n;\sigma+1)+xD_m(n;\sigma+1)\right]P_{n-m}(x), \tag{10}$$

in which

$$\beta_m(n)=\begin{cases}\alpha_m(n)+\dfrac{n}{n+\lambda-1}\alpha_{m-1}(n-1) & (m>0),\\ \alpha_0(n) & (m=0),\end{cases}$$

or, in view of (1.13),

$$\beta_m(n)=\begin{cases}n & (m=0),\\ -n & (m=1),\\ 0 & (m>1).\end{cases}$$

By virtue of Theorem 1.1, the second sum of (10) is zero, therefore the right-hand side of (9) reduces to

$$(\delta x-\varepsilon)n\left[P_n(x)-P_{n-1}(x)\right],$$

and Eq. (6) follows.

Formula (6) is a first-order difference equation with respect to $U_n(x) = xP_n'(x)$. Using the well-known formula for the general solution of such an equation and remembering that $U_0(x) = 0$ (see (1.1)), we get (7). □

Of course, Eqs. (6) and (7) provide a generalization of the classical differential-difference formulae for the Jacobi polynomials, see [7, p. 262].

Introducing the notation

$$Q_n(x) = (-1)^n (\lambda)_n P_n(x)/n!$$

yields somewhat simpler forms of (6) and (7):

$$x\left[Q_n'(x) - Q_{n-1}'(x)\right] = nQ_n(x) + (n + \lambda - 1)Q_{n-1}(x),$$

$$xQ_n'(x) = \sum_{k=0}^{n-1} (2k + \lambda)Q_k(x) + nQ_n(x).$$

Institute of Computer Science
University of Wrocław
51-151 Wrocław, Poland

1. W. N. BAILEY, "Contiguous hypergeometric functions of the type ${}_3F_2(1)$," *Proc. Glasgow Math. Assoc.*, v. 2, 1954, pp. 62–65.

2. S. LEWANOWICZ, *Recurrence Relations for Hypergeometric Functions of Unit Argument*, Report N-127, Inst. Computer Sci., Univ. of Wrocław, 1983. Also *Math. Comp.*, v. 45, 1985. (To appear.)

3. Y. L. LUKE, *The Special Functions and Their Approximations*, Academic Press, New York, 1969.

4. Y. L. LUKE, *Mathematical Functions and Their Approximations*, Academic Press, New York, 1975.

5. JET WIMP, "Recursion formulae for hypergeometric functions," *Math. Comp.*, v. 22, 1968, pp. 363–373.

6. JET WIMP, "Differential-difference properties of hypergeometric polynomials," *Math. Comp.*, v. 29, 1975, pp. 577–581.

7. E. D. RAINVILLE, *Special Functions*, Macmillan, New York, 1960.