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The Generalized Integro-Exponential Function

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Abstract. The generalized integro-exponential function is defined in terms of the exponential integral (incomplete gamma function) and its derivatives with respect to order. A compendium of analytic results is given in one section. Rational minimax approximations sufficient to permit the computation of the first six first-order functions are reported in another section.

1. Introduction. The generalized integro-exponential function plays an important role in the theory of transport processes and fluid flow, yet no unified summary of its properties exists in the literature. First introduced by Van de Hulst [12] who gave a power series and numerical tabulation for some special cases, the function was abstracted by Chandrasekhar [8] and recently reviewed by Van de Hulst [13]. In the interim, tabulated values were presented by Stankiewicz [26] and Gussmann [11]; the latter as well as Abu-Shumays [2] and Kaplan [16] independently summarized some analytic properties. Recent work by this author [20]–[22] has reasserted the significant role played by this function in transport theory, and, by invoking the theory of Meijer's G-functions [18], many general properties of this function have been unveiled which were previously unknown. These were summarized in an Appendix to [20] and in a Table in a conference proceedings [22].

Since these summaries appeared, the problem of accurately evaluating this function numerically has arisen. For small values of the real independent variable x, power series evaluations work reasonably well with coefficients obtainable from the formulas given in [20] and [22]. For intermediate values of x, power series break down due to cancellation of significant digits when evaluating differences between large numbers, and for large values of x, asymptotic series do not do justice to the inherent accuracy available with modern computers.

The purpose of this paper is then twofold:

(1) Reiterate, derive and present a unified summary of the analytic properties of these functions only some of which can be found elsewhere;

(2) Proffer some approximations on which the numerical evaluation of a large number of these functions can be based.

2. Analysis. Let s and z be continuous (complex) variables and n and j represent nonnegative integers. Define the "generalized integro-exponential function" $E_s^j(z)$ by

$$E_s^j(z) = \frac{(-1)^j}{j!}\frac{\partial^j}{\partial s^j}E_s(z), \tag{2.1}$$

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where $E_s(z)$ is the usual* exponential integral generalized to $n = s$, or in terms of incomplete gamma functions

$$E_s(z) = z^{s-1}\Gamma(1 - s, z).$$

We have the integral representation

$$E_s^0(z) \equiv E_s(z) = \int_1^\infty t^{-s}\exp(-zt)\,dt; \tag{2.2}$$

for the case $s = n$ this reduces to the exponential integral about which a considerable literature revolves [1], [3], [4], [6], [9], [14], [19], [23], [25], [27], [29]. Apply Eq. (2.1) to (2.2), interchange the order of integration and differentiation, and obtain the integral representation

$$E_s^j(z) = \frac{1}{\Gamma(j+1)}\int_1^\infty (\log t)^j t^{-s}\exp(-zt)\,dt \tag{2.3}$$

which, when $s = n$, reduces to Gussmann's [11] definition $E_{n,j}(z) = E_n^j(z)$, or that of van de Hulst [12] $E_1^{(j)}(z) = E_1^{j-1}(z)$. Equation (2.3) may be integrated by parts (integrate t^{-s} and differentiate the remaining factors) to obtain the recursion formula

$$E_s^j(z) = \left(zE_{s-1}^j(z) - E_s^{j-1}(z)\right)/(1-s), \qquad s \neq 1, j \geqslant 0, \tag{2.4}$$

defining $E_s^{-1}(z) = \exp(-z)$ for the case $j = 0$.

To obtain a unified theory of this function and its properties, start with the known identification of $\exp(-z)$ in terms of Meijer's G-function [18, No. 6.2.1(1) and 6.4(1)]

$$\exp(-z) = G_{0,1}^{1,0}\left(z\,\middle|\,{; \atop 0;}\right) = \frac{1}{2\pi i}\int_{L_0}\Gamma(-t)z^t\,dt, \tag{2.5}$$

substitute into Eq. (2.2), and use a known [18, No. 5.6.4(6)] integration formula to obtain

$$E_s(z) = G_{1,2}^{2,0}\left(z\,\middle|\,{;s \atop 0, s-1;}\right) = \frac{1}{2\pi i}\int_{L_0}\frac{\Gamma(-t)z^t}{s-1-t}\,dt \tag{2.6a}$$

$$= \exp(-z)z^{s-1}\psi(s, s; z), \tag{2.6b}$$

where L_0 is the negatively directed contour enclosing both the nonnegative t-axis, and the pole at $t = s - 1$. The third term in the equality (2.6a) is the contour integral representation [18, No. 5.2(1)] of the G-function according to prescription. In Eq. (2.6b), ψ is a confluent hypergeometric function of the second kind. Operating on Eq. (2.6a) as in Eq. (2.1), we obtain the contour integral representation

$$E_s^j(z) = \frac{1}{2\pi i}\int_{L_0}\frac{\Gamma(-t)z^t}{(s-1-t)^{j+1}}\,dt \tag{2.7a}$$

$$= G_{j+1,j+2}^{j+2,0}\left(z\,\middle|\,{;s,\ldots,s \atop 0, s-1,\ldots,s-1;}\right). \tag{2.7b}$$

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* "Exponential integral" (of order s) refers to the function $E_s(z)$. Some authors [13], [1] use this nomenclature to mean $E_n(z)$, while others [18] mean $E_1(z)$, or $\mathrm{Ei}(z) = -E_1(-z)$ or both.

The result (2.7b) is obtained by identifying the contour integral (2.7a) according to prescription [18, No. 5.2(1)]; Eq. (2.7a) provides a means of generalizing the function to noninteger values of j. Set $j = -1$ in Eq. (2.7a), and with reference to Eq. (2.5) obtain

$$E_s^{-1}(z) = \exp(-z), \tag{2.8}$$

from which it is easy to see how Eq. (2.4) reduces to a well-known result [1, No. 5.1.14] when $j = 0$ and $s = n$.

The power series follows immediately from Eq. (2.7a) by evaluating the residues of the simple poles of $\Gamma(-t)$ and the multi-poles of order $j + 1$ when $s \neq n$ or $j + 2$ if $s = n$. The results are

$$E_s^j(z) = \sum_{l=0}^{\infty} \frac{(-z)^l}{(s-1-l)^{j+1} l!} + \frac{z^{s-1}(-1)^j}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \log^{j-l} z\, \Gamma(1-s)^{(l)}, \qquad s \neq n, \tag{2.9}$$

and

$$E_n^j(z) = {\sum_{\substack{l=0 \\ l \neq n-1}}^{\infty}}' \frac{(-z)^l}{(n-1-l)^{j+1} l!} + \frac{z^{n-1}(-1)^{j+n}}{(j+1)!} \sum_{l=0}^{j+1} \binom{j+1}{l} \log^{1+j-l} z\, \Psi_{l,n}, \tag{2.10}$$

where $\Gamma(1-s)^{(l)}$ means the lth derivative of $\Gamma(1-s)$ with respect to s, and

$$\Psi_{l,n} = \lim_{t \to n-1} \frac{\partial^l}{\partial t^l} \left(\frac{\Gamma(t-n+2)\Gamma(n-t)}{\Gamma(1+t)} \right). \tag{2.11}$$

Analytic expressions for $\Psi_{l,n}$ $(0 \leqslant l \leqslant 8)$ have been obtained through the use of computer algebra [28]; they are given in Table 1. General, but complicated expressions can be found in [2, Eq. (4.12)]. The result (2.10) is also given in [11] and [2, Eq. (4.7)]. Similar coefficients may be obtained for the functions $E_{-n}^j(z)$; however an alternative method for the evaluation of the latter set of functions will be given shortly.

It is of some interest to obtain other integral representations. From Eq. (2.7b) we recognize the G-function as an Euler transform [18, No. 5.6.4(6)]:

$$E_s^j(z) = \int_1^{\infty} t^{-s} E_s^{j-1}(zt)\, dt, \qquad j \geqslant 0, \tag{2.12}$$

from which the name derives. The case $j = 1$, $s = 1$ has been obtained elsewhere [17] in the form of a power series; (2.12) may also be found in [2]. An alternate expression for Eq. (2.12) is the repeated integral form:

$$E_s^j(z) = \int_1^{\infty} dy_1 y_1^{-s} \int_1^{\infty} dy_2 \cdots \int_1^{\infty} dy_{j+1} y_{j+1}^{-s} \exp\left(-z \prod_{i=1}^{j+1} y_i \right). \tag{2.13}$$

From (2.2), (2.3) and (2.12), we find

$$E_s^j(z) = \frac{1}{\Gamma(j)} \int_1^{\infty} (\log t)^{j-1} t^{-s} E_s(zt)\, dt, \qquad j > 0, \tag{2.14a}$$

from which it may be shown that

$$\text{(2.14b)}\quad \int_1^\infty E_s^l(zt)t^{-s}\log^k t\,dt = \frac{\Gamma(k+1)}{\Gamma(l)}\int_1^\infty E_s^{k+1}(zt)t^{-s}\log^{l-1} t\,dt, \qquad l>0,\, k\geqslant 0.$$

Finally, using (2.14a) and (2.14b) we find the remarkable representation

$$\text{(2.14c)}\quad E_s^j(z) = \frac{1}{\Gamma(j-l)}\int_1^\infty E_s^l(zt)t^{-s}\log^{j-l-1} t\,dt \quad \forall\ 0\leqslant l\leqslant j-1$$

by a process of induction, employing (2.19). This generalizes both (2.12) and (2.14a).

Starting from Frullani's integral

$$\log t = \int_0^\infty dv(\exp(-v)-\exp(-tv))/v$$

and Eq. (2.3), the order of integration may, with care, be interchanged, giving the Laplace transform

$$\text{(2.15)}\qquad E_s^1(z) = \int_0^\infty dv\exp(-v)[E_s(z)-\exp(v)E_s(z+v)]/v.$$

After applying Eq. (2.1) to Eq. (2.15), a more general form is found:

$$\text{(2.16)}\quad E_s^{j+1}(z) = \frac{1}{(j+1)}\int_0^\infty dv\exp(-v)[E_s^j(z)-\exp(v)E_s^j(z+v)]/v, \qquad j\geqslant 0.$$

A number of G-function representations may also be derived. From Eq. (2.2) and known integration formula [18, No. 5.6.2(18)], we have

$$\text{(2.17)}\qquad E_s(z) = \exp(-z)G_{1,2}^{2,1}\left(z\,\middle|\,\begin{matrix}0;\\0,s-1;\end{matrix}\right)/\Gamma(s).$$

Starting with Eq. (2.7a), substitute $t := 2t$ and use the duplication formula for the gamma function to find [18, No. 5.2(1)]

$$\text{(2.18)}\quad E_s^j(z) = \frac{2^{-j-1}}{\sqrt{\pi}}G_{j+1,j+3}^{j+3,0}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix};(s+1)/2,\dots,(s+1)/2\\0,\frac{1}{2},(s-1)/2,\dots,(s-1)/2;\end{matrix}\right).$$

From Eq. (2.3) the differentiation rules are easy to uncover

$$\text{(2.19a)}\qquad \frac{\partial^k}{\partial z^k}E_s^j(z) = (-1)^k E_{s-k}^j(z),$$

$$\text{(2.19b)}\qquad \frac{\partial^m}{\partial s^m}E_s^j(z) = \frac{(-1)^m\Gamma(m+j+1)}{\Gamma(j+1)}E_s^{m+j}(z),$$

and from Eqs. (2.19) and (2.4) we discover that $E_s^j(z)$ satisfies

$$\text{(2.20a)}\qquad (s-1)\frac{\partial}{\partial s}E_s^j(z) - z\frac{\partial^2}{\partial s\partial z}E_s^j(z) + (j+1)E_s^j(z) = 0$$

as well as [18, No. 5.8(1)]

$$\text{(2.20b)}\qquad \left(1+\frac{\partial}{\partial z}\right)\left(z\frac{\partial}{\partial z}-s+1\right)^{j+1}E_s^j(z) = 0.$$

A short table of integrals is included as Appendix A.

In applications [27] we need to evaluate $E_{-n}^{j}(x)$, $n > 0$, $x = \mathrm{Re}(z) > 0$. Integrating by parts in Eq. (2.3) leads to the result

$$E_0^j(z) = E_1^{j-1}(z)/z, \qquad j \geqslant 0.$$

Repeated application of Eq. (2.4) gives

$$E_{-n}^{j}(z) = \frac{\Gamma(n+1)}{z^n}\left(\sum_{k=0}^{n-1} \frac{z^k E_{-k}^{j-1}(z)}{\Gamma(k+2)} + E_0^j(z)\right). \tag{2.21}$$

Similarly, repeated application of Eq. (2.21) leads to the simpler form

$$E_{-n}^{j}(z) = \frac{\Gamma(n+1)}{z^{n+1}}\left[\exp(-z)\sum_{l=0}^{n-j} \frac{z^l}{l!}\xi_{l,n}^j + \sum_{m=1}^{j} \xi_{0,n}^{m-1} E_1^{j-m}(z)\right], \tag{2.22}$$

where the (precomputed) constants $\xi_{l,n}^j$ are given analytically by

(2.23)

$$\xi_{l,n}^j = \sum_{k_j=0}^{n-j-l} \sum_{k_{j-1}=0}^{k_j} \cdots \sum_{k_1=0}^{k_2} \left(\frac{1}{k_j+l+j}\right)\left(\frac{1}{k_{j-1}+l+j-1}\right)\cdots\left(\frac{1}{k_1+l+1}\right),$$

$$\xi_{n-j,n}^j = \frac{(n-j)!}{n!}, \qquad \xi_{l,n}^0 = 1, \qquad \xi_{l,n}^j = 0 \quad \text{if } n < j + l.$$

The derivation of Eq. (2.22) comes by proving its truth for the cases $j = 1$ and $j = 2$ and proceeding by induction, using the identities

$$\xi_{0,n}^j = \sum_{k=0}^{n-1}\left(\frac{1}{k+1}\right)\xi_{0,k}^{j-1}, \tag{2.24a}$$

$$\xi_{l,n}^j = \sum_{k=0}^{n-l-j}\left(\frac{1}{k+l+j}\right)\xi_{l,k+l+j-1}^{j-1}. \tag{2.24b}$$

From Eq. (2.22) all the $E_{-n}^{j}(z)$ can be obtained for $n \geqslant 0$ if the functions $E_1^m(z)$ are known for $0 \leqslant m \leqslant j-1$. Approximations for $E_1^m(z)$ are given in Section 3. A lengthy form for the evaluation of $E_n^j(x)$ for $n > 1$ in terms of $E_1^j(x)$ is given by Gussmann [11, Eq. (A.18) to (A.21′)]. Note that Eq. (2.22) is numerically stable since all its terms are positive.

For large values of real z, the first few terms in the asymptotic series of $E_{-n}^{j}(x)$ are given by the first finite sum in Eq. (2.22). From Eq. (2.7b) and the theory [18, Section 5.10] of G-functions, the asymptotic series is theoretically known. The first terms are

$$E_s^j(z) \sim \exp(-z) z^{-(j+1)}[1 - (j+1)(j+2s)/(2z) + \cdots]. \tag{2.25}$$

A more general result can be obtained by integrating Eq. (2.12) repeatedly by parts:

$$E_n^{j+1}(z) = \frac{1}{z}\sum_{k=0}^{\infty} \frac{(-1)^k}{z^k}\frac{\Gamma(n+k)}{\Gamma(n)} E_{n+k+1}^j(z). \tag{2.26}$$

The claimed asymptotic formula is then ($\mathrm{Re}(z) \to +\infty$)

$$E_n^j(z) \sim \exp(-z) z^{-j-1} \sum_{l=0}^{P} \frac{(-1)^l}{z^l}\frac{\Gamma(n+l+j)}{\Gamma(n)}\xi_{n-1,n+l+j-1}^j, \tag{2.27}$$

where P is an integer. This can be proven by establishing its truth for the case $j = 0$ and $j = 1$, using in Eq. (2.26) the well-known [1, Eq. 5.1.51] result

$$E_{n+k+1}(z) \sim \exp(-z)z^{-1} \sum_{m=0}^{P} \frac{(-1)^m}{z^m} \frac{\Gamma(n+k+m+1)}{\Gamma(n+k+1)},$$

and proceeding by induction, employing the identity

$$\sum_{k=0}^{l} \frac{1}{(n+k)} \xi^j_{n+k,n+l+j} = \xi^{j+1}_{n-1,n+l+j} \tag{2.28}$$

in Eq. (2.27).

Finally, from Eq. (2.9) we have the special value

$$E_s^j(0) = \left(\frac{1}{s-1}\right)^{j+1}, \qquad \mathrm{Re}(s) > 1. \tag{2.29}$$

3. Rational Minimax Approximations to $E_1^j(x)$. With the aid of the computer code REMES2 [15], it proved possible to obtain rational minimax approximations to $E_1^j(x)$ for x real, $1 \leqslant j \leqslant 6$, and hence $E_{-n}^j(x)$ with $1 \leqslant j \leqslant 7$, $n > 0$ because of Eq. (2.22). The region $0 \leqslant x$ was conveniently divided into three ranges for the purpose of both evaluating the master functions and obtaining efficient fits.

3a. *Small* x. In this range, we evaluate $E_1^j(x)$ according to the power series (2.10) in the form

$$E_1^j(x) = P_k(x)/Q_l(x) + P_{1_j}(\log x), \tag{3.1}$$

where $P_k(x)/Q_l(x)$ is the (k, l) minimax fit to the infinite series plus the constant $(l = j + 1)$ term of the second sum in Eq. (2.10). $P_{1_j}(\log x)$ is a polynomial in $\log(e^\gamma x)$ of order j obtainable from Table 1 by means of computer algebra. Although no proof has been found, for $1 \leqslant j \leqslant 7$, this polynomial $P_{1_j}(\log x)$ can be written

$$P_{1_j} = \sum_{l=1}^{j+1} u_{j,l} v^l, \tag{3.2}$$

where

$$v = \log(e^\gamma x) \tag{3.3}$$

and

$$u_{j,l} = -u_{j-1,l-1}/l, \qquad l \geqslant 1, \tag{3.4}$$

with the constants $u_{j,0}$ given in Table 2, and γ being Euler's constant. The upper end of this range (X_U) was chosen to coincide with a point smaller than the first positive zero of $E_1^j(x) - P_{1_j}(\log x)$. All master function calculations were done in 120 bit double-precision arithmetic ($\sim$ 28 s.d.) and the approximation chosen was the lowest $(k + l)$ entry in the Walsh array with more than 14 s.d. accuracy. The coefficients of the (k, l) minimax fits are given in Table 3.

3b. *Intermediate* x. In this range, $X_U \leqslant x \leqslant 10$, the master functions were obtained from the power series (2.10), with an exponential weighting such that

$$E_1^j(x) = \exp(-x) P_k(x)/Q_l(x). \tag{3.5}$$

TABLE 1

Analytic expressions for $\Psi_{l,n}$ *in Eq.* (2.10)

```
  CPSI(1)=GAMIN    $    CPSI(2)=GAMIN*(-PSIN)

  CPSI(3)=+GAMIN*(PSIN**2-PSIN1+2.*ZETA2)

  CPSI(4)=+GAMIN*(3.*PSIN*PSIN1-6.*PSIN*ZETA2-PSIN**3-PSIN2)

  CPSI(5)=
1 +GAMIN*(4.*PSIN*PSIN2-6.*PSIN**2*PSIN1+12.*PSIN**2*ZETA2+PSIN**4
1 -12.*PSIN1*ZETA2+3.*PSIN1**2-PSIN3+12.*ZETA2**2+12.*ZETA4)

  CPSI(6)=
1 +GAMIN*(60.*PSIN*PSIN1*ZETA2-15.*PSIN*PSIN1**2+5.*PSIN*PSIN3-60.
1 *PSIN*ZETA2**2-60.*PSIN*ZETA4-10.*PSIN**2*PSIN2+10.*PSIN**3*PSIN
1 1-20.*PSIN**3*ZETA2-PSIN**5+10.*PSIN1*PSIN2-20.*PSIN2*ZETA2-PSIN
1 4)

  CPSI(7)=
1 +GAMIN*(-60.*PSIN*PSIN1*PSIN2+120.*PSIN*PSIN2*ZETA2+6.*PSIN*PSIN
1 4-180.*PSIN**2*PSIN1*ZETA2+45.*PSIN**2*PSIN1**2-15.*PSIN**2*PSIN
1 3+180.*PSIN**2*ZETA2**2+180.*PSIN**2*ZETA4+20.*PSIN**3*PSIN2-15.
1 *PSIN**4*PSIN1+30.*PSIN**4*ZETA2+PSIN**6+15.*PSIN1*PSIN3-180.*PS
1 IN1*ZETA2**2-180.*PSIN1*ZETA4+90.*PSIN1**2*ZETA2-15.*PSIN1**3+10
1 .*PSIN2**2-30.*PSIN3*ZETA2-PSIN5+360.*ZETA2*ZETA4+120.*ZETA2**3+
1 240.*ZETA6)

  CPSI(8)=
1 +GAMIN*(-105.*PSIN*PSIN1*PSIN3+1260.*PSIN*PSIN1*ZETA2**2+1260.*P
1 SIN*PSIN1*ZETA4-630.*PSIN*PSIN1**2*ZETA2+105.*PSIN*PSIN1**3-70.*
1 PSIN*PSIN2**2+210.*PSIN*PSIN3*ZETA2+7.*PSIN*PSIN5-2520.*PSIN*ZET
1 A2*ZETA4-840.*PSIN*ZETA2**3-1680.*PSIN*ZETA6+210.*PSIN**2*PSIN1*
1 PSIN2-420.*PSIN**2*PSIN2*ZETA2-21.*PSIN**2*PSIN4+420.*PSIN**3*PS
1 IN1*ZETA2-105.*PSIN**3*PSIN1**2+35.*PSIN**3*PSIN3-420.*PSIN**3*Z
1 ETA2**2-420.*PSIN**3*ZETA4-35.*PSIN**4*PSIN2+21.*PSIN**5*PSIN1-4
1 2.*PSIN**5*ZETA2-PSIN**7+420.*PSIN1*PSIN2*ZETA2+21.*PSIN1*PSIN4-
1 105.*PSIN1**2*PSIN2+35.*PSIN2*PSIN3-420.*PSIN2*ZETA2**2-420.*PSI
1 N2*ZETA4-42.*PSIN4*ZETA2-PSIN6)

  CPSI(9)=
1 +GAMIN*(-3360.*PSIN*PSIN1*PSIN2*ZETA2-168.*PSIN*PSIN1*PSIN4+840.
1 *PSIN*PSIN1**2*PSIN2-280.*PSIN*PSIN2*PSIN3+3360.*PSIN*PSIN2*ZETA
1 2**2+3360.*PSIN*PSIN2*ZETA4+336.*PSIN*PSIN4*ZETA2+8.*PSIN*PSIN6+
1 420.*PSIN**2*PSIN1*PSIN3-5040.*PSIN**2*PSIN1*ZETA2**2-5040.*PSIN
1 **2*PSIN1*ZETA4+2520.*PSIN**2*PSIN1**2*ZETA2-420.*PSIN**2*PSIN1*
1 *3+280.*PSIN**2*PSIN2**2-840.*PSIN**2*PSIN3*ZETA2-28.*PSIN**2*PS
1 IN5+10080.*PSIN**2*ZETA2*ZETA4+3360.*PSIN**2*ZETA2**3+6720.*PSIN
1 **2*ZETA6-560.*PSIN**3*PSIN1*PSIN2+1120.*PSIN**3*PSIN2*ZETA2+56.
1 *PSIN**3*PSIN4-840.*PSIN**4*PSIN1*ZETA2+210.*PSIN**4*PSIN1**2-70
1 .*PSIN**4*PSIN3+840.*PSIN**4*ZETA2**2+840.*PSIN**4*ZETA4+56.*PSI
1 N**5*PSIN2-28.*PSIN**6*PSIN1+56.*PSIN**6*ZETA2+PSIN**8-280.*PSIN
1 1*PSIN2**2+840.*PSIN1*PSIN3*ZETA2+28.*PSIN1*PSIN5-10080.*PSIN1*Z
1 ETA2*ZETA4-3360.*PSIN1*ZETA2**3-6720.*PSIN1*ZETA6-210.*PSIN1**2*
1 PSIN3+2520.*PSIN1**2*ZETA2**2+2520.*PSIN1**2*ZETA4-840.*PSIN1**3
1 *ZETA2+105.*PSIN1**4+56.*PSIN2*PSIN4+560.*PSIN2**2*ZETA2-840.*PS
1 IN3*ZETA2**2-840.*PSIN3*ZETA4+35.*PSIN3**2-56.*PSIN5*ZETA2-PSIN7
1 +13440.*ZETA2*ZETA6+10080.*ZETA2**2*ZETA4+1680.*ZETA2**4+5040.*Z
1 ETA4**2+10080.*ZETA8)
```

Note: CPSI(ℓ+1) = $\Psi_{\ell,n}$ ZETAℓ = $\zeta(\ell)$ PSIN = $\psi(n)$ GAMIN = $1./\Gamma(n)$ PSINℓ = $\psi^{\ell}(n)$

TABLE 2

Coefficients of the polynomials $P_{1_j}(\log x)$

j	coefficient $u_{j,0}$ (analytic)	(numerical)
−1	1	1
0	0	0
1	$\pi^2/12$	.82246 70334 24113
2	$-\zeta(3)/3$	−.40068 56343 86531
3	$\pi^4/160$	.60880 68189 62515
4	$-\zeta(3)\pi^2/36 - \zeta(5)/5$	−.53693 62760 78223
5	$\zeta(3)^2/18 + 61\pi^6/120960$	.56510 20417 78833
6	$-\zeta(3)\pi^4/480 - \zeta(5)\pi^2/60 - \zeta(7)/7$	−.55855 78221 73266
7	$\zeta(3)^2\pi^2/216 + \zeta(3)\zeta(5)/15 + 1261\pi^8/29030400$	.56127 49829 66818

also $P_{1_j} = \sum_{k=1}^{j+1} u_{j,k}v^k$ where $v = \log(e^{\gamma}x)$ and $u_{j,k} = -u_{j-1,k-1}/k$

example: $P_{13}(v) = \zeta(3)v/3 + \pi^2v^2/24 + v^4/24$

TABLE 3.1

```
E¹1(X)=P(X)/Q(X)*(X-X11)+P11(LOG(X)

   X11=.91446 48659 7277
IN THE RANGE  0.000.LE.X.LE. 1.000

RATIONAL APPROXIMATION IS R(3,4)
WITH PRECISION= 14.46 DIGITS

  P00  ( 6)  -.376341 57664 06520
  P01  ( 5)  -.635070 75949 96412
  P02  ( 4)  -.538768 20580 92452
  P03  ( 2)  -.584198 45557 93210
  Q00  ( 6)   .418437 62176 08333
  Q01  ( 6)   .121793 37377 23431
  Q02  ( 5)   .132630 73906 04789
  Q03  ( 3)   .551375 17596 10943
  Q04  ( 1)   .100000 00000 00000
```

```
FIT TO E¹1(X)=EXP(-X)*P(X)/Q(X)

IN THE RANGE  1.000.LE.X.LE.10.000

RATIONAL APPROXIMATION IS R(6,8)
WITH PRECISION= 14.09 DIGITS

  P00  ( 1)  -.338130 76971 20518
  P01  ( 0)   .891028 00665 24613
  P02  ( 3)   .211811 97726 44376
  P03  ( 3)   .283328 58929 93606
  P04  ( 3)   .120102 26074 96511
  P05  ( 2)   .192587 99072 06423
  P06  ( 0)   .999999 85034 28858
  Q00  ( 0)  -.400917 37819 68016
  Q01  ( 2)  -.106209 25463 66475
  Q02  ( 2)   .419023 64057 58286
  Q03  ( 3)   .569128 54287 46715
  Q04  ( 3)   .964803 08155 39357
  Q05  ( 3)   .616101 47030 82591
  Q06  ( 3)   .175879 22239 84149
  Q07  ( 2)   .222587 85540 75955
  Q08  ( 1)   .100000 00000 00000
```

```
E¹1(X)=EXP(-X)/X**2*P(Z)/Q(Z) ; Z=1/X

IN THE RANGE  0.000.LE.Z.LE.  .100

RATIONAL APPROXIMATION IS R(4,6)
WITH PRECISION= 15.13 DIGITS

  P00  ( -2)  .368896 16541 46809
  P01  (  0)  .111317 82274 12989
  P02  (  1)  .111573 42563 49138
  P03  (  1)  .430645 72608 32509
  P04  (  1)  .522913 13537 89389
  Q00  ( -2)  .368896 16541 46811
  Q01  (  0)  .122384 70770 37284
  Q02  (  1)  .144230 98012 73127
  Q03  (  1)  .747160 29601 50458
  Q04  (  2)  .168869 92689 15484
  Q05  (  2)  .135627 18800 19350
  Q06  (  1)  .100000 00000 00000
```

TABLE 3.2

FIT TO E^2_1(X)=P(X)/Q(X) +P12(LOG(X)

IN THE RANGE 0.000.LE.X.LE. .400

RATIONAL APPROXIMATION IS R(3,4)
WITH PRECISION= 16.20 DIGITS

P00	(6)	.216257 42293 79812
P01	(6)	-.493516 16951 20742
P02	(5)	-.784750 37376 89728
P03	(4)	-.381379 38950 85943
Q00	(6)	-.539718 43355 22324
Q01	(6)	-.115308 01225 47753
Q02	(4)	-.773816 80672 69915
Q03	(3)	-.122836 12906 79186
Q04	(1)	.100000 00000 00000

FIT TO E^2_1(X)=EXP(-X)*P(X)/Q(X)

IN THE RANGE .400.LE.X.LE. 2.000

RATIONAL APPROXIMATION IS R(5,8)
WITH PRECISION= 14.90 DIGITS

P00	(0)	.534274 07496 24268
P01	(2)	.172848 04128 51346
P02	(2)	.602765 67841 77444
P03	(2)	.582331 73035 43360
P04	(2)	.147552 63525 88969
P05	(0)	.999989 73572 67234
Q00	(-1)	.170899 18403 89456
Q01	(1)	.295254 33350 91985
Q02	(2)	.651380 89668 07230
Q03	(3)	.315340 67127 43816
Q04	(3)	.573546 38415 89567
Q05	(3)	.444897 80147 56000
Q06	(3)	.147788 79716 87537
Q07	(2)	.207544 71393 35389
Q08	(1)	.100000 00000 00000

FIT TO E^2_1(X)=EXP(-X)*P(X)/Q(X)

IN THE RANGE 2.000.LE.X.LE.10.000

RATIONAL APPROXIMATION IS R(5,8)
WITH PRECISION= 15.55 DIGITS

P00	(3)	.340433 44179 16653
P01	(3)	.876064 53575 85969
P02	(3)	.676598 41845 71783
P03	(3)	.203399 00769 19242
P04	(2)	.246893 21393 49231
P05	(0)	.999999 81273 60712
Q00	(2)	.206456 40742 55514
Q01	(4)	.120509 38588 74781
Q02	(4)	.511338 37482 96754
Q03	(4)	.778642 56924 73543
Q04	(4)	.547543 10587 36117
Q05	(4)	.194264 79236 68901
Q06	(3)	.352535 91968 35532
Q07	(2)	.306893 01010 78642
Q08	(1)	.100000 00000 00000

E^2_1(X)=EXP(-X)/X**3*P(Z)/Q(Z) ; Z=1/X

IN THE RANGE 0.000.LE.Z.LE. .100

RATIONAL APPROXIMATION IS R(5,5)
WITH PRECISION= 13.58 DIGITS

P00	(-3)	.325912 28045 69083
P01	(-2)	.976173 27349 42300
P02	(-1)	.894008 30387 59118
P03	(0)	.249087 11994 57106
P04	(-1)	.139742 26502 81342
P05	(-2)	-.401189 07964 43870
Q00	(-3)	.325912 28045 69169
Q01	(-1)	.117172 06417 64891
Q02	(0)	.148297 13910 05923
Q03	(0)	.802097 98704 24226
Q04	(1)	.174325 30019 86266
Q05	(1)	.100000 00000 00000

In the cases $j = 5$ and 6, multiple-precision (48 digit) arithmetic [7] was employed within the master functions to overcome large losses in accuracy near the upper part of the range, and again, the lowest $(k + l)$ approximation with more than 14 digit accuracy was chosen for inclusion in Table 3. In some cases, it was beneficial to split this range into two to obtain more efficient approximations.

3c. *Large x*. It proved fruitless to approximate $E_1^j(x)$ with the asymptotic formula (2.27) in the range $10 \leqslant x$. Accordingly, (k, l) minimax fits $R_j(1/x)$ were procured such that

$$E_1^j(x) = \exp(-x)x^{-j-1}R_j(1/x), \tag{3.6}$$

TABLE 3.3

FIT TO E^3_1(X)=P(X)/Q(X)+P13(LOG(X))

IN THE RANGE 0.000.LE.X.LE. .600

RATIONAL APPROXIMATION IS R(3,4)
WITH PRECISION= 15.65 DIGITS

P00	(6)	-.839003 82605 47848
P01	(7)	.119755 65275 64264
P02	(6)	.242909 62547 68230
P03	(5)	.108775 12253 31120
Q00	(7)	-.137811 17423 82510
Q01	(6)	-.296572 26101 03168
Q02	(5)	-.174055 92800 54023
Q03	(3)	-.157386 94906 03826
Q04	(1)	.100000 00000 00000

FIT TO E^3_1(X)=EXP(-X)*P(X)/Q(X)

IN THE RANGE .600.LE.X.LE.10.000

RATIONAL APPROXIMATION IS R(5,9)
WITH PRECISION= 13.37 DIGITS

P00	(2)	.157208 97422 57519
P01	(3)	.128368 95938 74056
P02	(3)	.270646 30105 87470
P03	(3)	.148490 63549 05473
P04	(2)	.228734 95971 24294
P05	(0)	.999998 89216 94849
Q00	(0)	.320385 15630 95068
Q01	(2)	.675468 89806 76635
Q02	(3)	.861412 12489 99020
Q03	(4)	.372041 91499 30951
Q04	(4)	.677596 67313 99825
Q05	(4)	.552376 35994 49519
Q06	(4)	.213346 91178 79909
Q07	(3)	.392231 03684 45133
Q08	(2)	.328733 75166 04005
Q09	(1)	.100000 00000 00000

E^3_1(X)=EXP(-X)/X**4*P(Z)/Q(Z) ; Z=1/X

IN THE RANGE 0.000.LE.Z.LE. .100

RATIONAL APPROXIMATION IS R(3,8)
WITH PRECISION= 14.31 DIGITS

P00	(-2)	-.174019 20356 94571
P01	(-1)	-.709402 14529 05107
P02	(0)	-.905606 10531 27851
P03	(1)	-.359745 25082 90533
Q00	(-2)	-.174019 20356 94563
Q01	(-1)	-.883421 34886 03280
Q02	(1)	-.164111 11311 14874
Q03	(2)	-.137785 23506 45778
Q04	(2)	-.514428 97426 13443
Q04	(2)	-.514428 97426 13443
Q05	(2)	-.685703 73818 80260
Q06	(2)	-.162601 49416 93719
Q07	(0)	-.957694 76113 15230
Q08	(1)	.100000 00000 00000

where $0 \leqslant 1/x \leqslant 0.1$. To evaluate the master functions, Eq. (2.12) was transformed into the integral form

$$E_1^j(x) = \exp(-x)x^{-j-1}\int_0^\infty dt\,\exp(-t)(1+t/x)^{-j-1}R_{j-1}\left(\frac{1}{t+x}\right), \tag{3.7}$$

where R_{j-1} was a 19 digit rational minimax fit to $x^j \exp(x) E_1^{j-1}(x)$ according to Eq. (3.6) obtained especially for that purpose. The integration was a 400-point Gauss-Laguerre calculation in double-precision arithmetic [5]. To begin the bootstrap, high-precision fits to $E_1(x)$ due to Cody and Thacher [9] in this range were employed. The fit in the Walsh table with smallest $(k + l)$ and greatest accuracy over 14 s.d. was chosen, and is reported in Table 3.

TABLE 3.4

FIT TO $E^{4}{}_{1}$(X)=P(X)/Q(X)+P14(LOG(X))

IN THE RANGE 0.000.LE.X.LE. .500

RATIONAL APPROXIMATION IS R(3,3)
WITH PRECISION= 15.13 DIGITS

P00	(6)	.921856 63967 05455
P01	(7)	-.164159 41458 60650
P02	(6)	-.112522 80878 61094
P03	(3)	-.607131 03056 26315
Q00	(7)	-.171688 27675 48834
Q01	(6)	-.140218 91431 54977
Q02	(4)	-.161995 44008 14657
Q03	(1)	.100000 00000 00000

FIT TO $E^{4}{}_{1}$(X)=EXP(-X)*P(X)/Q(X)

IN THE RANGE .500.LE.X.LE. 2.000

RATIONAL APPROXIMATION IS R(6,6)
WITH PRECISION= 13.75 DIGITS

P00	(-1)	.190007 72856 75969
P01	(0)	.145255 03101 02014
P02	(-1)	.812092 21046 70916
P03	(-2)	-.725683 52768 34330
P04	(-3)	.559368 64008 87053
P05	(-4)	-.321053 49168 10291
P06	(-6)	.996850 83917 42953
Q00	(-4)	.383398 00706 66617
Q01	(-1)	.950926 03238 66574
Q02	(1)	.175435 39262 79244
Q03	(1)	.827856 65799 05180
Q04	(2)	.117836 83566 14576
Q05	(1)	.609668 97953 70546
Q06	(1)	.100000 00000 00000

FIT TO $E^{4}{}_{1}$(X)=EXP(-X)*P(X)/Q(X)

IN THE RANGE 2.000.LE.X.LE.10.000

RATIONAL APPROXIMATION IS R(4,8)
WITH PRECISION= 13.93 DIGITS

P00	(2)	.353537 04817 63005
P01	(2)	.293820 63659 48671
P02	(2)	.145436 89922 68919
P03	(0)	.999939 71607 90340
P04	(-6)	.520751 56693 51248
Q00	(1)	-.223458 69506 60041
Q01	(3)	.204802 30991 30036
Q02	(4)	.183199 07073 94430
Q03	(4)	.333915 80415 91970
Q04	(4)	.276686 23525 75097
Q05	(4)	.128585 73754 40614
Q06	(3)	.297624 44755 03964
Q07	(2)	.295401 74603 13003
Q08	(1)	.100000 00000 00000

$E^{4}{}_{1}$(X)=EXP(-X)/X**5*P(Z)/Q(Z) ; Z=1/X

IN THE RANGE 0.000.LE.Z.LE. .100

RATIONAL APPROXIMATION IS R(5,7)
WITH PRECISION= 14.84 DIGITS

P00	(-5)	.395133 17174 71982
P01	(-3)	.204661 41023 99156
P02	(-2)	.346003 84617 74784
P03	(-1)	.204274 58543 62413
P04	(-1)	.211759 92902 12236
P05	(-2)	-.228430 28733 96219
Q00	(-5)	.395133 17174 71976
Q01	(-3)	.263931 38600 20275
Q02	(-2)	.672752 62012 18175
Q03	(-1)	.828969 69188 14699
Q04	(0)	.515915 51447 03679
Q05	(1)	.155462 70935 57096
Q06	(1)	.206789 80159 55259
Q07	(1)	.100000 00000 00000

In all three ranges, the function $E_1^j(x)$ computed by rational (k, l) fits in single precision was compared with the multiple-precision master routines for 20,000 pseudo-random values of x and $1/x$. The largest fractional error noted was 1.1×10^{-13}.

Finally, the functions $E^j_{-n}(x)$ were calculated for $1 \leqslant j \leqslant 7$, $0 \leqslant n \leqslant 7$ and pseudo-random values of x and $1/x$ according to Eq. (2.22). Comparisons were made with the same functions calculated via Eq. (2.3) using the single-precision routine DCADRE [10]. No significant discrepancies were found at the relative error level of 1×10^{-12}, and it was observed that rational minimax methods require $\sim$ 150 times less computation time than does DCADRE. For the case $j = 0$, a pre-existant library function based on [9] was used, although better methods are known [3]. Agreement with Gussman's tabulated values [11] was total.

TABLE 3.5

FIT TO E^5_1(X)=P(X)/Q(X)+P15(LOG(X))

IN THE RANGE 0.000.LE.X.LE. .300

RATIONAL APPROXIMATION IS R(3,2)
WITH PRECISION= 15.30 DIGITS

P00	(3)	.838743 38519 74414
P01	(4)	-.142760 40312 21125
P02	(2)	-.880503 85483 46258
P03	(0)	-.556431 96858 95740
Q00	(4)	.148423 35068 49911
Q01	(3)	.100211 06179 41411
Q02	(1)	.100000 00000 00000

FIT TO E^5_1(X)=EXP(-X)*P(X)/Q(X)

IN THE RANGE .300.LE.X.LE. 2.000

RATIONAL APPROXIMATION IS R(5,10)
WITH PRECISION= 15.44 DIGITS

P00	(-3)	.500522 58419 15039
P01	(0)	.179471 65122 08398
P02	(1)	.222234 03877 87114
P03	(1)	.379468 68767 06738
P04	(0)	.991769 17992 22935
P05	(-3)	.144274 11935 15928
Q00	(-4)	.322199 80730 27793
Q01	(-3)	.967191 28566 38765
Q02	(1)	.114091 98728 01536
Q03	(2)	.364550 65926 78242
Q04	(3)	.320690 08398 60978
Q05	(3)	.977517 80646 43698
Q06	(4)	.126115 41159 04314
Q07	(3)	.736669 39202 94918
Q08	(3)	.201906 35278 47405
Q09	(2)	.245286 16349 83681
Q10	(1)	.100000 00000 00000

FIT TO E^5_1(X)=EXP(-X)*P(X)/Q(X)

IN THE RANGE 2.000.LE.X.LE.10.000

RATIONAL APPROXIMATION IS R(4,9)
WITH PRECISION= 14.90 DIGITS

P00	(2)	.185231 13891 15313
P01	(2)	.169071 89408 09323
P02	(2)	.163822 45019 47477
P03	(0)	.999973 45809 73645
P04	(-6)	.178967 24703 86040
Q00	(0)	-.332104 62172 40376
Q01	(3)	.127144 45160 39896
Q02	(4)	.223717 64184 09195
Q03	(4)	.632007 63231 39911
Q04	(4)	.803518 43255 59348
Q05	(4)	.632932 10036 44392
Q06	(4)	.259400 04237 29686
Q07	(3)	.479992 93424 76704
Q08	(2)	.373802 66825 57554
Q09	(1)	.100000 00000 00000

E^5_1(X)=EXP(-X)/X**6*P(Z)/Q(Z) ; Z=1/X

IN THE RANGE 0.000.LE.Z.LE. .100

RATIONAL APPROXIMATION IS R(5,7)
WITH PRECISION= 14.05 DIGITS

P00	(-6)	.854783 10464 47894
P01	(-4)	.624139 17717 77879
P02	(-2)	.138649 42989 29153
P03	(-2)	.948336 88932 79712
P04	(-3)	-.423100 66195 40255
P05	(-4)	-.370718 93388 82177
Q00	(-6)	.854783 10464 47817
Q01	(-4)	.803643 62915 37229
Q02	(-2)	.279890 57603 97328
Q03	(-1)	.462603 61191 05448
Q04	(0)	.380244 88250 95635
Q05	(1)	.147127 35379 71468
Q06	(1)	.229031 45317 90874
Q07	(1)	.100000 00000 00000

Table 3.6

FIT TO $E^{6}{}_{1}$(X)=P(X)/Q(X)+P16(LOG(X))

IN THE RANGE 0.000.LE.X.LE. .100

RATIONAL APPROXIMATION IS R(2,2)
WITH PRECISION= 15.16 DIGITS

P00	(4)	-.828583 83172 51790
P01	(5)	.145297 02796 48708
P02	(3)	.486899 73737 68510
Q00	(5)	.148343 42996 06228
Q01	(3)	.545404 94731 07867
Q02	(1)	.100000 00000 00000

FIT TO $E^{6}{}_{1}$(X)=EXP(-X)*P(X)/Q(X)

IN THE RANGE .100.LE.X.LE. 2.000

RATIONAL APPROXIMATION IS R(6,12)
WITH PRECISION= 14.72 DIGITS

P00	(-3)	.271478 83310 87788
P01	(-1)	.195102 35413 21396
P02	(0)	.321565 23084 02674
P03	(1)	.138382 85866 93074
P04	(1)	.205248 36427 90331
P05	(0)	.989908 76978 43590
P06	(-3)	.167302 21625 00549
Q00	(-7)	-.771598 59706 76959
Q01	(-2)	.168613 64843 91203
Q02	(0)	.219806 16317 34924
Q03	(1)	.810987 78546 02619
Q04	(3)	.114098 17140 23387
Q05	(3)	.688229 02344 19855
Q06	(4)	.205074 91268 54280
Q07	(4)	.325821 03449 06946
Q08	(4)	.282942 94513 61645
Q09	(4)	.130505 73635 81354
Q10	(3)	.296709 78069 04683
Q11	(2)	.296971 60052 15741
Q12	(1)	.100000 00000 00000

FIT TO $E^{6}{}_{1}$(X)=EXP(-X)*P(X)/Q(X)

IN THE RANGE 2.000.LE.X.LE.10.000

RATIONAL APPROXIMATION IS R(7,7)
WITH PRECISION= 14.29 DIGITS

P00	(1)	.109840 75266 90525
P01	(-2)	-.628986 68217 39343
P02	(-3)	.352279 00224 82096
P03	(-4)	-.166654 15521 17304
P04	(-6)	.631863 11472 37745
P05	(-7)	-.177898 75053 81043
P06	(-9)	.327229 37007 14598
P07	(-11)	-.292513 40388 70733
Q00	(0)	.174130 46755 89622
Q01	(1)	.789054 90506 24093
Q02	(3)	.294284 05239 39408
Q03	(3)	.980479 98971 36415
Q04	(3)	.851988 71787 21233
Q05	(3)	.259397 92046 62330
Q06	(2)	.293770 63084 51261
Q07	(1)	.100000 00000 00000

$E^{6}{}_{1}$(X)=EXP(-X)/X**7*P(Z)/Q(Z) ; Z=1/X

IN THE RANGE 0.000.LE.Z.LE. .100

RATIONAL APPROXIMATION IS R(6,7)
WITH PRECISION= 14.98 DIGITS

P00	(-6)	.117928 10446 90701
P01	(-5)	.887532 43497 71136
P02	(-3)	.210604 43465 37132
P03	(-2)	.154902 34850 49091
P04	(-3)	-.627848 92801 53793
P05	(-3)	.202298 30636 59744
P06	(-4)	-.409048 68342 68896
Q00	(-6)	.117928 10446 90700
Q01	(-4)	.121773 11274 90613
Q02	(-3)	.487180 40530 95863
Q03	(-2)	.965568 34655 76481
Q04	(0)	.100200 50733 04699
Q05	(0)	.527459 47929 77240
Q06	(1)	.125439 02841 63380
Q07	(1)	.100000 00000 00000

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Appendix A

A Short Table of Integrals

From the G-function representation (2.7) and integrals listed in [21], it is possible to generate a table of very general integrals, which contain many useful results as special cases, all of which are believed to be new. A short list follows.

(A.1)
$$\int_0^\infty dt\, t^\gamma E_\mu^j(\alpha t) E_\nu^k(\beta t) = \beta^{-\gamma-1} G_{j+k+3,j+k+3}^{j+2,k+2}\left(\frac{\alpha}{\beta} \middle| \begin{matrix} -\gamma, 1-\nu-\gamma,\ldots,1-\nu-\gamma;\mu,\ldots,\mu \\ 0,\mu-1,\ldots,\mu-1;-\nu-\gamma,\ldots,-\nu-\gamma \end{matrix} \right).$$

Special case ($j = -1$)

(A.1a)
$$\int_0^\infty t^\gamma e^{-\alpha t} E_\nu^k(\beta t)\, dt = \beta^{-\gamma-1} \sum_{l=0}^{\infty} \frac{\Gamma(1+\gamma+l)}{\Gamma(1+l)} \left(\frac{1}{\nu+\gamma+l} \right)^{k+1} \left(\frac{-\alpha}{\beta} \right)^l \quad \text{if } |\alpha| \leqslant |\beta|.$$

Special case ($k = \gamma = 0$, $\nu = \alpha = \beta = 1$, $\mu = m$; [20], [24])

(A.1b)
$$\int_0^\infty E_1(t) E_m^j(t)\, dt = \frac{(-1)^j j!}{m^{j+1}} \left\{ \frac{1}{2} \sum_{l=0}^{j} \frac{(-1)^l}{l!} \left(\frac{m}{2} \right)^l \left[\psi^l\left(\frac{m+1}{2} \right) - \psi^l\left(\frac{m}{2} \right) \right] + \log 2 \right\}.$$

(A.2)
$$\int_0^\infty t^\gamma E_\nu^k(\beta t)\, dt = \beta^{-\gamma-1} \Gamma(1+\gamma)(\nu+\gamma)^{-k-1}.$$

(A.3)
$$\int_0^\infty t^{\gamma-1}(t+\beta)^{-\sigma} E_\nu^j(zt)\, dt = \frac{\beta^{\gamma-\sigma}}{\Gamma(\sigma)} G_{j+2,j+3}^{j+3,1}\left(\beta z \middle| \begin{matrix} 1-\gamma;\nu,\ldots,\nu \\ \sigma-\gamma,0,\nu-1,\ldots,\nu-1; \end{matrix} \right).$$

Special case ($\gamma = 1$, $\sigma = 1+\nu$, $j = 0$)

(A.3a)
$$\int_0^\infty (t+\beta)^{-\nu-1} E_\nu(zt)\, dt = \beta^{-\nu} \exp(\beta z) E_\nu(\beta z)/\nu.$$

Special case ($\sigma = 2$, $\gamma = 2-\nu$, $j = 0$)

(A.3b)
$$\int_0^\infty t^{1-\nu}(t+\beta)^{-2} E_\nu(zt)\, dt = \exp(\beta z) z^{\nu-1} \Gamma(2-\nu) E_{2-\nu}(\beta z),$$

(A.4)
$$\int_0^\infty {}_2F_1\left(\begin{matrix} \alpha,\beta \\ \gamma \end{matrix} \middle| -wt \right) t^\delta E_\nu^j(zt)\, dt = \frac{z^{-\delta}}{w} \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} G_{j+3,j+4}^{j+4,1}\left(\frac{z}{w} \middle| \begin{matrix} 0;\gamma-1,\nu+\delta,\ldots,\nu+\delta \\ \delta,\nu+\delta-1,\ldots,\nu+\delta-1,\alpha-1,\beta-1; \end{matrix} \right).$$

Special case ($\alpha = \beta = \delta = 1, \gamma = 2$)

(A.4a) $$\int_0^\infty \log(1 + wt)E_\nu^j(zt)\,dt = \frac{1}{z}G_{j+2,j+3}^{j+3,1}\left(\frac{z}{w}\middle|\begin{matrix}0;1+\nu,\ldots,1+\nu\\0,0,\nu,\ldots,\nu;\end{matrix}\right).$$

Special case ($\beta = \alpha - \frac{1}{2}, \gamma = 2\alpha, \delta = \alpha - 1 - \nu$)

(A.4b) $$\int_0^\infty [1 + \sqrt{1 + wt}\,]^{1-2\alpha}t^{\alpha-1-\nu}E_\nu^j(zt)\,dt$$
$$= \frac{z^{\nu+1-\alpha}\left(\alpha - \frac{1}{2}\right)}{w\sqrt{\pi}}G_{j+2,j+3}^{j+3,1}\left(\frac{z}{w}\middle|\begin{matrix}0;2\alpha-1,\alpha-1,\ldots,\alpha-1\\\alpha-1-\nu,\alpha-\frac{3}{2},\alpha-2,\ldots,\alpha-2;\end{matrix}\right),$$

(A.5) $$\int_0^\infty \cos(\alpha t)E_\nu^j(\beta t)\,dt$$
$$= \frac{2^{-j-1}}{\alpha}G_{j+2,j+2}^{j+2,1}\left(\frac{\beta^2}{\alpha^2}\middle|\begin{matrix}\frac{1}{2};(\nu+1)/2,\ldots,(\nu+1)/2\\\frac{1}{2},(\nu-1)/2,\ldots,(\nu-1)/2\end{matrix}\right)$$

(A.5a) $$= \frac{2^{-j-1}}{\beta}\sum_{l=0}^\infty\left(\frac{1}{\nu/2 + l}\right)^{j+1}\left(\frac{-\alpha^2}{\beta^2}\right)^l \quad \text{if } |\alpha| \leqslant |\beta|,$$

(A.6) $$\int_0^\infty \sin(\alpha t)E_\nu^j(\beta t)\,dt = \frac{2^{-j-1}}{\alpha}G_{j+2,j+2}^{j+2,1}\left(\frac{\beta^2}{\alpha^2}\middle|\begin{matrix}0;(\nu+1)/2,\ldots,(\nu+1)/2\\0,(\nu-1)/2,\ldots,(\nu-1)/2;\end{matrix}\right)$$

(A.6a) $$= \frac{2^{-j-1}\alpha}{\beta^2}\sum_{l=0}^\infty\left(\frac{1}{(1+\nu)/2 + l}\right)^{j+1}\left(\frac{-\alpha^2}{\beta^2}\right)^l \quad \text{if } |\alpha| \leqslant |\beta|.$$

(A.7) $$\int_1^\infty t^{-\alpha}(t-1)^{\alpha-\beta-1}E_s^j(zt)\,dt$$
$$= \Gamma(\alpha - \beta)G_{j+2,j+3}^{j+3,0}\left(z\middle|\begin{matrix};s,\ldots,s,\alpha\\0,\beta,s-1,\ldots,s-1;\end{matrix}\right).$$

Special case ($s = -n, \alpha = 1 - l, \beta = l$)

$$\int_1^\infty t^{l-1}E_{-n}^j(zt)\,dt$$

(A.7a) $$= \frac{\Gamma(n+1)}{z^{n+1}}\left[\sum_{k=0}^{n-j}\frac{z^k}{k!}\xi_{k,n}^j E_{n+2-k-l}(z) + E_{n+1-l}(z)\sum_{m=1}^{j}\xi_{0,n}^{j-m}\left(\frac{1}{n-l}\right)^m - \sum_{m=1}^{j}\sum_{i=0}^{m-1}(n-l)^{i-m}\xi_{0,n}^{j-m}E_1^i(z)\right]$$

if $l \neq n, n+1$,

(A.7b) $$\int_1^\infty t^{n-1}E_{-n}^j(zt)\,dt$$
$$= \frac{\Gamma(n+1)}{z^{n+1}}\left[\sum_{k=0}^{n-j}\frac{z^k}{k!}\xi_{k,n}^j E_{2-k}(z) + \sum_{m=1}^{j}\xi_{0,n}^{m-1}E_1^{j-m+1}(z)\right].$$

For the special case $l = n + 1$, see Eq. (2.12).

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