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# Some Periodic Continued Fractions With Long Periods

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**Abstract.** Let $p(D)$ be the period length of the continued fraction for $\sqrt{D}$. Under the extended Riemann Hypothesis for $\mathcal{Q}(\sqrt{D})$ one would expect that $p(D) = O(D^{1/2} \log\log D)$. In order to test this it is necessary to find values of $D$ for which $p(D)$ is large. This, in turn, requires that we be able to find solutions to large sets of simultaneous linear congruences. The University of Manitoba Sieve Unit (UMSU), a machine similar to D. H. Lehmer's DLS-127, was used to find such values of $D$. For example, if $D = 46257585588439$, then $p(D) = 25679652$. Some results are also obtained for the Voronoi continued fraction for $\sqrt[3]{D}$.

**1. Introduction.** Let $D$ be any positive integer. In Williams [7] it was pointed out that if $D$ is square-free, then $p(D)$, the period length of the continued fraction expansion of $\sqrt{D}$, should be bounded above by an expression of the form $cD^{1/2} \log\log D$. In fact, if

$$f(D) = \begin{cases} D^{1/2} \log\log D & \text{for } D \equiv 1 \pmod 8, \\ D^{1/2} \log\log 4D & \text{otherwise,} \end{cases}$$

we should have

$$G(D) = p(D)/f(D) < k + o(1) \tag{1.1}$$

under the extended Riemann Hypothesis for $\zeta_{\mathcal{K}}$ when $\mathcal{K} = \mathcal{Q}(\sqrt{D})$. Here $k = 3.7012$, but we expect by Lévy's Law that the smaller value $12e^{\gamma} \log 2/\pi^2 \approx 1.50103$ could be used for $k$. In [7] values of $D$ $(< 2 \times 10^9)$ were examined in order to find large values of $G(D)$. The largest value found was that of $G(D) = 1.040452$ for $D = 1492180699$. In this paper we describe a further attempt to find values of $D$ for which $G(D)$ is large. We also describe some analogous work done in the case of Voronoi's algorithm in $\mathcal{Q}(\sqrt[3]{D})$.

**2. Numerical Results.** A glance at the results and tables given in [7] reveals that, in order to find values of $D$ for which $G(D)$ is likely to be large, one should examine integers of the form $q$ or $2q$, where $q$ is a prime and $q \equiv -1 \pmod 4$. Further, if $r_i$ is the $i$th odd prime, one should also attempt to have the Legendre symbols

$$(D/r_i) = 1 \qquad (i = 1, 2, 3, \ldots, n) \tag{2.1}$$

for as large a value of $n$ as possible. Thus, for each such $r_i$ we would want $D$ to belong to one of the $(r_i - 1)/2$ congruence classes such that $(D/r_i) = 1$. To find such values of $D$ requires that we find solutions of large numbers of simultaneous

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linear congruences, a problem best solved by using a number sieve (see Lehmer [3]). In Patterson and Williams [5] a very fast version of such a device, called the University of Manitoba Sieve Unit (UMSU), is described. This machine will solve such systems of congruences at the rate of $1.33 \times 10^8$ trials at a solution per second.

We searched for values of $D$ of four different types:

$$\begin{array}{lll} \text{(i)} & D \equiv 3 \pmod 8 & D \text{ prime}, \\ \text{(ii)} & D \equiv 7 \pmod 8 & D \text{ prime}, \\ \text{(iii)} & D \equiv 6 \pmod 8 & D/2 \text{ prime}, \\ \text{(iv)} & D \equiv 1 \pmod 8 & D \text{ prime}. \end{array}$$

We examined values of $D$ of type (iv) to determine whether values of $G(D)$ would, as predicted by Shanks, tend to catch up to the larger values obtained for the other types of $D$. For each value of $n$ $(1, 2, 3, \ldots)$ UMSU was programmed to search for the first $m$ (at least) values of $D$ of a given type. For $n \leqslant 32$ $(r_{32} = 137)$, we used 50 as our value of $m$. Because of the amount of time needed to go farther, we cut this value down to 10 for $33 \leqslant n \leqslant 36$ $(r_{36} = 157)$. In addition, for $n = 37$ we used $m = 7$ and $m = 8$ for $D$ of type (i) and type (iii), respectively. For $D$ of type (iv) we used $m = 10$ for $n = 37$ and $m = 4$ with $n = 38$ for $D$ of type (ii).

After these numbers had been found, a job requiring many weeks of continuous use of UMSU, we computed the corresponding continued fraction period lengths $p(D)$ and the values of $G(D)$. We summarize our results in the four tables given below. We give only those numbers $D > 2 \times 10^9$. Also, we print $D$ and its corresponding $p$- and $G$-values only when $G(D)$ exceeds the value of $G(d)$ for all of our computed values of $d$ of the same type with $d < D$.

In Table 5 we present the values of $D$, from among those found by UMSU, with the largest $p(D)$ values. We give five such numbers for each $D$-type.

On examining these tables, we see that the values of $G(D)$ are certainly growing sufficiently slowly for (1.1) to hold. Further, the values of $G(D)$ for $D$ of type (iv) seem to be slowly catching up to those values for the other $D$-types.

TABLE 1. $D$ – Type (i)

| D | p(D) | G(D) |
|---|---|---|
| 2186009851 | 151838 | 1.037297 |
| 2287905811 | 155710 | 1.039131 |
| 7528121899 | 288198 | 1.043420 |
| 30738225571 | 603178 | 1.061828 |
| 614886781051 | 2794390 | 1.063448 |
| 1260977393659 | 4081590 | 1.076694 |
| 55400066448211 | 28076486 | 1.078532 |

In Table 6, we extend part of Table I of Lehmer, Lehmer and Shanks [4]. That is, for various values of $n$ we give the least prime $D \equiv 1 \pmod 8$ such that (2.1) holds. We also mention here that D. H. Lehmer had already found previously (but not published) the first six lines of this table.

TABLE 2. $D$ – Type (ii)

| D | p(D) | G(D) |
|---|---|---|
| 2763423391 | 170804 | 1.034456 |
| 4912298119 | 230048 | 1.036883 |
| 5097972751 | 234768 | 1.038196 |
| 12095524039 | 366384 | 1.040132 |
| 19672399231 | 471320 | 1.042810 |
| 24880707679 | 536964 | 1.053362 |
| 50151351559 | 772360 | 1.058250 |
| 62324011759 | 864408 | 1.059728 |
| 492210358039 | 2519212 | 1.074069 |
| 4944598510471 | 8181752 | 1.075383 |
| 22542868742839 | 17739532 | 1.076772 |
| 46257585588439 | 25679652 | 1.081244 |

TABLE 3. $D$ – Type (iii)

| D | p(D) | G(D) |
|---|---|---|
| 2340752254 | 157036 | 1.035754 |
| 7636279366 | 288766 | 1.037853 |
| 8813799094 | 312690 | 1.044133 |
| 8932573654 | 316434 | 1.049406 |
| 31416841054 | 611088 | 1.063790 |
| 6730689687166 | 9585044 | 1.076654 |
| 13518648471574 | 13732410 | 1.081381 |

TABLE 4. $D$ – Type (iv)

| D | p(D) | G(D) |
|---|---|---|
| 18901431649 | 433383 | .996329 |
| 22945498489 | 479525 | .997981 |
| 23258723401 | 483919 | 1.000142 |
| 28467424441 | 540685 | 1.007395 |
| 37312059409 | 625233 | 1.013966 |
| 40094470441 | 653345 | 1.021500 |
| 163965430561 | 1348681 | 1.024427 |
| 192052219969 | 1473213 | 1.032023 |
| 2570329924369 | 5552441 | 1.033038 |
| 2871842842801 | 5924695 | 1.041624 |
| 8103297298321 | 10135403 | 1.049695 |
| 457165855430761 | 79417945 | 1.055462 |

**3. Some Analogous Results for $\sqrt[3]{D}$.** It is well-known that the regular continued fraction expansion of $\sqrt[3]{D}$ is never periodic; however, Voronoi's [6] continued fraction is periodic for cubic irrationalities. Let $\mathcal{K} = \mathcal{Q}(\sqrt[3]{D})$ be the pure cubic field formed by adjoining $\sqrt[3]{D}$ to the rationals $\mathcal{Q}$, and let $\Delta$ be the discriminant of $\mathcal{K}$. Then, if $D$ is cube-free and $D = ab^2$ with $(a, b) = 1$, we have

$$\Delta = \begin{cases} -3a^2b^2 & \text{when } a^2 \equiv b^2 \pmod 9, \\ -27a^2b^2 & \text{otherwise.} \end{cases}$$

If $\varepsilon_0$ is the fundamental unit of $\mathcal{K}$, $R$ $(= \log \varepsilon_0)$ the regulator of $\mathcal{K}$, and $P$ the period of Voronoi's continued fraction, then by (8.3) of Williams [8], we get

$$R > [P/4] \log 2. \tag{3.1}$$

Unfortunately, we do not yet have a rule like Lévy's for this case, but it seems from empirical evidence that

$$R \approx vP, \tag{3.2}$$

TABLE 5

| Type | D | p(D) | G(D) |
|---|---|---|---|
| (i) | 152290419440611 | 46274886 | 1.062983 |
| | 165427035605659 | 48190146 | 1.061386 |
| | 206546921647291 | 54350198 | 1.069334 |
| | 215226414830491 | 54450146 | 1.049121 |
| | 300272328240091 | 65344634 | 1.063030 |
| (ii) | 133051755648751 | 42848636 | 1.054226 |
| | 142368153139039 | 44889152 | 1.067078 |
| | 146936775525439 | 45349180 | 1.060843 |
| | 166290530163319 | 48736480 | 1.070583 |
| | 174346066249111 | 49611996 | 1.063923 |
| (iii) | 246406633037854 | 57923528 | 1.041889 |
| | 256397742215806 | 60536004 | 1.067108 |
| | 285278695393246 | 64119584 | 1.070606 |
| | 301938138430366 | 64551980 | 1.047187 |
| | 350240722763374 | 70400728 | 1.059121 |
| (iv) | 229297977151681 | 54793321 | 1.034296 |
| | 259853252349289 | 58673599 | 1.039268 |
| | 273323976657169 | 60545353 | 1.045206 |
| | 366525636221761 | 69241975 | 1.029650 |
| | 457165855430761 | 79417945 | 1.055462 |

where $1.12 < v < 1.13$. Thus, if we can bound $R$, we can certainly get a result like (1.1).

If $h$ is the class number of $\mathscr{K}$, we have

$$hR = \frac{\sqrt{|\Delta|}}{2\pi}\Phi(1), \tag{3.3}$$

where

$$\Phi(1) = \lim_{s\to 1}\zeta_{\mathscr{K}}(s)/\zeta(s) = \prod_q f(q).$$

TABLE 6

| $r_n$ | Least D |
|---|---|
| 83 | 8114538721 |
| 89 | 9176747449 |
| 97, 101, 103 | 23616331489 |
| 107, 109, 113, 127 | 196265095009 |
| 131, 137, 139 | 2871842842801 |
| 149 | 26437680473689 |
| 151 | 89436364375801 |
| 157, 163, 167 | 112434732901969 |
| 173, 179 | 178936222537081 |

Here the (Euler) product is taken over all the primes $q$, and $f(q)$ is given below:

$$f(3) = \begin{cases} 3/2 & \text{when } a^2 \equiv b^2 \pmod 9, \\ 1 & \text{otherwise;} \end{cases}$$

$f(q) = 1$ when $q \mid ab$;

if $q \equiv -1 \pmod 3$ and $q \nmid ab$, then $f(q) = q^2/(q^2 - 1)$;

if $q \equiv 1 \pmod 3$ and $q \nmid ab$, then

$$f(q) = \begin{cases} q^2/(q-1)^2 & \text{when } (D/q)_3 = 1, \\ q^2/(q^2+q+1) & \text{otherwise.} \end{cases}$$

If we use the symbol $\prod_j^Q$ to denote the product over all primes less than or equal to $Q$ and $\equiv j \pmod 3$, and if we denote by $T(Q, D)$ the infinite product

$$\prod_{\substack{q>Q \\ q\equiv 1 \pmod 3}} f(q)$$

taken over all the primes exceeding $Q$ and $\equiv 1 \pmod 3$, then, since the infinite product

$$\prod_{q\equiv -1 \pmod 3} f(q)$$

taken over all the primes $\equiv -1 \pmod 3$ converges, we have

$$\Phi(1) = \left( f(3) \prod_{-1}^{Q} f(q) \prod_{1}^{Q} f(q) \right) T(Q, D)(1 + o(1)).$$

Now

$$\prod_1^Q f(q) \leqslant \prod_1^Q q^2/(q-1)^2;$$

hence

$$\prod_{-1}^{Q} f(q) \prod_{1}^{Q} f(q) \leqslant (2/3) \prod^{Q} q/(q-1) \prod^{Q} q/(q-\chi(q)),$$

where each of the products on the right-hand side is evaluated over all the primes $\leqslant Q$ and $\chi(q) = (-3/q)$. By Mertens' theorem

$$\prod^{Q} q/(q-1) = e^{\gamma} \log Q (1 + o(1)).$$

Also, since

$$L(1, \chi) = \prod_{q} q/(q - \chi(q)) = \frac{\pi}{3\sqrt{3}}$$

(the product taken over all the primes $q$), we get

$$\Phi(1) \leqslant \frac{2\pi e^{\gamma} f(3)}{9\sqrt{3}} (\log Q) T(Q, D)(1 + o(1)). \tag{3.4}$$

If $\mathscr{E}$ is the extension $\mathscr{K}(\omega)$ of $\mathscr{K}$, where $\omega^2 + \omega + 1 = 0$, then the discriminant $d$ of $\mathscr{E}$ is $3\Delta^2$ (see Barrucand and Cohn [1]). If we put

$$U(D) = T\big((\log d)^2, D\big),$$

then $U(D) < 1 + o(1)$ under the extended Riemann Hypothesis for $\zeta_{\mathscr{E}}$ (see, for example, Williams, Dueck and Schmid [9, pp. 282–283]). Combining this result and (3.4) with $Q = (\log d)^2$, we get

$$\Phi(1) < \frac{4\pi e^{\gamma} f(3)}{9\sqrt{3}} \log\log(3\Delta^2)(1 + o(1)).$$

It follows from (3.3) that

$$hR < \frac{2e^{\gamma} f(3)\sqrt{|\Delta|/3}}{9} \log\log(3\Delta^2)(1 + o(1)). \tag{3.5}$$

When, for example, $D$ is square-free, then

$$hR < \begin{cases} (1/3) e^{\gamma} D \log\log 3^3 D^4 (1 + o(1)) & \text{when } D \equiv \pm 1 \pmod 9, \\ (2/3) e^{\gamma} D \log\log 3^7 D^4 (1 + o(1)) & \text{otherwise.} \end{cases} \tag{3.6}$$

**4. Further Numerical Results.** From (3.3) we see that in order to maximize $R$ we must minimize $h$ and get $\Phi(1)$ as large as possible. Of the possibilities for $D$ square-free, $D \not\equiv \pm 1 \pmod 9$ and $3 \nmid h$ (see Honda [2]) we elected to examine prime values of $D \equiv 2$ or $5 \pmod 9$. If $r_i$ is the $i$th prime of the form $1 + 3t$, then the prime $D$ values which should give large $\Phi(1)$ values are those for which $(D/r_i)_3 = 1$ ($i = 1, 2, 3, \ldots, n$) for as large a value of $n$ as possible. We now encounter a difficulty, however. The determination of $P$ is very expensive for rather modest values of $D$ (say $\approx 200000$); thus, we decided to look at the values of $R$ instead. By using the methods described in [9] we can calculate $R$ much more rapidly than $P$; but, it still becomes very expensive to find $R$ when $D > 2 \times 10^9$. (It should, of course, be borne in mind that the discriminants for such values of $D$ are very large, exceeding $10^{20}$.)

UMSU was programmed to find the first 50 values of $D$ for each $n$ until a value of $n$ was reached for which the least of these 50 numbers exceeded $2^{31} - 1$, the word size of the machine used to compute $R$—an AMDAHL 470-V8. We then computed $R$ for each $D$ and $C(D) = R/(D \log\log(3^7 D^4))$. In the cubic case it takes very little time to find the $D$ values and a much larger amount of time to find the $R$ values, the reverse of the situation in the quadratic case.

Our results are summarized in the following tables. In Tables 7 and 8 we give only those values of $D$ for which $C(D)$ exceeds the $C$ values for any of the other numbers that we found which were less than $D$. In Tables 9 and 10 we give values of $D$ for which the corresponding regulator exceeds any of those previously found. Since $2e^\gamma/3 \approx 1.18738$, we have nothing here that comes near to violating the Riemann Hypothesis for $\zeta_{\mathscr{E}}$. Also the growth of $C(D)$ is slow and getting slower as $D$ increases.

TABLE 7. $D \equiv 2 \pmod 9$

| D | R | C(D) |
|---|---|---|
| 29 | 40.27082 | .454983 |
| 1721 | 3669.37913 | .588309 |
| 39521 | 92172.43814 | .596085 |
| 92009 | 218706.73901 | .597544 |
| 343433 | 895028.71553 | .640002 |
| 6616667 | 18089884.90792 | .642420 |
| 7202369 | 19994005.36092 | .651564 |
| 202306187 | 586455162.98256 | .653911 |
| 562788101 | 1689849729.97072 | .670149 |

TABLE 8. $D \equiv 5 \pmod 9$

| D | R | C(D) |
|---|---|---|
| 41 | 56.28937 | .440672 |
| 239 | 431.94224 | .533495 |
| 1301 | 2549.94344 | .545373 |
| 4523 | 9440.96250 | .560767 |
| 19391 | 42811.86808 | .572868 |
| 67829 | 154494.32105 | .575923 |
| 72617 | 168197.50896 | .584893 |
| 143879 | 361610.34278 | .626614 |
| 1145327 | 3021373.73848 | .635515 |
| 8596463 | 23331608.01905 | .635544 |
| 8666393 | 23925356.23751 | .646390 |
| 48487811 | 139358465.15040 | .658771 |
| 55570523 | 163251776.10755 | .672292 |
| 60435383 | 179011355.42037 | .677194 |

TABLE 9. $D \equiv 2 \pmod 9$

| D | R | C(D) |
|---|---|---|
| 689816063 | 1888303399.286361 | .609701902921 |
| 780923333 | 2040735586.012364 | .581327045940 |
| 807748787 | 2264449384.076498 | .623423186470 |
| 911130401 | 2663628567.917647 | .649341195701 |
| 947294867 | 2666732555.238140 | .625039976249 |
| 1039506833 | 2941248070.747570 | .627656937643 |
| 1090062947 | 3194601736.597826 | .649803055418 |
| 1250773679 | 3481668991.375506 | .616374865552 |
| 1345747619 | 3810698517.456939 | .626570574715 |
| 1411121837 | 3967734472.270628 | .621882987760 |
| 1627729013 | 4492140541.726865 | .609547346456 |
| 1695130949 | 5107010533.454052 | .665168464608 |
| 2044171163 | 5464205375.038442 | .589124377005 |

TABLE 10. $D \equiv 5 \pmod 9$

| D | R | C(D) |
|---|---|---|
| 78446831 | 195588785.889993 | .568325353629 |
| 85474661 | 229039192.818766 | .610210176803 |
| 140795537 | 399133674.604591 | .641989245305 |
| 172132241 | 499066717.859134 | .655156175458 |
| 226496759 | 590664701.273444 | .587554486086 |
| 230154107 | 639837059.815727 | .626247127877 |
| 246667721 | 681500286.460078 | .621912563185 |
| 258947807 | 705470723.780494 | .612941643650 |
| 262192559 | 708275870.201634 | .607683348911 |
| 267667889 | 739791831.127160 | .621604001102 |
| 313154087 | 892931349.895030 | .640243198576 |
| 613655951 | 1737626269.841794 | .631422697999 |
| 641290649 | 1782982379.770936 | .619710166881 |
| 671319221 | 1958397914.780726 | .649933701273 |
| 736002077 | 2030844203.759158 | .614180052846 |
| 784288049 | 2197173781.242724 | .623178495983 |
| 789581183 | 2210040440.336767 | .622584176723 |
| 792812201 | 2226806639.150783 | .624725493228 |
| 860248787 | 2595846960.356864 | .670626975448 |
| 914070821 | 2652552259.996093 | .644540413185 |
| 948371243 | 2660317585.609716 | .622821760536 |
| 957302429 | 2743025183.193182 | .636135024907 |
| 1400879507 | 4155081949.704781 | .656054383521 |
| 1617735209 | 4322176122.273822 | .590142344485 |
| 1632061859 | 4549031363.516906 | .615612610599 |
| 1827261311 | 4810329644.671832 | .580807712394 |
| 1831479161 | 5515724098.441698 | .664430018640 |
| 2108312123 | 5713478707.454342 | .597085411322 |
| 2124689657 | 6127255313.478815 | .635344945017 |

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