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Supplement to
The Convergence of Galerkin Approximation Schemes for Second-Order Hyperbolic Equations With Dissipation

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2. Convergence Estimates for Semi-discrete Approximations

PROPOSITION 1. *If* $U(t)$ *satisfies* (1.6) *and* $U_h(t)$ *satisfies* (1.10) *with* $U_h(o) = P_h U_o$, $U_o \in \mathcal{H}^{q+1}(\Omega) \times \mathcal{H}^q(\Omega)$ *then*

$$|||U(t) - U_h(t)|||_o \le c(t^*,\alpha)\, h^{q-1} |||U_o|||_q, \quad 1 \le q \le r,\ 0 \le t \le t^*.$$

Proof. Firstly for q=1, we have

$$\begin{aligned} |||U(t) - U_h(t)|||_o &\le |||U(t)|||_o + |||U_h(t)|||_o \\ &\le |||U_o|||_o + |||P_h U_o|||_o \\ &\le 2|||U_o|||_o \end{aligned}$$

by (1.4), (1.12) and since P_h is the projection with respect to $((\cdot,\cdot))_o$. For $2 \le q \le r$, we write

$$U(t) - U_h(t) = (U(t) - P_h U(t)) + (P_h U(t) - U_h(t)).$$

By (1.16),

$$\begin{aligned} |||U(t) - P_h U(t)|||_o &\le ch^{q-1} |||U(t)|||_{q-1} \\ &\le ch^{q-1} |||U_o|||_{q-1} \\ &\le ch^{q-1} |||U_o|||_q, \end{aligned}$$

we have to show that

$$|||P_h U(t) - U_h(t)|||_o \le ch^{q-1} |||U_o|||_q.$$

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From the identity

$$J_hD_t\,U(t) + U(t) + \alpha T\,U(t) = (J_h - J)\,D_t\,U(t),$$

it follows that

$$J_hD_t\mathbf{P}_h\,U(t) + \mathbf{P}_h\,U(t) + \alpha T\mathbf{P}_h\,U(t) = (J_h - J)\,D_t\,U(t) + J_h(\mathbf{P}_h - I)\,D_t\,U(t)$$
$$+ (\mathbf{P}_h - I)\,U(t) + \alpha T(\mathbf{P}_h - I)\,U(t)$$
$$= (J - J_h)\,\Lambda U(t) + J_h(I - \mathbf{P}_h)\,\Lambda U(t) + (\mathbf{P}_h - I)\,U(t)$$
$$+ \alpha\{(J - J_h)\,TU(t) + J_h(I - \mathbf{P}_h)\,TU(t) + T(\mathbf{P}_h - I)\,U(t)\}$$
$$= \rho_h(t) + \alpha\{JT - J_h\mathbf{P}_hT + T(\mathbf{P}_h - I)\}\,U(t),$$

where $\rho_h(t)$ is as in [5], (2.3).
Simplification of the last factor yields

$$J_hD_t\mathbf{P}_h\,U(t) + \mathbf{P}_h\,U(t) + \alpha T\mathbf{P}_h\,U(t) = \rho_h(t) + \alpha[(T - T_h)\,P_h^0\dot{u},0]^T:$$

$U_h(t)$ satisfies

$$J_hD_t\,U_h(t) + U_h(t) + \alpha T_h\,U_h(t) = 0, \quad U_h(0) = \mathbf{P}_hU_o,$$

and setting $E_h^*(t) = \mathbf{P}_h\,U(t) - U_h(t)$, we obtain

$$J_hD_tE_h^*(t) + E_h^*(t) = \rho_h(t) + \alpha[(T - T_h)\,P_h^0\dot{u},0]^T - \alpha T\mathbf{P}_h\,U(t) + \alpha T_h\,U_h(t)$$
$$= \rho_h(t) + \alpha[T_h(\dot{u}_h - P_h^0\dot{u}),0]^T.$$
$$E_h^*(0) = 0.$$

We form the energy inner product with $D_tE_h^*(t)$, and use the skew-adjointness of J_h to obtain

$$((E_h^*(t),D_tE_h^*(t)))_o = ((\rho_h(t),D_tE_h^*(t)))_o + \alpha(([T_h(\dot{u}_h - P_h^0\dot{u}),o],\ D_tE_h^*(t)))_o$$

i.e. $\frac{1}{2}\frac{d}{dt}|||E_h^*(t)|||_o^2 = ((\rho_h(t),\ D_tE_h^*(t)))_o + \alpha a(T_h(\dot{u}_h - P_h^0\dot{u}),\ P_h^1\dot{u} - \dot{u}_h)$.

For the final term here we have

$$\alpha a(T_h(\dot{u}_h - P_h^0\dot{u}),\ P_h^1\dot{u} - \dot{u}_h) = \alpha(\dot{u}_h - P_h^0\dot{u},\ P_h^1\dot{u} - \dot{u}_h)$$
$$= \alpha(\dot{u}_h - P_h^0\dot{u},\ P_h^0\dot{u} - \dot{u}_h) + \alpha(\dot{u}_h - P_h^0\dot{u},\ P_h^1\dot{u} - P_h^0\dot{u})$$
$$\le \alpha(\dot{u}_h - P_h^0\dot{u},\ P_h^1\dot{u} - P_h^0\dot{u}),$$

so that

$$\frac{d}{dt}|||E_h^*(t)|||_o^2 \le 2((\rho_h(t),\ D_tE_h^*(t)))_o + 2\alpha(\dot{u}_h - P_h^0\dot{u},\ P_h^1\dot{u} - P_h^0\dot{u}).$$

Exactly as in [5] Proposition 1, or [1] Theorem 2.1, we derive the inequality

$$(2.1)\quad \frac{1}{2}\sup_{0\le t\le t^*}|||E_h^*(t)|||_o^2 \le 4\sup_{0\le t\le t^*}\{|||\rho_h(t)|||_o^2 + (t^*)^2|||D_t\rho_h(t)|||_o^2\}$$
$$+ \alpha\sup_{0\le t\le t^*}\{\tfrac{1}{4\alpha}\|\dot{u}_h - P_h^0\dot{u}\|_o^2 + 4\alpha(t^*)^2\|P_h^1\dot{u} - P_h^0\dot{u}\|_o^2\}.$$

Now $\qquad |||\rho_h(t)|||_o \le ch^{q-1}|||U_o|||_{q-1}$

from [5], Proposition 1.
By (1.2), we have

$$D_t\rho_h(t) = \{(J_h - J)\,\Lambda^2U(t) + J_h(\mathbf{P}_h - I)\,\Lambda^2\,U(t) + (I - \mathbf{P}_h)\,\Lambda U(t)\}$$
$$+ \alpha\{(J_h - J)\,\Lambda TU(t) + J_h(\mathbf{P}_h - I)\,\Lambda TU(t) + (I - \mathbf{P}_h)\,TU(t)\}$$
$$= [D_t\rho_h(t)]_I + \alpha[D_t\rho_h(t)]_{II}.$$

The first term is exactly as in [5] and therefore

$$|||[D_t\rho_h(t)]_I|||_o \le ch^{q-1}|||U_o|||_q.$$

For the second term, using (1.16), (1.17) and (1.18), we obtain

$$\begin{aligned} |||[D_t\rho_h(t)]_{II}|||_o &\le |||(J_h - J)\ \Lambda T\ U(t)|||_o + |||(\mathbf{P}_h - I)\ \Lambda T\ U(t)|||_{-1,h} \\ &\quad + |||(I - \mathbf{P}_h)\ T\ U(t)|||_o \\ &\le ch^{q-1}|||TU(t)|||_{q-1} \\ &= ch^{q-1}|||[0,\dot{u}(t)]|||_{q-1} \\ &\le ch^{q-1}|||U(t)|||_{q-1} \\ &\le ch^{q-1}|||U_o|||_{q-1}\ . \end{aligned}$$

Combining the above inequalities, we obtain

$$\begin{aligned} |||D_t\rho_h(t)|||_o &\le ch^{q-1}|||U_o|||_q + \alpha ch^{q-1}|||U_o|||_{q-1} \\ &\le c(\alpha)h^{q-1}|||U_o|||_q\ . \end{aligned}$$

For the remaining terms of (2.1) we have the estimates

$$\|(\dot{u}_h - P_h^o\dot{u})(t)\|_o \le |||E_h^*(t)|||_o$$

and

$$\begin{aligned} \|(P_h^1 - P_h^o)\dot{u}(t)\|_o &\le \|(P_h\dot{u} - \dot{u})(t)\|_o + \|(\dot{u} - P_h\dot{u})(t)\|_o \\ &\le ch^{q-1}\|\dot{u}(t)\|_{q-1} \\ &\le ch^{q-1}|||U(t)|||_{q-1} \\ &\le ch^{q-1}|||U_o|||_{q-1}. \end{aligned}$$

Therefore from (2.1)

$$\sup_{t\le t^*} |||E_h^*(t)|||_o^2 \le c(\alpha,t^*)\ h^{2(q-1)}|||U_o|||_q^2 + \frac{1}{2}\sup_{t\le t^*} |||E_h^*(t)|||_o^2$$

and for $0\le t\le t^*$, we have

$$|||E_h^*(t)|||_o \le c(\alpha,t^*)\ h^{q-1}|||U_o|||_q,$$

and the proposition has been established.

PROPOSITION 2. *If* $U(t)$ *satisfies* (1.6) *and* $U_h(t)$ *satisfies* (1.10) *with* $U_h(0) = \mathbf{P}_hU_o$ *and* $U_o \in \dot{H}^{q+1}(\Omega) \times \dot{H}^q(\Omega)$, $1 \le q \le r$, *then*

$$|||U(t) - U_h(t)|||_{-p,h} \le c(t^*,\alpha)h^{p+q-1}|||U_o|||_q, \quad 1 \le p \le r-1.$$

Proof. As in the previous proposition, we write

$$J_hD_t\ U(t) + U(t) + \alpha T\ U(t) = (J_h - J)\ D_t\ U(t),$$

so that

$$J_h^{p+1}D_t\ U(t) + J_h^p\ U(t) + \alpha J_h^p\ TU(t) = J_h^p(J_h - J)\ D_t\ U(t).$$

We also have

$$J_h^{p+1}D_t\ U_h(t) + J_h^p\ U_h(t) + \alpha J_h^pT_h\ U_h(t) = 0.$$

so that with

$$E_h(t) = U(t) - U_h(t),$$

it follows that

$$\begin{aligned} J_h^{p+1}D_tE_h(t) + J_h^pE_h(t) &= J_h^p(J_h - J)\ D_tU(t) - \alpha J_h^pT\ U(t) + \alpha J_h^pT_h\ U_h(t) \\ &= J_h^p(J - J_h)\Lambda U(t) + \alpha J_h^p(J - J_h)TU(t) - \alpha J_h^pTU(t) \\ &\quad + \alpha J_h^pT_h\ U_h(t) \\ &= J_h^p(J - J_h)\Lambda U(t) + \alpha J_h^p[T_h(\dot{u}_h - \dot{u}),0]^T. \end{aligned}$$

Set

$$\sigma_h^1(t) = (J - J_h)\Lambda\, U(t),$$

$$\sigma_h^2(t) = [T_h(\dot{u}_h - \dot{u}), 0]^T,$$

and form the energy inner product with $J_h^p D_t E_h(t)$.

Then

$$\frac{1}{2}\frac{d}{dt}|||J_h^p E_h(t)|||_o^2 = ((J_h\sigma_h^1(t), J_h^p D_t E_h(t)))_o + \alpha((J_h^p\sigma_h^2(t), J_h^p D_t E_h(t)))_o$$

The latter term is treated as follows:

$$\begin{aligned}((J_h^p\sigma_h^2(t), J_h^p D_t E_h(t)))_o &= (-1)^p ((J_h^{2p}\sigma_h^2(t), D_t E_h(t)))_o \\ &= (([T_h^{p+1}(\dot{u}_h - \dot{u}), 0]^T, [\dot{u} - \dot{u}_h, D_t\dot{u} - D_t\dot{u}_h]^T))_o \\ &= a(T_h^{p+1}(\dot{u}_h - \dot{u}), \dot{u} - \dot{u}_h) \\ &= a(T_h^{p+1}(\dot{u}_h - \dot{u}), P_h^o\dot{u} - \dot{u}_h) + a(T_h^{p+1}(\dot{u}_h - \dot{u}), \dot{u} - P_h^o\dot{u}) \\ &= (T_h^p(\dot{u}_h - \dot{u}), P_h^o\dot{u} - \dot{u}_h) + a(T_h^{p+1}(\dot{u}_h - \dot{u}), P_h^1(\dot{u} - P_h^o\dot{u})) \\ &= (T_h^p(\dot{u}_h - \dot{u}), \dot{u} - \dot{u}_h) + (T_h^p(\dot{u}_h - \dot{u}), P_h^1\dot{u} - P_h^1 P_h^o\dot{u}) \\ &\le (\dot{u}_h - \dot{u}, P_h^1\dot{u} - P_h^1 P_h^o\dot{u})_{-p,h} \\ &= (\dot{u}_h - \dot{u}, P_h^1\dot{u} - P_h^o\dot{u})_{-p,h} \qquad (2.2)\end{aligned}$$

since $P_h^1 P_h^o = P_h^o$.

$$(a(P_h^1 P_h^o u, \varphi) = a(P_h^1 P_h^o u, P_h^1\varphi) = a(P_h^o u, P_h^1\varphi) = a(P_h^o u, \varphi), \quad \varphi \in \dot{H}^1(\Omega)).$$

Then

$$\frac{d}{dt}|||E_h(t)|||_{-p,h}^2 \le 2((J_h^p\sigma_h^1(t), J_h^p D_t E_h(t)))_o + 2\alpha(\dot{u}_h - \dot{u}, (P_h^1 - P_h^o)\dot{u})_{-p,h},$$

and, as in the proof of Proposition 1,

$$\begin{aligned}\sup_{0<t<t^*}|||E_h(t)|||_{-p,h}^2 \le c \sup_{0\le t\le t^*}\{|||\sigma_h^1(t)|||_{-p,h}^2 &+ (t^*)^2|||D_t\sigma_h^1(t)|||_{-p,h}^2 \\ &+ |||E_h(0)|||_{-p,h}^2 + |||\sigma_h^1(0)|||_{-p,h}^2\} \\ + \alpha \sup_{0\le t\le t^*}\{\frac{1}{4\alpha}\|\dot{u}_h - \dot{u}\|_{-p,h}^2 &+ 4\alpha(t^*)^2\|(P_h^1 - P_h^o)\dot{u}\|_{-p,h}^2\}.\end{aligned}$$

Therefore, since $\|\dot{u}_h - \dot{u}\|_{-p,h} \le |||E_h(t)|||_{-p,h}$ we have

$$\begin{aligned}\sup_{0\le t\le t^*}|||E_h(t)|||_{-p,h} \le c(t^*)\{\sup_{0\le t\le t^*} |||\sigma_h^1(t)|||_{-p,h} &+ |||D_t\sigma_h^1(t)|||_{-p,h} \\ &+ |||E_h(0)|||_{-p,h} + \alpha\|(P_h^1 - P_h^o)\dot{u}\|_{-p,h}\}, \qquad (2.3)\end{aligned}$$

which may be compared with (2.11) of [5], and

$$|||\sigma_h^1(t)|||_{-p,h} \le ch^{p+q-1}|||U_o|||_{q-1}.$$

Also

$$\begin{aligned}D_t\sigma_h^1(t) &= (J_h - J)\,\Lambda^2 U(t) + \alpha(J_h - J)\,IU(t) \\ &= [D_t\sigma_h^1(t)]_I + \alpha[D_t\sigma_h^1(t)]_{II}.\end{aligned}$$

The first term here corresponds to $D_t\sigma_h(t)$ of [5], Proposition 2, and so

$$|||D_t\sigma_h^1(t)|||_{-p,h} \le ch^{p+q-1}|||U_o|||_q,$$

while

$$\begin{aligned}|||[D_t\sigma_h^1(t)]_{II}|||_{-p,h} &\le ch^{p+q-1}|||IU(t)|||_{q-2} \\ &\le ch^{p+q-1}|||U(t)|||_{q-2} \\ &\le ch^{p+q-1}|||U_o|||_q.\end{aligned}$$

As in [5], we have the estimate

$$|||E_h(0)|||_{-p,h} \le ch^{p+q-1}|||U_0|||_q.$$

Finally

$$\|(P_h^1 - P_h^0)\dot{u}\|_{-p,h} \le \|P_h^1\dot{u} - \dot{u}\|_{-p,h} + \|\dot{u} - P_h^0\dot{u}\|_{-p,h}$$

$$\le c\{\|P_h^1\dot{u} - \dot{u}\|_{-p} + h^p\|P_h^1\dot{u} - \dot{u}\|_0 + \|P_h^0\dot{u} - \dot{u}\|_{-p} + h^p\|P_h^0\dot{u} - \dot{u}\|_0\}$$

$$\le ch^{p+q-1}\|\dot{u}(t)\|_{q-1} \qquad (2.4)$$

$$\le ch^{p+q-1}|||U(t)|||_{q-1}$$

$$\le ch^{p+q-1}|||U_0|||_q$$

and the proposition has been established .

Remark. The optimal L^2-estimate for $u_h(t) - u(t)$ follows from the case p=1:

$$\|u(t) - u_h(t)\|_0 \le c(t^*,\alpha)\, h^r|||U_0|||_r.$$

Propositions 1 and 2 and (1.18) yield our main result.

THEOREM 1. *If* $U(t)$ *is the solution of* (1.6) and $U_h(t)$ *is the solution of* (1.10) *with* $U_h(0) = \mathbf{P}_hU_0$, $U_0 \in \mathring{H}^{q+1}(\Omega) \times \mathring{H}^q(\Omega)$, $1 \le q \le r$, *then for* $0 \le t \le t^*$,

$$|||U(t) - U_h(t)|||_{-p} \le c(t^*,\alpha)\, h^{p+q-1}|||U_0|||_q,$$

$$0 \le p \le r-1.$$

We refer the reader to the corresponding result for the non-dissipative case, Theorem 1 and Remarks 1 and 2 of [5].

4. Estimates for higher order time derivatives

THEOREM 4. *Assume* $U_0 \in \mathring{H}^{s+q+1}(\Omega) \times \mathring{H}^{s+q}(\Omega)$, $s \ge 1$, $1 \le q \le r$. *Then*

$$|||D_t^s\,U(t) - D_t^s\,U_h(t)|||_{-p} \le c(t^*,\alpha)\, h^{p+q-1}|||U_0|||_{s+q}, \quad 0 \le p \le r-1.$$

Proof. The energy estimate is derived first. We have

$$D_t^s\,U(t) - D_t^s\,U_h(t) = (-1)^s\,\{\Lambda_\alpha^s\,U(t) - \Lambda_{\alpha,h}^s\,U_h(t)\}$$

$$= (-1)^s\,\{\Lambda_\alpha^s\,U(t) - \Lambda_{\alpha,h}\,J_{\alpha,h}^2\,\Lambda_\alpha^{s+1}\,U(t)\}$$

$$+ (-1)^s\,\{\Lambda_{\alpha,h}\,J_{\alpha,h}^2\,\Lambda_\alpha^{s+1}\,U(t) - \Lambda_{\alpha,h}^s\,U_h(t)\}$$

$$= (-1)^s\,\{E_h^{**}(t) + E_h^*(t)\}\ .$$

Now

$$|||E_h^{**}(t)|||_0 = |||\Lambda_\alpha^s\,U(t) - \Lambda_{\alpha,h}\,J_{\alpha,h}^2\,\Lambda_\alpha^{s+1}\,U(t)|||_0$$

$$= |||(J_\alpha - \Lambda_{\alpha,h}\,J_{\alpha,h}^2)\,\Lambda_\alpha^{s+1}\,U(t)|||_0\ , \qquad (4.4)$$

and one can show that

$$J_\alpha - \Lambda_{\alpha,h}\,J_{\alpha,h}^2 = J - \Lambda_h\,J_h^2 + \alpha\begin{bmatrix} T-T_h & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore,

$$|||E_h^{**}(t)|||_0 \le |||(J - \Lambda_h\,J_h^2)\,\Lambda_\alpha^{s+1}\,U(t)|||_0 + \alpha|||[(T - T_h)(\Lambda_\alpha^{s+1}\,U(t))^I,0]^T|||_0.$$

Here $(Z)^I$ denotes the first component of $Z = [(Z)^I,(Z)^{II}]^T$. From (4.5) of [5] and our estimate (4.3), we have immediately that

$$|||(J - \Lambda_h\,J_h^2)\,\Lambda_\alpha^{s+1}\,U(t)|||_0 \le ch^{q-1}|||\Lambda_\alpha^{s+1}\,U(t)|||_{q-2}$$

$$\le c(\alpha)h^{q-1}|||U(t)|||_{s+q-1}$$

$$\le c(\alpha)h^{q-1}|||U_0|||_{s+q-1}.$$

Also, by (1.9) and (4.3),

$$|||[(T - T_h)(\Lambda_\alpha^{s+1} U(t))^I, 0]^T|||_o = \|(T - T_h)(\Lambda_\alpha^{s+1} U(t))^I\|_1$$
$$\le ch^{q-1}\|(\Lambda_\alpha^{s+1} U(t))^I\|_{q-2}$$
$$\le ch^{q-1}|||\Lambda_\alpha^{s+1} U(t)|||_{q-3}$$
$$\le c(\alpha)\ h^{q-1}|||U(t)|||_{s+q-2}$$
$$\le c(\alpha)\ h^{q-1}|||U_o|||_{s+q-1},$$

so that

(4.5) $$|||E_h^{**}(t)|||_o \le c(\alpha)\ h^{q-1}|||U_o|||_{s+q-1}.$$

Using (4.1), we can write

$$J_{\alpha,h} D_t E_h^*(t) + E_h^*(t) = (J_{\alpha,h} - J_\alpha)\ D_t \Lambda_\alpha^s U(t) + J_{\alpha,h} D_t(\Lambda_{\alpha,h} J_{\alpha,h}^2 \Lambda_\alpha^{s+1} U(t)$$
$$- \Lambda_\alpha^s U(t)) + (\Lambda_{\alpha,h} J_{\alpha,h}^2 \Lambda_\alpha^{s+1} U(t) - \Lambda_\alpha^s U(t))$$
$$\equiv \tilde{\rho}_h(t),$$

and $$E_h^*(0) = \Lambda_{\alpha,h} J_{\alpha,h}^2 \Lambda_\alpha^{s+1} U_o - \Lambda_{\alpha,h}^s J_{\alpha,h}^{s+1} \Lambda_\alpha^{s+1} U_o$$
$$= 0.$$

Since $J_{\alpha,h} = J_h + \alpha T_h^*$, we have

$$J_h D_t E_h^*(t) + E_h^*(t) = \tilde{\rho}_h(t) - \alpha T_h^* D_t E_h^*(t)$$
$$E_h^*(0) = 0.$$

Taking the energy inner product with $D_t E_h^*(t)$ and using the skew-adjointness of J_h yields

$$\frac{1}{2}\frac{d}{dt}|||E_h^*(t)|||_o^2 = ((\tilde{\rho}_h(t), D_h E_h^*(t)))_o - \alpha((T_h^* D_t E_h^*(t), D_t E_h^*(t)))_o.$$

Now, for any $Z \in S_h^r(\Omega) \times L^2(\Omega)$,

$$((T_h^* Z, Z))_o = (([T_h(Z)^I, 0], [(Z)^I, (Z)^{II}]))_o$$
$$= a(T_h(Z)^I, (Z)^I)$$
$$= ((Z)^I, (Z)^I)$$
$$\ge 0 .$$

The term

$$D_t E_h^*(t) = \Lambda_{\alpha,h}^{s+1} U_h(t) - \Lambda_{\alpha,h} J_{\alpha,h}^2 \Lambda_\alpha^{s+2} U(t)$$
$$= \Lambda_{\alpha,h}^{s+1} U_h(t) - J_{\alpha,h} \Lambda_\alpha^{s+2} U(t)$$
$$\in S_h^r(\Omega) \times L^2(\Omega).$$

In fact, for $U_o \in \mathring{H}^{s+q+1}(\Omega) \times \mathring{H}^{s+q}(\Omega)$, $\Lambda_\alpha^{s+2} U(t) \subset L^2(\Omega) \times \mathring{H}^{-1}(\Omega)$ and therefore, $J_{\alpha,h} \Lambda_\alpha^{s+2} U(t) \in S_h^r(\Omega) \times L^2(\Omega)$. Also, $U_h(t) \in S_h^r(\Omega) \times S_h^r(\Omega)$ and $\Lambda_{\alpha,h}^{s+1} U_h(t) \in S_h^r(\Omega) \times S_h^r(\Omega)$.
Thus,

$$((T_h^* D_t E_h^*(t), D_t E_h^*(t)))_o \ge 0$$

and

$$\frac{1}{2}\frac{d}{dt}|||E_h^*(t)|||_o^2 \le ((\tilde{\rho}_h(t), D_t E_h^*(t)))_o .$$

As in the proofs of Propositions 1 and 2, this leads to

$$\sup_{0<t\le t^*} |||E_h^*(t)|||_o \le c(t^*) \sup_{0<t\le t^*} \{|||\tilde{\rho}_h(t)|||_o + |||D_t \tilde{\rho}_h(t)|||_o\},$$

which may be compared with (4.8) of [5].

Now

$$D_t\, \tilde{\rho}_h(t) = (J_{\alpha,h} - J_\alpha)\, \Lambda_\alpha^{s+2}\, U(t) + J_{\alpha,h}(\Lambda_{\alpha,h}\, J_{\alpha,h}^2\, \Lambda_\alpha^{s+3}\, U(t) - \Lambda_\alpha^{s+2}\, U(t))$$
$$+ (\Lambda_\alpha^{s+1}\, U(t) - \Lambda_{\alpha,h}\, J_{\alpha,h}^2\, \Lambda_\alpha^{s+2}\, U(t)),$$

and these terms are now estimated. We have

$$|||(J_{\alpha,h} - J_\alpha)\, \Lambda_\alpha^{s+2}\, U(t)|||_o = |||(J_h - J)\, \Lambda_\alpha^{s+2}\, U(t) + \alpha(T_h^* - T^*)\, \Lambda_\alpha^{s+2}\, U(t)|||_o$$
$$\le ch^{q-1}|||\Lambda_\alpha^{s+2}\, U(t)|||_{q-2}$$
$$+ \alpha|||[(T_h - T)(\Lambda_\alpha^{s+2}\, U(t))^I, 0]|||_o$$
$$\le c(\alpha)\{h^{q-1}|||U(t)|||_{s+q}$$
$$+ \|(T_h - T)(\Lambda_\alpha^{s+2}\, U(t))^I\|_1\}$$
$$\le c(\alpha)\{h^{q-1}|||U_o|||_{s+q} + h^{q-1}\|(\Lambda_\alpha^{s+2}\, U(t)^I\|_{q-2}\}$$
$$\le c(\alpha)\, h^{q-1}\{|||U_o|||_{s+q} + |||\Lambda_\alpha^{s+2}\, U(t)|||_{q-3}\}$$
$$\le c(\alpha)\, h^{q-1}|||U_o|||_{s+q}. \tag{4.6}$$

From (4.4) - (4.5), we obtain

$$|||(\Lambda_\alpha^{s+1} - \Lambda_{\alpha,h}\, J_{\alpha,h}^2\, \Lambda_\alpha^{s+2})\, U(t)|||_o = |||(J_\alpha - \Lambda_{\alpha,h}\, J_{\alpha,h}^2)\, \Lambda_\alpha^{s+2}\, U(t)|||_o$$
$$\le c(\alpha)\, h^{q-1}|||U_o|||_{s+q}, \tag{4.7}$$

whereas

$$|||J_{\alpha,h}(\Lambda_{\alpha,h}\, J_{\alpha,h}^2\, \Lambda_\alpha^{s+3}\, U(t) - \Lambda_\alpha^{s+2}\, U(t))|||_o$$
$$\le |||J_h(\Lambda_{\alpha,h}\, J_{\alpha,h}^2\, \Lambda_\alpha^{s+3}\, U(t) - \Lambda_\alpha^{s+2}\, U(t))|||_o$$
$$+ |||\alpha T_h^*(\Lambda_{\alpha,h}\, J_{\alpha,h}^2\, \Lambda_\alpha^{s+3}\, U(t) - \Lambda_\alpha^{s+2}\, U(t))|||_o$$
$$\le c(\alpha)\; |||(\Lambda_{\alpha,h}\, J_{\alpha,h}^2\, \Lambda_\alpha^{s+3} - \Lambda_\alpha^{s+2})\, U(t)|||_{-1,h}\, ,$$

since $|||T_h^* Z|||_o \le |||Z|||_{-1,h}$.

(In fact, $|||T_h^* Z|||_o^2 = \|T_h(Z)^I\|_1^2 \le \|T_h(Z)^I\|_1^2 + \|T_h(Z)^{II}\|_o^2$
$= |||J_h^2\, Z|||_o^2 = |||Z|||_{-2,h}^2$
$\le |||Z|||_{-1,h}^2$).

Therefore,

$$|||J_{\alpha,h}(\Lambda_{\alpha,h}\, J_{\alpha,h}^2 - J_\alpha)\, \Lambda_\alpha^{s+3}\, U(t)|||_o \le c(\alpha)\{|||(\Lambda_{\alpha,h}\, J_{\alpha,h}^2 - J_\alpha)\, \Lambda_\alpha^{s+3}\, U(t)|||_{-1}$$
$$+ h|||(\Lambda_{\alpha,h}\, J_{\alpha,h}^2 - J_\alpha)\, \Lambda_\alpha^{s+3}\, U(t)|||_o\}.$$

As in the proofs of (4.4) and (4.5),

$$|||(\Lambda_{\alpha,h}\, J_{\alpha,h}^2 - J_\alpha)\, \Lambda_\alpha^{s+3}\, U(t)|||_{-1} \le c(\alpha)h^{q-1}|||U_o|||_{s+q} \quad \text{and}$$
$$|||(\Lambda_{\alpha,h}\, J_{\alpha,h}^2 - J_\alpha)\, \Lambda_\alpha^{s+3}\, U(t)|||_o \le c(\alpha)\, h^{q-2}|||U_o|||_{s+q}\, ;$$

so that

$$|||J_{\alpha,h}(\Lambda_{\alpha,h}\, J_{\alpha,h}^2 - J_\alpha)\, \Lambda_\alpha^{s+3}\, U(t)|||_o \le c(\alpha)\, h^{q-1}|||U_o|||_{s+q}\, . \tag{4.8}$$

Then (4.6) - (4.8) yield

$$|||D_t\, \tilde{\rho}_h(t)|||_o \le c(\alpha)\, h^{q-1}|||U_o|||_{s+q}\, .$$

Similarly,

$$|||\tilde{\rho}_h(t)|||_o \le c(\alpha)\, h^{q-1}|||U_o|||_{s+q-1}\, .$$

The latter two inequalities lead to the estimate

(4.9) $$|||E_h^*(t)|||_o \le c(t^*,\alpha)\, h^{q-1} |||U_o|||_{s+q}, \quad 0 \le t \le t^*,$$

which together with (4.5), gives

$$|||D_t^s\, U(t) - D_t^s\, U_h(t)|||_o \le c(t^*,\alpha)\, h^{q-1} |||U_o|||_{s+q},$$

$$0 \le t \le t^*, \quad 2 \le q \le r, \quad s \ge 1.$$

To establish negative norm estimates, we need only show that

$$|||D_t^s\, U(t) - D_t^s\, U_h(t)|||_{-p,h} \le c(\alpha,t^*)\, h^{p+q-1} |||U_o|||_{s+q}, \quad 1 \le p \le r-1 .$$

Since

$$J_\alpha\, D_t\, \Lambda_\alpha^s\, U(t) + \Lambda_\alpha^s\, U(t) = 0,$$

we have

$$J_{\alpha,h}\, D_t\, \Lambda_\alpha^s\, U(t) + \Lambda_\alpha^s\, U(t) = (J_{\alpha,h} - J_\alpha)\, D_t\, \Lambda_\alpha^s\, U(t),$$

and

$$J_h^p\, J_{\alpha,h}\, D_t\, \Lambda_\alpha^s\, U(t) + J_h^p\, \Lambda_\alpha^s\, U(t) = J_h^p(J_{\alpha,h} - J_\alpha)\, D_t\, \Lambda_\alpha^s\, U(t) .$$

This can be expressed as

$$J_h^p(J_h + \alpha T_h^*)\, D_t\, \Lambda_\alpha^s\, U(t) + J_h^p\, \Lambda_\alpha^s\, U(t) = J_h^p(J_{\alpha,h} - J_\alpha)\, D_t\, \Lambda_\alpha^s\, U(t)$$

or

$$J_h^{p+1}\, D_t\, \Lambda_\alpha^s\, U(t) + J_h^p\, \Lambda_\alpha^s\, U(t) = J_h^p(J_{\alpha,h} - J_\alpha - \alpha T_h^*)\, D_t\, \Lambda_\alpha^s\, U(t) .$$

On the other hand,

$$J_{\alpha,h}\, D_t\, \Lambda_{\alpha,h}^s\, U_h(t) + \Lambda_{\alpha,h}^s\, U_h(t) = 0,$$

so that

$$J_h^p\, J_{\alpha,h}\, D_t\, \Lambda_{\alpha,h}^s\, U_h(t) + J_h^p\, \Lambda_{\alpha,h}^s\, U_h(t) = 0,$$

which we write as

$$J_h^p(J_h + \alpha T_h^*)\, D_t\, \Lambda_{\alpha,h}^s\, U_h(t) + J_h^p\, \Lambda_{\alpha,h}^s\, U_h(t) = 0$$

or

$$J_h^{p+1}\, D_t\, \Lambda_{\alpha,h}^s\, U_h(t) + J_h^p\, \Lambda_{\alpha,h}^s\, U_h(t) = -\alpha J_h^p\, T_h^*\, D_t\, \Lambda_{\alpha,h}^s\, U_h(t) .$$

Set $E_h(t) = \Lambda_\alpha^s\, U(t) - \Lambda_{\alpha,h}^s\, U_h(t)$. Then

$$J_h^{p+1}\, D_t\, E_h(t) + J_h^p\, E_h(t) = J_h^p(J_{\alpha,h} - J_\alpha)\, D_t\, \Lambda_\alpha^s\, U(t) - \alpha J_h^p\, T_h^*\, D_t\, E_h(t)$$

$$\equiv J_h^p\, \delta_h(t) - \alpha J_h^p\, T_h^*\, D_t\, E_h(t) .$$

Take the energy inner product with $J_h^p\, D_t\, E_h(t)$:

$$\frac{1}{2}\frac{d}{dt} |||J_h^p\, E_h(t)|||_o^2 = ((J_h^p\, \tilde{\delta}_h(t), J_h^p\, D_t\, E_h(t)))_o - \alpha((J_h^p\, T^* D_t\, E_h(t), J_h^p\, D_t\, E_h(t)))_o .$$

Now

$$-((J_h^p\, T_h^*\, D_t\, E_h(t), J_h^p\, D_t\, E_h(t)))_o$$

$$= (-1)^{p+1}((J_h^{2p}\, T_h^*\, D_t\, E_h(t), D_t\, E_h(t)))_o$$

$$= -(([T_h^p(T_h^*\, D_t\, E_h(t))^I, T_h^p(T_h^*\, D_t\, E_h(t))^{II}], D_t\, E_h(t)))_o$$

$$= -(([T_h^{p+1}(D_t\, E_h(t)^I, 0], D_t\, E_h(t)))_o$$

$$= -a(T_h^{p+1}(D_t\, E_h(t))^I, (D_t\, E_h(t))^I)$$

$$\le ((D_t\, E_h(t))^I, (P_h^1 - P_h^o)(\Lambda_\alpha^{s+1}\, U(t))^I)_{-p,h}$$

parallel to the derivation of (2.2), Proposition 2.

As in (2.3) of that proposition, we obtain for $0 \le t \le t^*$,

$$|||E_h(t)|||_{-p,h} \le c(t^*)\{|||\tilde{\delta}_h(t)|||_{-p,h} + |||D_t\, \tilde{\delta}_h(t)|||_{-p,h} + |||E_h(0)|||_{-p,h}$$

$$+ \alpha\|(P_h^1 - P_h^o)(\Lambda_\alpha^{s+1}\, U(t))^I\|_{-p,h}\} .$$

By (4.6) we have

$$|||E_h(0)|||_{-p,h} = |||(J_\alpha - J_{\alpha,h})\, \Lambda_\alpha^{s+1}\, U_o|||_{-p,h}$$

$$\le c(\alpha)\, h^{p+q-1} |||U_o|||_{s+q-1}$$

and

$$|||D_t \tilde{\delta}_h(t)|||_{-p,h} = |||(J_{\alpha,h} - J_\alpha)\Lambda_\alpha^{s+2} U(t)|||_{-p,h}$$

$$\le c(\alpha)\, h^{p+q-1} |||U_o|||_{s+q},$$

as in (4.6), and also

$$|||\tilde{\delta}_h(t)|||_{-p,h} \le c(\alpha)\, h^{p+q-1} |||U_o|||_{s+q-1}.$$

By the analysis of (2.4), Proposition 2, and using (4.3),

$$\|(P_h^1 - P_h^0)(\Lambda_\alpha^{s+1} U(t))^I\|_{-p,h} \le ch^{p+q-1}\|(\Lambda_\alpha^{s+1} U(t)^I\|_{q-1}$$

$$\le ch^{p+q-1}|||\Lambda_\alpha^{s+1} U(t)|||_{q-2}$$

$$\le c(\alpha)\, h^{p+q-1}|||U(t)|||_{s+q-1}$$

$$\le c(\alpha)\, h^{p+q-1}|||U_o|||_{s+q-1}\ .$$

We have thus obtained

$$|||E_h(t)|||_{-p,h} \le c(t^*,\alpha)\, h^{p+q-1}|||U_o|||_{s+q} \qquad \text{for } 0 \le t \le t^*, \quad 1 \le p \le r-1$$

or

$$|||D_t^s U(t) - D_t^s U_h(t)|||_{-p,h} \le c(t^*,\alpha)\, h^{p+q-1}|||U_o|||_{s+q}.$$