On the Rate of Convergence
for the Approximation of Nonlinear Problems

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Abstract. This paper shows how to obtain from estimates on linear problems error bounds in various norms for the approximation of nonlinear problems. The theory developed in this paper is applied to finite element methods for approximating the problem $-\Delta u = \lambda e^u$ and the Navier-Stokes equations.

1. Introduction. The aim of this paper is to present an abstract theory to obtain error estimates in various norms for the approximation of solution branches of nonlinear equations with the aid of known estimates for corresponding linear problems. We give a general analysis similar to that of Brezzi-Rappaz-Raviart [2] and apply it to the study of the convergence of finite element methods for nonlinear elliptic problems. Our analysis applies to regular points and simple limit points but not to bifurcation points.

In order to illustrate our results, we consider the following model problem. Let Ω be a bounded convex domain in $\mathbf{R}^2$ with sufficiently smooth boundary $\partial\Omega$. We are interested in approximating the solution of the boundary value problem

$$(1.1) \qquad -\Delta u = \lambda e^u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where λ is a real parameter. It is well-known (see Amann [1], for instance) that there exists a maximum value λ^* of the parameter λ such that Problem (1.1) has at least one solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$; moreover, there exists a unique solution $u^* \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (1.1) for $\lambda = \lambda^*$, and (u^*, λ^*) is a turning point.

In order to parametrize the solution branch of Problem (1.1) which passes through (u^*, λ^*), a new variable t and a normalization equation are introduced as in Keller [8]. To this end we take a nontrivial function $\varphi^* \in H_0^1(\Omega)$ such that

$$(1.2) \qquad -\Delta\varphi^* = \lambda^* e^{u^*}\varphi^* \quad \text{in } \Omega,$$

and we choose a continuous linear functional φ on $L^2(\Omega)$ with $\varphi(\varphi^*) \neq 0$. Then there exist $t_0 > 0$ and a unique continuous mapping $t \in (-t_0, t_0) \to (u(t), \lambda(t)) \in (H_0^1(\Omega) \cap L^\infty(\Omega)) \times \mathbf{R}$ such that for $|t| < t_0$:

$$(1.3) \qquad \begin{aligned} -\Delta u(t) &= \lambda(t) e^{u(t)} \quad \text{in } \Omega, \\ \varphi(u(t) - u^*) &= t, \quad u(0) = u^*, \quad \lambda(0) = \lambda^*. \end{aligned}$$

In order to compute an approximation of $\Gamma = \{(u(t), \lambda(t)): |t| < t_0\}$, we consider a finite element method for discretizing Problem (1.1). Let V_h be the finite element subspace of $H_0^1(\Omega) \cap L^\infty(\Omega)$ of piecewise linear polynomials with respect to a triangulation of Ω with mesh size $h > 0$. For the sake of simplicity we assume here that Ω is a polygonal domain; in the case of a curved boundary $\partial\Omega$ we use an appropriate modification of the functions of V_h in the boundary triangles. An approximation $(u_h, \lambda) \in V_h \times \mathbf{R}$ of solutions of (1.1) is defined by

$$(1.4) \qquad D(u_h, v_h) = \lambda(e^{u_h}, v_h)_0 \quad \text{for all } v_h \in V_h,$$

where $D(\cdot, \cdot)$ and $(\cdot, \cdot)_0$ denote the Dirichlet integral and the $L^2(\Omega)$-scalar product, respectively.

Using the general results of Brezzi-Rappaz-Raviart [2], it is possible to prove that for $h \leqslant h_0$ and $\varepsilon > 0$ small enough, there exists a unique continuous mapping $t \in (-t_0, t_0) \to (u_h(t), \lambda_h(t)) \in V_h \times \mathbf{R}$ such that for $|t| < t_0$:

$$(1.5) \qquad D(u_h(t), v_h) = \lambda_h(t)(e^{u_h(t)}, v_h)_0 \quad \forall v_h \in V_h,$$

$$\varphi(u_h(t) - u^*) = t, \quad \|u(t) - u_h(t)\|_{H^1(\Omega)} + |\lambda(t) - \lambda_h(t)| < \varepsilon.$$

Moreover, error estimates for $|\lambda(t) - \lambda_h(t)|$ and $\|u(t) - u_h(t)\|_{H^1(\Omega)}$ are obtained; but by using this theory it is not possible to get optimal error estimates for $u(t) - u_h(t)$ in the L^2-norm or the L^∞-norm, for example.

In Section 2 we give an abstract setting which permits us to obtain error estimates in various norms for the approximation of the solutions of nonlinear equations. In order to justify our formalism we return to the above example by setting $V = H_0^1(\Omega) \cap L^\infty(\Omega)$, $W = L^2(\Omega)$. If T and T_h are continuous linear operators from W into V defined, for $f \in W$, by

$$Tf = u \in V \quad \text{if } D(u, v) = (f, v)_0 \text{ for all } v \in V,$$

and

$$T_h f = u_h \in V_h \quad \text{if } D(u_h, v_h) = (f, v_h)_0 \text{ for all } v_h \in V_h,$$

if $G: V \times \mathbf{R} \to W$ is the nonlinear mapping given by $G(u, \lambda) = -\lambda e^u$, then Problems (1.1) and (1.4) are, respectively, equivalent in finding pairs (u, λ) and (u_h, λ) in $V \times \mathbf{R}$ such that

$$(1.6) \qquad F(u, \lambda) := u + TG(u, \lambda) = 0,$$

$$(1.7) \qquad F_h(u_h, \lambda) := u_h + T_h G(u_h, \lambda) = 0.$$

We remark that if $(u_h, \lambda) \in V \times \mathbf{R}$ is a solution of (1.7), then u_h belongs to the range of T_h and we have $u_h \in V_h$. Result (1.5) is a direct consequence of the work of Brezzi-Rappaz-Raviart [2] and of the fact that $\lim_{h \to 0} \|T - T_h\|_{\mathscr{L}(W, V)} = 0$. Moreover, we have the error estimate

$$(1.8) \qquad |\lambda(t) - \lambda_h(t)| + \|u(t) - u_h(t)\|_V \leqslant C\|F_h(\lambda(t), u(t))\|_V,$$

with C independent of $h \leqslant h_0$ and $|t| < t_0$, which leads to an optimal error estimate $\|u(t) - u_h(t)\|_{H^1(\Omega)} \leqslant Ch$. If we want to obtain optimal L^2-error estimates from the same theory, we would like to set Problems (1.6), (1.7) in $H \times \mathbf{R}$ instead of $V \times \mathbf{R}$, with $H = L^2(\Omega)$; unfortunately we cannot define G on $H \times \mathbf{R}$, and, in Section 2, we develop an abstract setting for replacing V by H in the error estimate (1.8).

In Section 3 we apply this theory to the above example and prove quasi-optimal L^2- and L^∞-estimates for the conforming as well as for a mixed finite element method. Section 4 is devoted to the conforming finite element method for the Navier-Stokes problem, using the "stream-function formulation" with optimal error estimates in the H^1-norm.

2. Abstract Error Estimates. In this section, V and W will represent real Banach spaces; $\mathscr{L}(W, V)$ is the space of bounded linear operators from W to V. If there is no danger of confusion, the different norms $\| \cdot \|_V, \| \cdot \|_W, \ldots$ will be denoted simply by $\| \cdot \|$. The norm of a product space of the form $V \times \mathbf{R}$ will be defined by $\|(v, \lambda)\|_{V \times \mathbf{R}} := \|v\|_V + |\lambda|$.

Let T and T_h belong to $\mathscr{L}(W, V)$, and let $G: V \times \mathbf{R} \to W$ be a nonlinear C^p-mapping with $p \geqslant 2$; h denotes a positive parameter, the values of which have an accumulation point at 0.

In a neighborhood of a point $(u_0, \lambda_0) \in V \times \mathbf{R}$, we consider the nonlinear equations

$$(2.1) \qquad F(u, \lambda) = 0, \qquad F_h(u, \lambda) = 0,$$

where F and F_h: $V \times \mathbf{R} \to V$ are nonlinear mappings defined by

$$(2.2) \qquad F(u, \lambda) = u + TG(u, \lambda), \qquad F_h(u, \lambda) = u + T_h G(u, \lambda).$$

For an intuitive meaning of V, W, G, T, T_h, F, and F_h, the reader may refer to the example of Section 1.

We first suppose:

(2.3) (a) $F(u_0, \lambda_0) = 0,$

(2.4) (b) T is a compact operator,

(2.5) (c) $\lim_{h \to 0} \|T - T_h\|_{\mathscr{L}(W, V)} = 0.$

Denoting by $F'(u, \lambda) \in \mathscr{L}(V \times \mathbf{R}, V)$, by $D_u F(u, \lambda) \in \mathscr{L}(V, V)$, and by $D_\lambda F(u, \lambda) \in \mathscr{L}(\mathbf{R}, V)$, respectively, the total derivative of F at (u, λ) and the partial derivatives of F with respect to u and λ, we remark that Hypothesis (2.4) implies that $D_u F(u, \lambda)$ is a Fredholm operator of index 0 and, consequently, that $F'(u, \lambda)$ is a Fredholm operator of index 1.

We next suppose either that $D_u F(u_0, \lambda_0)$ is an isomorphism from V into itself, or that $D_u F(u_0, \lambda_0)$ has a kernel of dimension 1 and $D_\lambda(u_0, \lambda_0)$ does not belong to the range of $D_u F(u_0, \lambda_0)$; in the first case, (u_0, λ_0) is a "regular point"; in the second case, (u_0, λ_0) is a "simple limit point". This assumption can be written simply as

(2.6) (d)   Range $F'(u_0, \lambda_0) = V.$

The following result can be found in Descloux-Rappaz [4].

THEOREM 2.1. *Under Hypotheses* (2.1)–(2.6), *there exist* $h_0 > 0$ *and a neighborhood of* $(u_0, \lambda_0) \in V \times \mathbf{R}$ *such that, for* $h \leqslant h_0$, *and in this neighborhood, each of the equations* $F(u, \lambda) = 0$ *and* $F_h(u, \lambda) = 0$ *possesses a unique branch of solutions. These branches can be parametrized as* $(u(t), \lambda(t))$, $(u_h(t), \lambda_h(t))$, $|t| \leqslant t_0$, $t_0 > 0$, *with the following properties*:

(α) $(u(t), \lambda(t))$ *and* $(u_h(t), \lambda_h(t))$ *are of class* C^p; $(u(0), \lambda(0)) = (u_0, \lambda_0)$; $u'(0) \neq 0$;

(β) $\lim_{h \to 0} \sup_{|t| \leqslant t_0} \{\|u^{(k)}(t) - u_h^{(k)}(t)\|_V + |\lambda^{(k)}(t) - \lambda_h^{(k)}(t)|\} = 0$, $0 \leqslant k \leqslant p - 1$, *where* $u^{(k)}, \lambda^{(k)}, \ldots$ *are the kth derivatives of* $u, \lambda, \ldots$ *with respect to t*;

(γ) *there exists a constant C such that, for* $|t| \leqslant t_0, h \leqslant h_0, 0 \leqslant k \leqslant p - 1$, *we have*

$$\|u^{(k)}(t) - u_h^{(k)}(t)\|_V + |\lambda^{(k)}(t) - \lambda_h^{(k)}(t)| \leqslant C \sum_{l=0}^{k} \left\| \frac{d^l}{dt^l} F_h(u(t), \lambda(t)) \right\|_V. \qquad \square$$

The purpose of this section is to derive error estimates for $u^{(k)}(t) - u_h^{(k)}(t)$ in a norm different from $\|\cdot\|_V$. Let H be a Banach space for which we suppose

        (e) $V \subset H$ with continuous injection;

(2.7)   (f) there exists a constant C such that along the solution branch of the exact problem $(u(t), \lambda(t))$ defined by Theorem 2.1, we have

$$(2.8) \qquad \left\| D_\lambda^{k-l} D_u^l G(u(t), \lambda(t))[v_1, \ldots, v_l] \right\|_W \leqslant C \|v_l\|_H \prod_{i=1}^{l-1} \|v_i\|_V$$

for $|t| \leqslant t_0, 1 \leqslant k \leqslant p - 1, 1 \leqslant l \leqslant k$, and for all $v_1, \ldots, v_l \in V$.

Our main abstract result is contained in the following

THEOREM 2.2. *We assume that Hypotheses (2.1)–(2.8) are satisfied. Then, for the exact and the approximate branches of solutions defined by Theorem 2.1, there exist constants* t_0, h_0, *and C and parametrizations* $(u(t), \lambda(t)), (u_h(t), \lambda_h(t))$, *such that, for* $0 \leqslant k \leqslant p - 2, |t| \leqslant t_0, h \leqslant h_0$, *we have*

$$\|u^{(k)}(t) - u_h^{(k)}(t)\|_H + |\lambda^{(k)}(t) - \lambda_h^{(k)}(t)|$$

$$\leqslant C \left\{ \sum_{l=0}^{k} \left\| \frac{d^l}{dt^l} F_h(u(t), \lambda(t)) \right\|_H + \|F_h(u(t), \lambda(t))\|_V^2 \right\}. \qquad \square$$

Remark 2.1. In Theorems 2.1 and 2.2, the parametrizations of the exact (respectively, of the approximate) solution branch are not necessarily identical; however, in the proofs we shall show that it is possible to choose the same ones.

Remark 2.2. In general, Theorem 2.2 gives better bounds for $|\lambda^{(k)}(t) - \lambda_h^{(k)}(t)|$ than Theorem 2.1 does.

Remark 2.3. In many examples, Hypothesis (2.8) will be verified by using regularity properties of the solutions of the exact problem.

Remark 2.4. Hypotheses (2.4) and (2.5) can be weakened by using results of Descloux-Rappaz [4].

We first prove the following auxiliary result.

LEMMA 2.1. *Let* $R \in \mathscr{L}(W, V), S \in \mathscr{L}(V, W), a \in W, \alpha \in \mathbf{R}, \varphi \in \mathscr{L}(V, \mathbf{R})$; *let* $A \in \mathscr{L}(V \times \mathbf{R}, V \times \mathbf{R})$ *and* $B \in \mathscr{L}(W \times \mathbf{R}, W \times \mathbf{R})$ *be the linear operators defined by*

$$A(v, \lambda) = (v + RSv + \lambda Ra, \varphi(v) + \alpha\lambda),$$

$$B(w, \lambda) = (w + SRw + \lambda a, \varphi(Rw) + \alpha\lambda).$$

We suppose that A is an isomorphism. Then B is an isomorphism and

$$\|B^{-1}\| \leqslant 1 + (1 + \|R\|)(1 + \|S\| + \|a\|)\|A^{-1}\|.$$

Proof. We only verify the surjectivity of B. Let $g \in W, \gamma \in \mathbf{R}$ be given; setting $(v, \lambda) = A^{-1}(Rg, \gamma), w = g - \lambda a - Sv$, if follows that $B(w, \lambda) = (g, \gamma)$. From the expressions of w and λ, one deduces easily the bound for $\|B^{-1}\|$. $\square$

In the following, we shall suppose that Hypotheses (2.3)–(2.8) are satisfied. By (2.6), since $F'(u_0, \lambda_0)$ is a Fredholm operator of index 1, it follows that the kernel of $F'(u_0, \lambda_0)$ has dimension 1. Let $(z, \omega) \in V \times \mathbf{R}$ be a nonvanishing element of the kernel of $F'(u_0, \lambda_0)$. By (2.7) and the Hahn-Banach theorem, there exist $\varphi \in \mathscr{L}(H, \mathbf{R})$ and $\alpha \in \mathbf{R}$ such that

$$(2.9) \qquad \varphi(z) + \alpha\omega \neq 0.$$

Let us define the mappings $\mathscr{F}: V \times \mathbf{R} \times \mathbf{R} \to V \times \mathbf{R}$ and $\mathscr{G}: W \times \mathbf{R} \times \mathbf{R} \to W \times \mathbf{R}$ by

$$(2.10) \quad \mathscr{F}(u, \lambda, t) = \left(u + TG(u, \lambda), \varphi(u - u_0) + \alpha(\lambda - \lambda_0) - t\right),$$

$$(2.11) \quad \mathscr{G}(w, \lambda, t) = \left(w + G(Tw, \lambda), \varphi(Tw - u_0) + \alpha(\lambda - \lambda_0) - t\right).$$

Furthermore, we set $w_0 = -G(u_0, \lambda_0)$ and notice that, by (2.3), we have $u_0 = Tw_0$ and, consequently, $\mathscr{F}(u_0, \lambda_0, 0) = 0$, $\mathscr{G}(w_0, \lambda_0, 0) = 0$. (2.8) and (2.9) imply that $D_{(u, \lambda)}\mathscr{F}(u_0, \lambda_0, 0) \in \mathscr{L}(V \times \mathbf{R}, V \times \mathbf{R})$ is an isomorphism; setting, in Lemma 2.1, $R = T$, $S = D_uG(u_0, \lambda_0)$, $a = D_\lambda G(u_0, \lambda_0)$, we see that $A = D_{(u, \lambda)}\mathscr{F}(u_0, \lambda_0, 0)$ and that $B = D_{(w, \lambda)}\mathscr{G}(w_0, \lambda_0, 0) \in \mathscr{L}(W \times \mathbf{R}, W \times \mathbf{R})$ is an isomorphism. We next define $\mathscr{F}_h: V \times \mathbf{R} \times \mathbf{R} \to V \times \mathbf{R}$ and $\mathscr{G}_h: W \times \mathbf{R} \times \mathbf{R} \to W \times \mathbf{R}$ by the right-hand members of (2.10) and (2.11) when T is replaced by T_h. By (2.5) there exists a neighborhood $\mathscr{V}$ of $(u_0, \lambda_0, 0) \in V \times \mathbf{R} \times \mathbf{R}$ and a neighborhood $\mathscr{W}$ of $(w_0, \lambda_0, 0) \in W \times \mathbf{R} \times \mathbf{R}$ such that, for $0 \leqslant k \leqslant p - 1$, $\lim_{h \to 0} \mathscr{F}_h^{(k)} = \mathscr{F}^{(k)}$ uniformly in $\mathscr{V}$ and $\lim_{h \to 0} \mathscr{G}_h^{(k)} = \mathscr{G}^{(k)}$ uniformly in $\mathscr{W}$. By the classical implicit function theorem applied to $\mathscr{F}$ and $\mathscr{G}$ and by its special version given in Brezzi-Rappaz-Raviart [2] applied to $\mathscr{F}_h$ and $\mathscr{G}_h$, we obtain

LEMMA 2.2. (α) *There exist positive constants t_0, h_0, and C_0, and for $|t| \leqslant t_0$, $h \leqslant h_0$ there exist unique elements $(u(t), \lambda(t)) \in V \times \mathbf{R}$, $(w(t), \mu(t)) \in W \times \mathbf{R}$, $(u_h(t), \lambda_h(t)) \in V \times \mathbf{R}$, $(w_h(t), \mu_h(t)) \in W \times \mathbf{R}$ such that*

$$\mathscr{F}(u(t), \lambda(t), t) = 0, \qquad \|u(t) - u_0\|_V + |\lambda(t) - \lambda_0| \leqslant C_0,$$

$$\mathscr{G}(w(t), \mu(t), t) = 0, \qquad \|w(t) - w_0\|_W + |\mu(t) - \lambda_0| \leqslant C_0,$$

$$\mathscr{F}_h(u_h(t), \lambda_h(t), t) = 0, \qquad \|u_h(t) - u_0\|_V + |\lambda_h(t) - \lambda_0| \leqslant C_0,$$

$$\mathscr{G}_h(w_h(t), \mu_h(t), t) = 0, \qquad \|w_h(t) - w_0\|_W + |\mu_h(t) - \lambda_0| \leqslant C_0;$$

moreover, all functions $u(t), \lambda(t), w(t), \mu(t), \ldots$ are C^p-mappings.

(β) *For $0 \leqslant k \leqslant p - 1$ we have*

$$\lim_{h \to 0} \sup_{|t| \leqslant t_0} \left\{ \left\|u^{(k)}(t) - u_h^{(k)}(t)\right\|_V + \left|\lambda^{(k)}(t) - \lambda_h^{(k)}(t)\right| \right.$$

$$\left. + \left\|w^{(k)}(t) - w_h^{(k)}(t)\right\|_W + \left|\mu^{(k)}(t) - \mu_h^{(k)}(t)\right| \right\} = 0.$$

(γ) *There exists a constant C such that, for $0 \leqslant k \leqslant p - 1$, $|t| \leqslant t_0$, and $h \leqslant h_0$, we have*

$$\left\|u^{(k)}(t) - u_h^{(k)}(t)\right\|_V + \left|\lambda^{(k)}(t) - \lambda_h^{(k)}(t)\right| \leqslant C \sum_{l=0}^{k} \left\|\frac{d^l}{dt^l}\mathscr{F}_h(u(t), \lambda(t), t)\right\|_{V \times \mathbf{R}},$$

$$\left\|w^{(k)}(t) - w_h^{(k)}(t)\right\|_W + \left|\mu^{(k)}(t) - \mu_h^{(k)}(t)\right| \leqslant C \sum_{l=0}^{k} \left\|\frac{d^l}{dt^l}\mathscr{G}_h(w(t), \mu(t), t)\right\|_{W \times \mathbf{R}}. \quad \square$$

Note that, in Lemma 2.2, t is a pseudo-arclength parameter, as in Keller [8]. Clearly Theorem 2.1 is a direct consequence of Lemma 2.2, and we now proceed to the

Proof of Theorem 2.2. We remark that $\mathscr{G}(-G(u(t), \lambda(t)), \lambda(t), t) = 0$ and $\mathscr{G}_h(-G(u_h(t), \lambda_h(t)), \lambda_h(t), t) = 0$ for $|t| \leqslant t_0$; by the uniqueness of the quantities $u(t), \lambda(t), \ldots$ in Lemma 2.2, we deduce the following key relations:

$$(2.12) \qquad \begin{cases} w(t) = -G(u(t), \lambda(t)), \qquad u(t) = Tw(t), \\ w_h(t) = -G_h(u_h(t), \lambda_h(t)), \qquad u_h(t) = T_h w_h(t), \\ \mu(t) = \lambda(t), \mu_h(t) = \lambda_h(t) \quad \text{for } |t| \leqslant t_0, h \leqslant h_0. \end{cases}$$

In the following, C will denote a generic constant.

Since G is a map of class C^p, we can assume, without loss of generality, that for $|t| \leqslant t_0$, $h \leqslant h_0$, $(Tw(t), \lambda(t)) = (u(t), \lambda(t))$, and $(T_h w(t), \lambda(t))$ belong to a convex neighborhood $\mathscr{V} \subset V \times \mathbf{R}$ of (u_0, λ_0) for which we have

$$(2.13) \qquad \sup_{(v, \lambda) \in \mathscr{V}} \left\| G^{(k)}(v, \lambda) \right\| \leqslant C, \qquad 0 \leqslant k \leqslant p,$$

where $\| \cdot \|$ represents the norm of the k-multilinear operators from $(V \times \mathbf{R})^k$ into W.

Since $\mathscr{G}(w(t), \mu(t), t) = 0$, we obtain, by Lemma 2.2 and (2.12) for $0 \leqslant k \leqslant p - 1$, $|t| \leqslant t_0, h \leqslant h_0$:

$$\begin{aligned}
& \left\| w^{(k)}(t) - w_h^{(k)}(t) \right\|_W + \left| \lambda^{(k)}(t) - \lambda_h^{(k)}(t) \right| \\
(2.14) \qquad & \leqslant C \sum_{l=0}^{k} \left\{ \left\| \frac{d^l}{dt^l} \left(G(T_h w(t), \lambda(t)) - G(Tw(t), \lambda(t)) \right) \right\|_W \right. \\
& \qquad\qquad\qquad \left. + \left\| (T_h - T) w^{(l)}(t) \right\|_H \right\},
\end{aligned}$$

where we have used Hypothesis (2.7) and the fact that $\varphi \in \mathscr{L}(H, \mathbf{R})$. Furthermore, by (2.7), (2.12) and the uniform boundedness of the operators $T_h \in \mathscr{L}(W, V)$, we have

$$\begin{aligned}
(2.15) \qquad \left\| u^{(k)}(t) - u_h^{(k)}(t) \right\|_H & \leqslant \left\| (T_h - T) w^{(k)}(t) \right\|_H + \left\| T_h \left(w^{(k)}(t) - w_h^{(k)}(t) \right) \right\|_H \\
& \leqslant \left\| (T_h - T) w^{(k)}(t) \right\|_H + C \left\| w^{(k)}(t) - w_h^{(k)}(t) \right\|_W.
\end{aligned}$$

Clearly, the relation $\mathscr{F}(u(t), \lambda(t), t) = 0$ implies $F(u(t), \lambda(t)) = 0$ (see (2.2)), and by (2.12) we have $F_h(u(t), \lambda(t)) = (T - T_h)w(t)$; consequently, by (2.14) and (2.15), the proof of Theorem 2.2 will be completed if we establish that, for $0 \leqslant k \leqslant p - 2, |t| \leqslant t_0$, and $h \leqslant h_0$, we have

$$\begin{aligned}
(2.16) \qquad & \sum_{l=0}^{k} \left\| \frac{d^l}{dt^l} \left(G(T_h w(t), \lambda(t)) - G(Tw(t), \lambda(t)) \right) \right\|_W \\
& \leqslant C \left\{ \sum_{l=0}^{k} \left\| (T_h - T) w^{(l)}(t) \right\|_H + \left\| (T_h - T) w(t) \right\|_V^2 \right\}.
\end{aligned}$$

For the sake of simplicity, we verify (2.16) only for $k = 0$ and $k = 1$. We shall use the notations $\underline{V} = V \times \mathbf{R}$, $\underline{H} = H \times \mathbf{R}$, $G(\underline{v}) = G(v, \lambda)$ for $\underline{v} = (v, \lambda) \in \underline{V}$, and we

set $\underline{u}(t) = (u(t), \lambda(t)) = (Tw(t), \lambda(t))$, $\underline{v}_h(t) = (T_h w(t), \lambda(t))$. For $k = 1$, $k = 2$, Hypothesis (2.8) implies

$$
\begin{aligned}
\|G'(\underline{u}(t))\underline{v}\|_W &\leqslant C\|\underline{v}\|_H, \\
\|G''(\underline{u}(t))[\underline{v}_1, \underline{v}_2]\|_W &\leqslant C\|\underline{v}_1\|_V\|\underline{v}_2\|_H
\end{aligned}
\tag{2.17}
$$

for all $\underline{v}, \underline{v}_1, \underline{v}_2 \in \underline{V}$ and all $|t| \leqslant t_0$. By (2.13), (2.17), and Taylor's formula, we get

$$
\|G(\underline{v}_h) - G(\underline{u})\|_W \leqslant C\{\|\underline{v}_h - \underline{u}\|_H + \|\underline{v}_h - \underline{u}\|_V^2\},
\tag{2.18}
$$

where, from now on, we simply write $\underline{u}^{(k)}$ and $\underline{v}_h^{(k)}$ for $\underline{u}^{(k)}(t)$ and $\underline{v}_h^{(k)}(t)$. (2.18) proves (2.16) for $k = 0$. With the same arguments we obtain

$$
\begin{aligned}
\left\|\frac{d}{dt}(G(\underline{v}_h) - G(\underline{u}))\right\|_W &= \|G'(\underline{v}_h)\underline{v}_h' - G'(\underline{u})\underline{u}'\|_W \\
&\leqslant \|(G'(\underline{v}_h) - G'(\underline{u}))\underline{v}_h'\|_W + \|G'(\underline{u})(\underline{v}_h' - \underline{u}')\|_W \\
&\leqslant \|G''(\underline{u})[\underline{v}_h', \underline{v}_h - \underline{u}]\|_W + C\|\underline{v}_h'\|_V\|\underline{v}_h - \underline{u}\|_V^2 + \|G'(\underline{u})(\underline{v}_h' - \underline{u}')\|_W \\
&\leqslant C\{\|\underline{v}_h - \underline{u}\|_H + \|\underline{v}_h' - \underline{u}'\|_H + \|\underline{v}_h - \underline{u}\|_V^2\},
\end{aligned}
\tag{2.19}
$$

where we have used the boundedness of $\|\underline{v}_h'\|_V$ and relation (2.13) for $k = 1, 2, 3$; (2.19) proves relation (2.16) for $k = 1$.

3. Application I: A Second-Order Boundary Value Problem. Let $\Omega \subset \mathbf{R}^2$ be a bounded convex domain with sufficiently smooth boundary $\partial\Omega$. We consider the nonlinear boundary value problem

$$
-\Delta u = \lambda e^u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.
\tag{3.1}
$$

We will show that the abstract results can be applied in the context of approximation of branches of solutions of (3.1) by conforming and nonconforming finite element methods.

First we will consider the case of conforming finite elements. We use the standard notation of Sobolev spaces $H_0^1(\Omega)$, $H^m(\Omega)$, $W^{m,p}(\Omega), \ldots$ and set $V := H_0^1(\Omega) \cap L^\infty(\Omega)$, $W := L^2(\Omega)$. A weak formulation of (3.1) is
    Find $(u, \lambda) \in V \times \mathbf{R}$ with

$$
D(u, v) = \lambda(e^u, v)_0 \quad \text{for all } v \in V,
\tag{3.2}
$$

where $D(\cdot, \cdot)$ and $(\cdot, \cdot)_0$ denote Dirichlet's integral and the $L^2(\Omega)$-scalar product, respectively.

We define the operator $T \in \mathscr{L}(W, V)$ by

$$
Tf = u \quad \text{if } D(u, v) = (f, v)_0 \text{ for all } v \in V
$$

holds, and the nonlinear operator $G \colon V \times \mathbf{R} \to W$ is given by $G(u, \lambda) := -\lambda e^u$. Then (3.2) is equivalent to the following problem:
    Find $(u, \lambda) \in V \times \mathbf{R}$ with

$$
F(u, \lambda) := u + TG(u, \lambda) = 0.
\tag{3.3}
$$

To determine approximate solutions of (3.3) we consider subspaces $V_h \subset V$, which consist of continuous, piecewise linear functions with respect to regular triangulations of Ω with mesh-size $h > 0$. (In the case of a curved boundary $\partial\Omega$ we use an appropriate modification of the functions of V_h in the boundary triangles. See

Zlamal [16], for example.) With respect to the approximation properties of the spaces V_h we refer to Ciarlet [3], for example. We define $T_h \in \mathscr{L}(W, V_h)$ by

$$T_h f = u_h \quad \text{if } D(u_h, \varphi) = (f, \varphi)_0 \text{ for all } \varphi \in V_h.$$

The following estimates are known:

$$(3.4) \qquad \|(T - T_h)f\|_{L^2(\Omega)} + h\|(T - T_h)f\|_{H_0^1(\Omega)} \leqslant Ch^2\|f\|_{L^2(\Omega)};$$

$$(3.5) \qquad \|(T - T_h)f\|_{L^\infty(\Omega)} \leqslant Ch\|f\|_{L^2(\Omega)};$$

$$(3.6) \quad \|(T - T_h)f\|_{L^\infty(\Omega)} \leqslant Ch^2|\ln h| \, \|Tf\|_{W^{2,\infty}(\Omega)} \quad \text{if } Tf \in W^{2,\infty}(\Omega) \text{ holds.}$$

Here C denotes numerical constants which are independent of f and h. For proofs of (3.4) and (3.6) we refer to Nitsche [10], [11]. Estimate (3.5) follows from (3.4) and inverse estimates. An immediate consequence of (3.4) and (3.5) is

$$(3.7) \qquad \|T - T_h\|_{\mathscr{L}(W, V)} \leqslant Ch.$$

The Galerkin problem to approximate the solution of (3.3) is formulated as follows:
Find $(u_h, \lambda) \in V \times \mathbf{R}$ with

$$(3.8) \qquad F_h(u_h, \lambda) := u_h + T_h G(u_h, \lambda) = 0.$$

Note that if (u_h, λ) is a solution of (3.8), then u_h belongs to the range of T_h and we have $u_h \in V_h$.

Assume $(u_0, \lambda_0) \in V \times \mathbf{R}$ fulfills (2.3) and (2.6). First of all, by Theorem 2.1, we have:

For $|t| \leqslant t_0$, $h \leqslant h_0$, there exist branches $(u(t), \lambda(t))$, $(u_h(t), \lambda_h(t))$ of solutions of (3.3) and (3.8), respectively, with $(u(0), \lambda(0)) = (u_0, \lambda_0)$ such that, for $k \in \mathbf{N}$,

$$(3.9) \qquad \left\|u^{(k)}(t) - u_h^{(k)}(t)\right\|_{H_0^1(\Omega) \cap L^\infty(\Omega)} + \left|\lambda^{(k)}(t) - \lambda_h^{(k)}(t)\right| \leqslant C_k h$$

holds, where C_k is independent of h and t.

The proof of (3.9) is an immediate consequence of (3.7) and the fact that G is of class C^∞. Applying Theorem 2.2, we get

THEOREM 3.1. *For the branches* $(u(t), \lambda(t))$, $(u_h(t), \lambda_h(t))$ *as above, the estimate*

$$(3.10) \qquad \left\|u^{(k)}(t) - u_h^{(k)}(t)\right\|_{L^2(\Omega)} + \left|\lambda^{(k)}(t) - \lambda_h^{(k)}(t)\right| \leqslant C_k h^2$$

holds for all $k = 0, 1, 2, \ldots$. *If, moreover, u is a C^p-mapping of $]-t_0, t_0[$ into $W^{2,\infty}(\Omega)$, $p \geqslant 2$, then, for $0 \leqslant k \leqslant p - 2$,*

$$(3.11) \qquad \left\|u^{(k)}(t) - u_h^{(k)}(t)\right\|_{L^\infty(\Omega)} \leqslant C_k h^2|\ln h|$$

holds. Here C_k denotes constants which are independent of t and h.

Proof. First we set $H := L^2(\Omega)$. Then Assumption (2.7) is trivial; to apply Theorem 2.2 we have to prove (2.8). Since we can write $G(u, \lambda) = \lambda G(u)$ with $G(u) = e^u$, we only have to show

$$(3.12) \qquad \left\|D^l G(u)[v_1, \ldots, v_l]\right\|_W \leqslant C\|v_l\|_H \prod_{i=1}^{l-1} \|v_i\|_V$$

with $W = L^2(\Omega)$, $V = H_0^1(\Omega) \cap L^\infty(\Omega)$. Since $D^l G(u)[v_1, \ldots, v_l] = e^u \cdot v_1 \cdots v_l$ holds, (3.12) follows with

$$C = \exp\left(\|u\|_{L^\infty(\Omega)}\right).$$

Now (3.10) is a direct consequence of Theorem 2.2 and estimate (3.4). To show (3.11) we set $H := L^\infty(\Omega)$. Then (2.7) and (2.8) are trivial and the result follows by combining Theorem (2.2) and estimates (3.4) and (3.6). $\square$

Next we will apply a mixed finite element method to get approximations of a branch of solutions of Problem (3.1). In this case we set $V := L^\infty(\Omega) \times L^2(\Omega)^2$ and $W := L^2(\Omega)$. Furthermore, let $H(\text{div}; \Omega)$ be the subspace of those elements $z \in L^2(\Omega)^2$ with div $z \in L^2(\Omega)$. Using these notations, another weak formulation of (3.1) is:

Find $((u_1, u_2), \lambda) \in V \times \mathbf{R}$ with

(3.13)
$$-(u_1, \text{div } z)_0 = (u_2, z)_0 \qquad \text{for all } z \in H(\text{div}; \Omega),$$
$$-(\text{div } u_2, w)_0 = \lambda(e^{u_1}, w)_0 \quad \text{for all } w \in L^2(\Omega),$$

where $(\cdot, \cdot)_0$ denotes the scalar product in $L^2(\Omega)$ as well as in $L^2(\Omega)^2$.

In this case the operator $T: W \to V$ is defined as follows. For $f \in W$ denote by $u = (u_1, u_2) \in L^2(\Omega) \times H(\text{div}; \Omega)$ the solution of

$$-(u_1, \text{div } z)_0 = (u_2, z)_0 \quad \text{for all } z \in H(\text{div}; \Omega),$$
$$-(\text{div } u_2, w)_0 = (f, w)_0 \quad \text{for all } w \in L^2(\Omega).$$

Using shift theorems and Sobolev's imbedding theorem, we get $Tf = u \in V$.

Further, let $G: V \times \mathbf{R} \to W$ be given by $G(u, \lambda) := -\lambda e^{u_1}$. Then Problem (3.13) is equivalent to:

Find $(u, \lambda) \in V \times \mathbf{R}$ with

(3.14)
$$F(u, \lambda) := u + TG(u, \lambda) = 0.$$

To determine approximations of the solution of (3.14) by a mixed finite element method, we use the subspaces introduced by Raviart-Thomas [12]. With respect to a quasi-uniform triangulation of Ω with mesh size $h > 0$, we denote by S_h the space of piecewise linear functions without any boundary conditions on $\partial\Omega$ or continuity requirements across interelement boundaries. H_h are subspaces of $H(\text{div}; \Omega)$ which consist of pairs $\zeta = (\zeta_1, \zeta_2)$ such that ζ_1 and ζ_2 are special polynomials of degree 2. We set $V_h := S_h \times H_h$; the operator $T_h: W \to V_h$ is defined by $T_h f = u_h = (u_{1,h}, u_{2,h})$ if

$$-(u_{1,h}, \text{div } \zeta)_0 = (u_{2,h}, \zeta)_0 \quad \text{for all } \zeta \in H_h,$$
$$-(\text{div } u_{2,h}, \omega)_0 = (f, \omega)_0 \quad \text{for all } \omega \in S_h.$$

The following estimates for $u = Tf$ and $u_h = T_h f$ are known:

(3.15)
$$\|u_1 - u_{1,h}\|_{L^2(\Omega)} + h\|u_2 - u_{2,h}\|_{L^2(\Omega)^2} \leqslant Ch^2\|f\|_{L^2(\Omega)},$$

(3.16)
$$\|u_1 - u_{1,h}\|_{L^\infty(\Omega)} \leqslant Ch\|f\|_{L^2(\Omega)}.$$

For a proof of (3.15) see Raviart-Thomas [12], Scholz [13]; (3.16) follows from (3.15) and inverse estimates. A direct consequence of (3.15) and (3.16) is

(3.17)
$$\|T - T_h\|_{\mathcal{L}(W, V)} \leqslant Ch.$$

The Galerkin problem corresponding to (3.14) is defined by

(3.18)
$$F_h(u_h, \lambda) := u_h + T_h G(u_h, \lambda) = 0.$$

If (2.3) and (2.6) are satisfied, for $|t| \leqslant t_0$, $h \leqslant h_0$, $k \in N$, according to Theorem 2.1 there exist branches $((u_1(t), u_2(t)), \lambda(t))$ and $((u_{1,h}(t), u_{2,h}(t)), \lambda_h(t))$ with

$$
(3.19) \quad \begin{aligned}
\left\| u_1^{(k)}(t) - u_{1,h}^{(k)}(t) \right\|_{L^\infty(\Omega)} + \left\| u_2^{(k)}(t) - u_{2,h}^{(k)}(t) \right\|_{L^2(\Omega)^2} \\
+ \left| \lambda^{(k)}(t) - \lambda_h^{(k)}(t) \right| \leqslant C_k h,
\end{aligned}
$$

where C_k is independent of t and h.

To get a better order of convergence for the approximation of u_1 in the $L^2(\Omega)$-norm, we need

LEMMA 3.1. *Let $f \in W$ be given and set $u = (u_1, u_2) = Tf$ and $u_h = (u_{1,h}, u_{2,h}) = T_h f$. Then the estimate*

$$
(3.20) \qquad \left\| u_2 - u_{2,h} \right\|_{H^{-1}(\Omega)^2} \leqslant Ch^2 \|f\|_{L^2(\Omega)},
$$

with $H^{-1}(\Omega) = H_0^1(\Omega)^*$, holds, where C is independent of f and h.

Proof. By definition we have

$$
\left\| u_2 - u_{2,h} \right\|_{H^{-1}(\Omega)^2} = \sup \frac{(u_2 - u_{2,h}, z)_0}{\|z\|_{H_0^1(\Omega)^2}},
$$

where the supremum has to be taken over all $z \in H_0^1(\Omega)^2$. Let z be fixed. In Raviart-Thomas [12] a linear operator $\Pi_h\colon H^1(\Omega)^2 \to H_h$ has been constructed with

$$
\| z - \Pi_h z \|_{L^2(\Omega)^2} \leqslant Ch \|z\|_{H^1(\Omega)^2}
$$

and div $\Pi_h z = P_h$ div z, where $P_h\colon L^2(\Omega) \to S_h$ denotes the $L^2(\Omega)$-projection. Using the defining relations of T and T_h, we therefore get

$$
\left| (u_2 - u_{2,h}, z)_0 \right| \leqslant \left| (u_2 - u_{2,h}, z - \Pi_h z)_0 \right| + \left| (u_1 - u_{1,h}, P_h \operatorname{div} z)_0 \right|
$$

$$
\leqslant C \left\{ h \| u_2 - u_{2,h} \|_{L^2(\Omega)^2} + \| u_1 - u_{1,h} \|_{L^2(\Omega)} \right\} \cdot \| z \|_{H^1(\Omega)^2}.
$$

Together with (3.15) the assertion follows.   □

Using this result, we get

THEOREM 3.2. *With the same notations as above, the estimate*

$$
(3.21) \quad \begin{aligned}
\left\| u_1^{(k)}(t) - u_{1,h}^{(k)}(t) \right\|_{L^2(\Omega)} + \left\| u_2^{(k)}(t) - u_{2,h}^{(k)}(t) \right\|_{H^{-1}(\Omega)^2} \\
+ \left| \lambda^{(k)}(t) - \lambda_h^{(k)}(t) \right| \leqslant C_k h^2
\end{aligned}
$$

holds, C_k independent of h and t.

Proof. We set $H := L^2(\Omega) \times H^{-1}(\Omega)^2$ and again apply Theorem 2.2. By (3.15) and (3.20) we get

$$
\| (T - T_h)f \|_H \leqslant Ch^2 \|f\|_W
$$

for $f \in W$, $W = L^2(\Omega)$. The inclusion (2.7) is trivial, and (2.8) is shown in the same way as in the proof of Theorem 3.1; so the assertion follows.   □

Remark 3.1. If the solution of (3.14) is sufficiently smooth, an estimate analogous to (3.11) can be proved, using L^∞-estimates for the mixed finite element method. (See Johnson-Thomée [7], Scholz [14].)

4. Application II: The Navier-Stokes Problem. In this section we consider the Navier-Stokes equation (stream-function formulation) for an incompressible viscous fluid. Let $\Omega \subset \mathbf{R}^2$ be a convex polygonal domain. For $g \in H^{-2}(\Omega)$ we consider the problem

$$
(4.1) \qquad
\begin{aligned}
\nu \Delta^2 \psi + \psi_x \Delta \psi_y - \psi_y \Delta \psi_x &= g \quad \text{in } \Omega, \\
\psi = \psi_n &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
$$

where $\nu > 0$ is the viscosity coefficient; $\Delta\psi$ is the vorticity. (Again we use standard notations of Sobolev spaces; especially, we denote $H^{-k}(\Omega) := H_0^k(\Omega)^*$ and $W^{-k,p}(\Omega) := W_0^{k,q}(\Omega)^*$, $k \in \mathbf{N}$, $1 < p < \infty$, $1/p + 1/q = 1$.) A weak formulation of Problem (4.1) is:

Find $\psi \in H_0^2(\Omega)$ such that

$$
(4.2) \qquad \nu\left(\Delta\psi, \Delta v\right) + \left(\psi_x \Delta\psi_y - \psi_y \Delta\psi_x, v\right) = (g, v) \quad \text{for all } v \in H_0^2(\Omega),
$$

where $(\cdot, \cdot)$ denotes the $L^2(\Omega)$-scalar product of the pairing between elements of $W^{-k,p}(\Omega)$ and $W_0^{k,q}(\Omega)$. (For details we refer to Girault-Raviart [5] or Teman [15], for example.)

For functions $u, v \in C_0^\infty(\Omega)$ we find, by

$$
u_x \Delta u_y - u_y \Delta u_x = \left(u_x \Delta u\right)_y - \left(u_y \Delta u\right)_x
$$

and integration by parts, that

$$
(4.3) \qquad \left(u_x \Delta u_y - u_y \Delta u_x, v\right) = \left(v_x u_y - v_y u_x, \Delta u\right);
$$

this means for $u \in H_0^2(\Omega)$ we have $u_x \Delta u_y - u_y \Delta u_x \in W^{-1,4/3}(\Omega)$. Since the operator $T: H^{-2}(\Omega) \to H_0^2(\Omega)$, defined by

$$
Tf = u \quad \text{if } \left(\Delta u, \Delta v\right) = (f, v) \text{ for all } v \in H_0^2(\Omega),
$$

is also continuous from $W^{-1,4/3}(\Omega)$ into $W^{3,4/3}(\Omega)$ (see Grisvard [6]), the solution of the Navier-Stokes problem fulfills the regularity condition $\psi \in H_0^2(\Omega) \cap W^{3,4/3}(\Omega)$ for $g \in W^{-1,4/3}(\Omega)$. If, moreover, $g \in H^{-1}(\Omega)$ holds, the regularity result $\psi \in H_0^2(\Omega) \cap H^3(\Omega)$ follows by Sobolev's imbedding theorem and the fact that T is also continuous as an operator from $H^{-1}(\Omega)$ to $H_0^2(\Omega) \cap H^3(\Omega)$. (Consult also Kellogg-Osborn [9].)

We want to show that the general theory of Section 2 can be applied if we choose $W := H^{-2}(\Omega)$, $V := W^{1,4}(\Omega)$, and $H := H_0^1(\Omega)$, and if the right-hand side g of Problem (4.1) is regular enough. The crucial point for further analysis is the following

LEMMA 4.1. *Let $u, v \in C_0^\infty(\Omega)$ be given. Then*

$$
(4.4) \qquad \left(u_x \Delta u_y - u_y \Delta u_x, v\right) = \left(u_x u_y, v_{yy} - v_{xx}\right) + \left(u_x^2 - u_y^2, v_{xy}\right)
$$

holds.

Proof. First we use (4.3) and get

$$
\left(u_x \Delta u_y - u_y \Delta u_x, v\right) = \left(v_x u_y - v_y u_x, \Delta u\right).
$$

Now we write

$$\left(v_x u_y, \Delta u\right) = \left(u_y u_{xx} + u_y u_{yy}, v_x\right) = \left(\left(u_x u_y\right)_x - \tfrac{1}{2} u_{x|y}^2 + \tfrac{1}{2} u_{y|y}^2, v_x\right)$$

$$= -\left(u_x u_y, v_{xx}\right) + \tfrac{1}{2}\left(u_x^2 - u_y^2, v_{xy}\right).$$

By the same argument we get

$$\left(v_y u_x, \Delta u\right) = -\left(u_x u_y, v_{yy}\right) + \tfrac{1}{2}\left(u_y^2 - u_x^2, v_{xy}\right),$$

and the result follows. □

Let $g \in H^{-2}(\Omega)$ be fixed. We get $G: V \times \mathbf{R} \to W$ with

$$G(u, \lambda) := \lambda\left(u_x \Delta u_y - u_y \Delta u_x - g\right).$$

Using Lemma 4.1, we see that G is well defined. Further, Problem (4.2) is equivalent to

(4.5) $$\psi + TG(\psi, \lambda) = 0, \qquad \lambda = 1/\nu.$$

By the regularity results mentioned above, we find $G(\psi, \lambda) \in W^{-1,4/3}(\Omega)$ for $g \in W^{-1,4/3}(\Omega)$, and $G(\psi, \lambda) \in H^{-1}(\Omega)$ for $g \in H^{-1}(\Omega)$.

Let V_h, $0 < h \leqslant h_0$, be a family of finite-dimensional subspaces of $H_0^2(\Omega)$ with the following approximation property:

For $u \in H_0^2(\Omega) \cap W^{3,q}(\Omega)$ there exists $u_h \in V_h$ with

(4.6) $$\|u - u_h\|_{W^{k,p}(\Omega)} \leqslant Ch^{3-k-2(1/q-1/p)}\|u\|_{W^{3,q}(\Omega)},$$

$0 \leqslant k \leqslant 2, 1 < q \leqslant p < \infty$, where C is independent of u and h.

Remark 4.1. Approximation property (4.6) is typical for conforming finite element subspaces with respect to regular triangulations of Ω with mesh size $h > 0$. (See Ciarlet [3], for example.)

The operator $T_h: W \to V_h$ is defined by

$$T_h f = u_h \in V_h \quad \text{if } \left(\Delta u_h, \Delta \varphi\right) = (f, \varphi) \text{ for all } \varphi \in V_h.$$

The discrete analogue of Problem (4.5) is

(4.7) $$\psi_h + T_h G(\psi_h, \lambda) = 0.$$

To apply the general theory of Section 2 we need the following lemmas.

LEMMA 4.2. *The estimate*

(4.8) $$\|T - T_h\|_{\mathscr{L}(W,V)} \leqslant Ch^{1/2}$$

holds with C independent of h.

Proof. Let $f \in W$ be given. By definition we have

$$\|(T - T_h)f\|_V = \sup_{z \in W^{-1,4/3}(\Omega)} \frac{(u - u_h, z)}{\|z\|_{W^{-1,4/3}(\Omega)}}.$$

For $z \in W^{-1,4/3}(\Omega)$ fixed there exists $w \in H_0^2(\Omega) \cap W^{3,4/3}(\Omega)$ such that $\Delta^2 w = z$ and

(4.9) $$\|w\|_{W^{3,4/3}(\Omega)} \leqslant C\|z\|_{W^{-1,4/3}(\Omega)}$$

holds with C independent of z. We get

$$\left(u - u_h, z\right) = \left(\Delta(u - u_h), \Delta w\right) = \left(\Delta(u - u_h), \Delta(w - \omega)\right), \qquad \omega \in V_h,$$

where we used the defining equations of u and u_h, respectively. If $\omega \in V_h$ is especially chosen as the orthogonal projection of w onto V_h with respect to the scalar product $(\Delta \cdot, \Delta \cdot)$, we get

$$(\Delta(u - u_h), \Delta(w - \omega)) = (\Delta u, \Delta(w - \omega)) \leq C\|u\|_{H_0^2(\Omega)} \inf_{\omega \in V_h} \|w - \omega\|_{H_0^2(\Omega)}.$$

Using the approximation property (4.6) with $k = 2$, $q = 4/3$, and $p = 2$, we find

$$\inf_{\omega \in V_h} \|w - \omega\|_{H_0^2(\Omega)} \leq Ch^{1/2}\|w\|_{W^{3,4/3}(\Omega)}.$$

By (4.9) and the a priori estimate

$$\|u\|_{H_0^2(\Omega)} \leq C\|f\|_{H^{-2}(\Omega)} = C\|f\|_W,$$

we finally get

$$\|(T - T_h)f\|_V \leq Ch^{1/2}\|f\|_W,$$

and (4.8) is shown. □

Remark 4.2. With the same duality arguments, for $f \in H^{-1}(\Omega)$ one can prove

$$(4.10) \qquad \|(T - T_h)f\|_V \leq Ch^{3/2}\|f\|_{H^{-1}(\Omega)}$$

and

$$(4.11) \qquad \|(T - T_h)f\|_H \leq Ch^2\|f\|_{H^{-1}(\Omega)},$$

$V = W_0^{1,4}(\Omega)$ and $H = H_0^1(\Omega)$.

LEMMA 4.3. *Let ψ be a solution of (4.5) and assume $g \in W^{-1,4/3}(\Omega)$: i.e., $\psi \in H_0^2(\Omega) \cap W^{3,4/3}(\Omega)$. For all $u \in H_0^1(\Omega)$ the estimate*

$$(4.12) \qquad \|D_\psi G(\psi, \lambda)u\|_W \leq C|\lambda| \, \|\psi\|_{W^{3,4/3}(\Omega)}\|u\|_H$$

holds, C independent of ψ, λ and u.

Proof. Let $v \in H_0^2(\Omega)$ be fixed. Using Lemma 4.1 we have

$$(G(\psi, \lambda), v) = \lambda\left[(\psi_x\psi_y, v_{yy} - v_{xx}) + (\psi_x^2 - \psi_y^2, v_{xy}) - (g, v)\right],$$

and, consequently, we find

$$(D_\psi G(\psi, \lambda)u, v) = \lambda(\psi_x u_y + \psi_y u_x, v_{yy} - v_{xx}) + 2\lambda(\psi_x u_x - \psi_y u_y, v_{xy}).$$

We therefore get

$$(4.13) \qquad \left|(D_\psi G(\psi, \lambda)u, v)\right| \leq C|\lambda| \, \|\psi\|_{W^{1,\infty}(\Omega)}\|v\|_{H_0^2(\Omega)}\|u\|_{H_0^1(\Omega)}.$$

By Sobolev's imbedding theorem we find

$$\|\psi\|_{W^{1,\infty}(\Omega)} \leq C\|\psi\|_{W^{3,4/3}(\Omega)};$$

together with (4.13) we get (4.12). □

Now let (ψ_0, λ_0) be a solution of (4.5) such that (2.3) and (2.6) hold, and let $(\psi(t), \lambda(t))$ and $(\psi_h(t), \lambda_h(t))$ be branches of solutions of (4.5) and (4.7), respectively, according to Theorem 2.1. For simplicity we assume $g \in H^{-1}(\Omega)$, i.e., $\psi(t) \in H_0^2(\Omega) \cap H^3(\Omega)$. Then we get

THEOREM 4.1. *For $|t| \leq t_0$, $h \leq h_0$ the estimate*

$$(4.14) \qquad \|\psi^{(k)}(t) - \psi_h^{(k)}(t)\|_{W_0^{1,4}(\Omega)} + |\lambda^{(k)}(t) - \lambda_h^{(k)}(t)| \leq C_k h^{3/2},$$

$k \in \mathbf{N}$, *holds. Moreover, for* $k = 0$,

(4.15) $$\|\psi(t) - \psi_h(t)\|_{H_0^1(\Omega)} + |\lambda(t) - \lambda_h(t)| \leqslant Ch^2$$

holds.

Proof. (4.14) is a consequence of Theorem 2.1, Lemma 4.2, and estimate (4.10). Lemma 4.3 shows that Assumption (2.8) of the general theory is fulfilled for $p = 2$; (4.15) therefore follows by Theorem 2.2, (4.10), and (4.11). □

Remark 4.3. Estimates analogous to (4.14) and (4.15) can be derived if only $g \in W^{-1,4/3}(\Omega)$ is assumed.

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