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Supplement to Boundary Value Techniques for Initial Value Problems in Ordinary Differential Equations

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APPENDIX

In [1], the following Theorem 6 was presented for the error of a Galerkin method for initial value problems. Here we present a proof.

THEOREM. *Let* U *be the solution of* (3.2) *where* (3.3), (3.4) *in* [1] *are satisfied. Then the Galerkin solution* $\tilde{U}$, *in the space of piecewise polynomial continuous functions of degree* p, *defined by* (3.5a,b,c) *in* [1] *satisfies*

$$\| U-\tilde{U} \| = O(h^{p+\nu})\{\|\varepsilon U\|^2_{p+2} + \|U\|^2_{p+1}\}^{\frac{1}{2}}, \quad h \to 0,$$

where $\nu = 1$ *if* $p = 1$, $1 \geq \nu \geq \frac{1}{2}$ *if* $p = 3,5,\ldots$ *and* $\nu = 0$ *if* p *is even, and*

$$\| v \|^2 = \tfrac{1}{2}(\varepsilon V(T),V(T)) - \int_{t_0}^{T} \rho(t)\|V(t)\|^2 dt.$$

PROOF. Our first objective is to derive an estimate of θ. We have $\theta(t_0) = \eta(t_0) = 0$ and

$$(1) \qquad a(\tilde{U};\theta) - a(U_I;\theta) = \int_{t_0}^{T} \{(\varepsilon\dot{\theta},\theta) - (F(t,\tilde{U})-F(t,U_I),\theta)\}dt.$$

From

$$\int_{t_0}^{T} (\varepsilon\dot{\theta},\theta)dt = -\int_{t_0}^{T} (\dot{\theta},\varepsilon\theta)dt + [(\varepsilon\theta,\theta)]_{t_0}^{T},$$

if follows that

$$\int_{t_0}^{T} (\varepsilon\dot{\theta},\theta)dt = \tfrac{1}{2}(\varepsilon\theta(T),\theta(T)).$$

Hence by (3.3),

(2) $$a(\tilde{U};\theta) - a(U_I;\theta) = \frac{1}{2}(\varepsilon\theta(T),\theta(T)) - \int_{t_0}^{T} (F(t,\tilde{U})-F(t,U_I),\theta)dt$$
$$\geq \frac{1}{2}(\varepsilon\theta(T),\theta(T)) - \int_{t_0}^{T} \rho(t)\|\theta(t)\|^2 dt := \|\theta\|^2.$$

Further we get from (3.6) and the Lipschitz continuity (3.4), in [1]

(3) $$a(\tilde{U};\theta) - a(U_I;\theta) = a(\tilde{U};\theta) - a(U;\theta) + a(U;\theta) - a(U_I;\theta) =$$
$$= a(U;\theta) - a(U_I;\theta) \leq |\int_{t_0}^{T} (\varepsilon\dot{\eta},\theta)dt| + C\int_{t_0}^{T} \|\eta\|\|\theta\| dt.$$

First we shall estimate the term $\int_{t_0}^{T} (\varepsilon\dot{\eta},\theta)dt$. Since ε is a diagonal matrix, we may consider each component individually, i.e., $\int_{t_0}^{T} \dot{\eta}_k\theta_k dt$, where η_k,θ_k are scalar functions. Since $\theta_k \in \mathring{S}_h$, we have

$$\theta_k = \sum_{i=0}^{N-1}\sum_{j=1}^{p} \gamma_{i,j}\tilde{\phi}_{i,j}$$

where $\{\tilde{\phi}_{i,j}\}$, $i = 0,1,\ldots,N-1$, $j = 1,2,\ldots,p$ is a set of basis functions (test functions) spanning $\mathring{S}_h$, but not necessarily equal to $\phi_{i,j}$. We write $\eta_k = \tilde{\eta}_k + \eta_R$, where $\tilde{\eta}_k$ is the leading (polynomial) term in an expansion of the interpolation error (for more details, see e.g. Axelsson and Gustafsson [2]). For instance, if we use piecewise linear basis functions, then at t_i, $i = 1,3,\ldots,N-1$ we have

$$\tilde{\eta}_k(t) = \begin{cases} \frac{1}{2}\ddot{U}_k(t_i)(t-t_{i-1})(t-t_i), & t_{i-1} \leq t \leq t_i \\ \frac{1}{2}\ddot{U}_k(t_i)(t-t_i)(t-t_{i+1}), & t_i < t \leq t_{i+1} \end{cases}.$$

(For simplicity, we may assume that N is even.) With the piecewise polynomial basis functions of degree p, an easy calculation shows that

(4) $$\left\{\int_{t_0}^{T} \dot{\tilde{\eta}}_k^2 dt\right\}^{\frac{1}{2}} = O(h^p)\|U_k\|_{p+1}, \left\{\int_{t_0}^{T} \dot{\eta}_R^2 dt\right\}^{\frac{1}{2}} = O(h^{p+1})\|U_k\|_{p+2}, \quad h \to 0.$$

We shall prove that because of cancellation

(5) $$|\int_{t_0}^{T} \dot{\eta}_k\theta_k dt| \leq \rho_0\int_{t_0}^{T} \theta_k^2 dt + \frac{1}{2}\theta_k^2(T) + O(h^{2(p+\nu_k)})\|U_k\|_{p+2}^2, \begin{cases} \nu_k=1, & p=1 \\ \nu_k \geq \frac{1}{2}, & p=3,5,\ldots \end{cases}$$

i.e.,

$$|\int_{t_0}^{T} (\varepsilon\dot{\eta},\theta)dt| \leq \rho_0\int_{t_0}^{T} \|\theta(t)\|^2 dt + \frac{1}{2}(\varepsilon\theta(T),\theta(T)) + O(h^{2(p+\nu_k)})\|\varepsilon U\|_{p+2}^2.$$

We have

(6) $$\int_{t_0}^{T} \dot{\eta}_k\theta_k dt = \int_{t_0}^{T} \dot{\tilde{\eta}}_k\theta_k dt + \int_{t_0}^{T} \dot{\eta}_R\theta_k dt.$$

From (4) there follows

$$|\int_{t_0}^{T} \dot{\eta}_R\theta_k dt| \leq O(h^{p+1})\left\{\int_{t_0}^{T} \theta_k^2 dt\right\}^{\frac{1}{2}}\|U_k\|_{p+2} \leq \rho_0\int_{t_0}^{T} \theta_k^2 dt + O(h^{2(p+1)})\|U_k\|_{p+2}^2,$$

and it remains to consider the first term in (6). We have

$$\int_{t_0}^{T} \dot{\tilde{\eta}}_k\theta_k dt = -\int_{t_0}^{T} \tilde{\eta}_k\dot{\theta}_k dt.$$

As test functions $\tilde{\phi}_{i,j}$ we choose the basis functions of degree q, $q = 1,2,\ldots,p$, at each t_i, $i = 1,2,\ldots,N$, with support on (t_{i-1},t_{i+1}), see Fig.1.

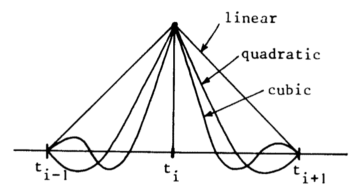

Figure 1 Test functions $\tilde{\phi}_{i,j}$, $j = 1,2,\ldots,p$ (p=3).

We have

$$(7)\qquad \int_{t_0}^{T} \tilde{\eta}_k \dot{\theta}_k dt = \sum_{i,j} \int_{t_0}^{T} \gamma_{i,j} \tilde{\eta}_k \dot{\tilde{\phi}}_{i,j} dt =$$

$$= \sum_{i=1}^{N-1} \sum_{j=1}^{p} \gamma_{i-1,j} \int_{t_{i-1}}^{t_{i+1}} \tilde{\eta}_k \dot{\tilde{\phi}}_{i-1,j} dt + \sum_{j=1}^{p} \gamma_{N-1,j} \int_{t_{N-1}}^{t_N} \tilde{\eta}_k \dot{\tilde{\phi}}_{N-1,j} dt.$$

Note that $\tilde{\eta}_k$ and $\dot{\theta}_k$ are polynomials of degree p + 1 and p − 1, respectively on each subinterval. Also note that $\tilde{\eta}_k$ is zero at the (p+1) Lobatto points. Hence, if we apply Lobatto quadrature we get (from the error term of the quadrature),

$$\int_{t_{i-1}}^{t_i} \tilde{\eta}_k \dot{\tilde{\phi}}_{i-1,j} dt = O\left(h^{2p+1}\right) \frac{\partial^{2p}}{\partial t^{2p}} \left(\tilde{\eta}_k \dot{\tilde{\phi}}_{i-1,j}\right)$$

$$= O\left(h^{2p+1}\right)\binom{2p}{p+1} \frac{\partial^{p+1}}{\partial t^{p+1}} \left(\tilde{\eta}_k\right) \frac{\partial^p}{\partial t^p}\left(\tilde{\phi}_{i-1,j}\right) = O\left(h^{p+1}\right).$$

The loss of p in the exponent is due to the fact that

$$\frac{\partial^p}{\partial t^p}\left(\tilde{\phi}_{i-1,j}\right) = O\left(h^{-p}\right).$$

Note however that for p odd this derivative, which is piecewise constant, appears with the same numerical value but with opposite signs in the two adjacent intervals (t_{i-1}, t_i) and (t_i, t_{i+1}). Hence

$$\int_{t_{i-1}}^{t_{i+1}} \tilde{\eta}_k \dot{\tilde{\phi}}_{i-1,j} dt = 0, \quad i = 1,2,\ldots,N-1,$$

and by (7)

$$(8)\qquad \int_{t_0}^{T} \tilde{\eta}_k \dot{\theta}_k dt = \sum_{j=1}^{p} \gamma_{N-1,j} \int_{t_{N-1}}^{t_N} \tilde{\eta}_k \dot{\tilde{\phi}}_{N-1,j} dt.$$

Consider first the case of piecewise linear basis functions (p=1). Then we only have one test function at t_N. With $\tilde{\phi}_{N-1,1} = \phi_N$, $\gamma_{N-1,1} = \gamma_N = \theta_k(T)$, we get

$$\left|\int_{t_0}^{T} \tilde{\eta}_k \dot{\theta}_k dt\right| = |\gamma_N \int_{t_{N-1}}^{t_N} \tilde{\eta}_k \dot{\phi}_N dt| \le |\theta_k(T)| \max_{t_{N-1} \le t \le t_N} |\tilde{\eta}_k(t)|$$

$$= |\theta_k(T)|\,\left|O\left(h^{p+1}\right)\right| \|U_k\|_{p+2}.$$

Here we have used the Sobolev inequality,

$$\max_{t_0 \le t \le T} |U_k^{(p+1)}| \le C\|U_k\|_{p+1}, \quad U_k \in H^{p+2}(t_0, T).$$

Hence (5) follows for p = 1. For the general case, p > 1, we choose $\tilde{\phi}_{N-1,j} = \phi_{N-1,j}$ and we then get

$$(9)\qquad \left|\int_{t_0}^{T} \tilde{\eta}_k \dot{\theta}_k dt\right| = \left|\sum_{j=1}^{p} \gamma_{N-1,j} \int_{t_{N-1}}^{t_N} \tilde{\eta}_k \dot{\tilde{\phi}}_{N-1,j} dt\right| =$$

$$\le C \sum_{j=1}^{p} |\gamma_{N-1,j}| \max_{t_{N-1} \le t \le t_N} |\tilde{\eta}_k| \le C'\left\{h^{-1} \int_{t_{N-1}}^{t_N} \theta_k^2 dt\right\}^{\frac{1}{2}} \max_{t_0 \le t \le t_N} |\tilde{\eta}_k|.$$

Here we have used the fact that

$$\sum_{j=1}^{p} |\gamma_{N-1,j}| \text{ and } \left\{h^{-1} \int_{t_{N-1}}^{t_N} \theta_k^2 dt\right\}^{\frac{1}{2}}$$

are both norms in a finite dimensional space ($\mathbb{R}^r$), and hence equivalent (uniformly in h). Finally we get from (9),

$$\left|\int_{t_0}^{T} \tilde{\eta}_k \dot{\theta}_k dt\right| = C''\left\{\int_{t_0}^{T} \theta_k^2 dt\right\}^{\frac{1}{2}} O\left(h^{p+\nu_k}\right) \|U_k\|_{p+1}$$

where

$$(10)\qquad \nu_k = \min\left\{1, \tfrac{1}{2}\left[1 + \left[\log\left(\int_{t_0}^{T} \theta_k^2 dt \Big/ \int_{t_{N-1}}^{t_N} \theta_k^2 dt\right)\right] / \log h^{-1}\right]\right\}.$$

Together with (3.14) and (3.15) this implies (3.12a) for p = 3,5,... , where

$$(11)\qquad 1 \geq \nu = \min_{1\leq k\leq m} \nu_k \geq \frac{1}{2}.$$

Finally, from (2), (3) and (5) we get

$$\|\theta\|^2 \leq O(h^{2(p+\nu)})\|\varepsilon U\|^2_{p+2} + \|\eta\|^2$$

$$\leq O(h^{2(p+\nu)})\left\{\|\varepsilon U\|^2_{p+2} + \|U\|^2_{p+1}\right\}^{\frac{1}{2}}, \; p = 1,3,\dots .$$

Clearly, this estimate with $\nu = 0$ is also valid for p even. We now obtain the Galerkin error

$$\|U-\tilde{U}\| \leq \|U-U_I\| + \|U_I-\tilde{U}\| = \|\eta\| + \|\theta\| \leq$$

$$O(h^{p+\nu})\left\{\|\varepsilon U\|^2_{p+2} + \|U\|^2_{p+1}\right\}^{\frac{1}{2}}, \; h \to 0. \qquad \square$$

Note that if U is smooth, $\tilde{U}$ and U_I will be smooth, so

$$\int_{t_0}^{T} \theta_k^2 dt / \int_{t_{N-1}}^{T} \theta_k^2 dt = O(h^{-1}) .$$

Hence it follows from (10), (11) that ν will be close to 1.

REFERENCES

[1] AXELSSON, A.O.H. and J.G. VERWER, *Boundary value techniques for initial value problems in ordinary differential equations*, this issue.

[2] AXELSSON, A.O.H. and I. GUSTAFSSON, *Quasi optimal finite element approximation of first order hyperbolic and of convection-dominated convection diffusion equations*, in: *Analytical and numerical approaches to asymptotics in analysis*, eds. O. Axelsson, L.S. Frank and A. van der Sluis, North-Holland Mathematics Studies 47, North-Holland Publishing Company, Amsterdam - New York - Oxford, 1981.