MATHEMATICS OF COMPUTATION
VOLUME 45, NUMBER 171
JULY 1985, PAGES S5–S16

Supplement to Constructing Integral Lattices With Prescribed Minimum. I

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The Gram matrices of all lattices occuring in Figure 1 can be obtained from the following three Gram matrices :

$$Q_{23} = \begin{pmatrix}
3 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & -1 & -1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 3 & -1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
1 & -1 & 3 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & -1 & 3 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & 3 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
-1 & -1 & -1 & -1 & 1 & 3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & 1 & 1 & 3 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 3 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & 1 & 1 & 1 & 0 & -1 & 1 & 3 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & 3 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 3 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 3 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 3 & 0 & -1 & -1 & -1 & 0 & 0 \\
1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 3 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 3 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 3 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & 3 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 1 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 3
\end{pmatrix}$$

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$Q'_{19} =$

$$\begin{pmatrix}
3 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 3 & -1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
1 & -1 & 3 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 1 \\
-1 & -1 & -1 & -1 & 3 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & 1 & 3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
-1 & -1 & -1 & -1 & 1 & 1 & 3 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 3 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & 1 & 1 & 1 & 0 & -1 & 1 & 3 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & 3 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 3 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 3 & 0 & 0 & -1 & 1 \\
-1 & -1 & -1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 3 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 3
\end{pmatrix}$$

$Q''_{23} =$

$$\begin{pmatrix}
3 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & 3 & -1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
1 & -1 & 3 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\
1 & 1 & -1 & 3 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & -1 & -1 & -1 & 3 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\
-1 & -1 & -1 & -1 & 1 & 3 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
-1 & -1 & -1 & -1 & 1 & 1 & 3 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 3 & 1 & -1 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 3 & 1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 1 \\
-1 & -1 & -1 & 0 & 1 & 1 & 1 & 0 & -1 & 1 & 3 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & 3 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\
-1 & -1 & -1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & -1 & 3 & -1 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -1 & 3 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & -1 & 1 & -1 & 0 & 0 & 3 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & 0 & -1 & 0 & 0 & -1 \\
-1 & 0 & -1 & 0 & 1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 3 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 3 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 3 & 0 & -1 \\
0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 3 & 0 \\
-1 & -1 & -1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 3
\end{pmatrix}$$

(i) Λ_{23} has a basis $\underline{e}_1,\ldots,\underline{e}_{23}$ such that $Q_{23} = (<\underline{e}_i,\underline{e}_j>)$.
For the i-dimensional lattices $\Gamma_i = <\underline{e}_1,\ldots,\underline{e}_i>_{\mathbb{Z}}$ $(1 \le i \le 23)$ we find

$\Lambda_i = \Gamma_i \quad (i = 1,2,\ldots,8,9,12,23)$,

$\Lambda_i^a = \Gamma_i \quad (i = 10,11,13,14,\ldots,22)$,

$\Lambda_{10}^b = <\Gamma_9,\underline{e}_{11}>_{\mathbb{Z}}$, $\Lambda_{11}^b = <\Gamma_{10},\underline{e}_{12}>_{\mathbb{Z}}$,

$\Lambda_{13}^b = <\Gamma_{12},\underline{e}_{15}>_{\mathbb{Z}}$, $\Lambda_{14}^b = <\Gamma_{13},\underline{e}_{15}>_{\mathbb{Z}}$,

$\Lambda_{14}^c = <\Gamma_{13},\underline{e}_{17}>_{\mathbb{Z}}$, $\Lambda_{15}^b = <\Gamma_{14},\underline{e}_{17}>_{\mathbb{Z}}$, $\Lambda_{16}^b = <\Gamma_{15},\underline{e}_{17}>_{\mathbb{Z}}$.

(ii) Λ_{19}^a has a basis $\underline{f}_1,\ldots,\underline{f}_{19}$ such that $Q'_{19} = (<\underline{f}_i,\underline{f}_j>)$.
(The transformation $(\underline{f}_1,\ldots,\underline{f}_{19}) \rightarrow (\underline{e}_1,\ldots,\underline{e}_{19})$ is given below.) For the i-dimensional lattices $\Gamma'_i = <\underline{f}_1,\ldots,\underline{f}_i>_{\mathbb{Z}}$ $(1 \le i \le 19)$ we obtain ($\simeq$ denotes isometry),

$\Lambda_{12} \simeq \Gamma'_{12}, \Lambda_{13}^a \simeq \Gamma'_{13}, \Lambda_{14}^d = \Gamma'_{14}, \Lambda_{15}^c = \Gamma'_{15}$,

$\Lambda_{16}^c = \Gamma'_{16}, \Lambda_{17}^b = \Gamma'_{17}, \Lambda_{18}^b = \Gamma'_{18}, \Lambda_{19}^a = \Gamma'_{19}$.

Let $\underline{y}_1 = -2\underline{f}_1+3\underline{f}_2+5\underline{f}_3+6\underline{f}_4+\underline{f}_5+2\underline{f}_6+5\underline{f}_7+2\underline{f}_8+\underline{f}_9+2\underline{f}_{10}$
$-2\underline{f}_{11}+2\underline{f}_{12}+\underline{f}_{13}+\underline{f}_{14}+\underline{f}_{15}+\underline{f}_{16}+\underline{f}_{18}+\underline{f}_{19}$,

$\underline{y}_2 = -4\underline{f}_1+5\underline{f}_2+9\underline{f}_3+10\underline{f}_4+2\underline{f}_5+4\underline{f}_6+7\underline{f}_7+3\underline{f}_8+2\underline{f}_9+3\underline{f}_{10}$
$-3\underline{f}_{11}+3\underline{f}_{12}+2\underline{f}_{13}+\underline{f}_{14}+\underline{f}_{15}+\underline{f}_{16}+\underline{f}_{17}+\underline{f}_{18}+\underline{f}_{19}$,

of length 3 respectively,

then

$\Lambda_{15}^d = <\Gamma'_{14}, \underline{f}_{16}>_{\mathbb{Z}}$, $\Lambda_{16}^d = <\Gamma'_{15}, \underline{y}_1>_{\mathbb{Z}}$

(add (0 0 0 0 1 -1 0 0 1 0 0 0 -1 1 0 3) as row (column) to the Gram matrix of Γ'_{15} to obtain the Gram matrix of Λ^d_{16}),

$\Lambda^c_{17} = \langle \Lambda^d_{16}, \underline{y}_2 \rangle_{\mathbb{Z}}$ (add (0 0 0 0 1 0 -1 0 1 0 -1 1 0 -1 0 1 3) to the Gram matrix of Λ^d_{16}),

$\Lambda^d_{17} = \langle \Gamma'_{16}, \underline{y}_1 \rangle_{\mathbb{Z}}$ with $\langle \underline{y}_1, \underline{f}_{16} \rangle = 1$,

$\Lambda^e_{17} = \langle \Gamma'_{16}, f_{18} \rangle_{\mathbb{Z}}$, $\Lambda^c_{18} = \langle \Gamma'_{17}, \underline{y}_1 \rangle_{\mathbb{Z}}$ with $\langle \underline{y}_1, \underline{f}_{17} \rangle = -1$.

Let

$\underline{y}_3 = -2\underline{f}_1+3\underline{f}_2+4\underline{f}_3+5\underline{f}_4+\underline{f}_5+2\underline{f}_6+5\underline{f}_7+2\underline{f}_8+\underline{f}_9+2\underline{f}_{10} - 2\underline{f}_{11} +2\underline{f}_{12}+2\underline{f}_{13}+\underline{f}_{14}+2\underline{f}_{16}+\underline{f}_{18}$,

$\underline{y}_4 = -\underline{f}_1+\underline{f}_2+3\underline{f}_3+3\underline{f}_4+\underline{f}_5+\underline{f}_6+2\underline{f}_7+\underline{f}_8+\underline{f}_9+\underline{f}_{10} - \underline{f}_{11} +\underline{f}_{13}+\underline{f}_{14}\ \underline{f}_{17}$, of length 3 respectively,

then

$\Lambda^b_{13} \simeq \langle \Gamma'_{12}, \underline{f}_{15} \rangle_{\mathbb{Z}}$, $\Lambda^e_{14} = \langle \Gamma'_{12}, \underline{f}_{15}, \underline{y}_3 \rangle_{\mathbb{Z}}$ (add (0 0 0 -1 1 0 0 0 1 1 0 1 -1 3) to the Gram matrix of Λ^b_{13}),

$\Lambda^e_{15} = \langle \Lambda^e_{14}, \underline{f}_{13} \rangle_{\mathbb{Z}}$ with $\langle \underline{y}_3, \underline{f}_{13} \rangle = 0$,

$\Lambda^e_{16} = \langle \Lambda^e_{15}, \underline{f}_{16} \rangle_{\mathbb{Z}}$ with $\langle \underline{y}_3, \underline{f}_{16} \rangle = 1$,

$\Lambda^f_{17} = \langle \Lambda^e_{16}, \underline{y}_4 \rangle_{\mathbb{Z}}$ (add (1 -1 1 0 0 0 0 0 0 0 -1 0 0 0 1 3) to the Gram matrix of Λ^e_{16}),

$\Lambda^g_{17} = \langle \Lambda^e_{16}, \underline{f}_{19} \rangle_{\mathbb{Z}}$ with $\langle \underline{f}_{19}, \underline{y}_3 \rangle = 0$,

$\Lambda^d_{18} = \langle \Lambda^f_{17}, \underline{f}_{19} \rangle_{\mathbb{Z}}$ with $\langle \underline{f}_{19}, \underline{y}_4 \rangle = 0$.

(iii) Λ_{23} has a basis $\underline{g}_1, \ldots, \underline{g}_{23}$ such that $Q''_{23} = (\langle \underline{g}_i, \underline{g}_j \rangle)$. (The transformation $(\underline{g}_1, \ldots, \underline{g}_{23}) \to (\underline{e}_1, \ldots, \underline{e}_{23})$ is given below.)

For the i-dimensional lattice $\Gamma''_i = \langle \underline{g}_1, \ldots, \underline{g}_i \rangle_{\mathbb{Z}}$ $(1 \le i \le 23)$ we have

$\Lambda^b_{13} \simeq \Gamma''_{13}$, $\Lambda^e_{14} \simeq \Gamma''_{14}$, $\Lambda^f_{15} = \Gamma''_{15}$, $\Lambda^f_{16} = \Gamma''_{16}$,

$\Lambda^h_{17} = \Gamma''_{17}$, $\Lambda^e_{18} = \Gamma''_{18}$, $\Lambda^b_i = \Gamma''_i$ $(19 \le i \le 22)$,

$\Lambda^f_{18} = \langle \Gamma''_{17}, \underline{g}_{19} \rangle_{\mathbb{Z}}$, $\Lambda^d_{19} = \langle \Lambda^f_{18}, \underline{g}_{21} \rangle_{\mathbb{Z}}$,

$\Lambda^c_{20} = \langle \Gamma''_{19}, \underline{g}_{21} \rangle_{\mathbb{Z}}$.

With the embedding of Γ''_{18} into Γ''_{20} given here Λ^c_{19} is not an intermediate lattice. To obtain the Gram matrix of Λ^c_{19} we must add (0 0 0 0 0 0 0 0 0 0 0 0 0 -1 -1 0 0 -1 3) to the Gram matrix of Λ^e_{18}. Finally, to get from Λ^c_{19} to Λ^b_{20} one has to add (0 0 0 0 0 0 0 0 1 0 -1 -1 0 0 -1 -1 0 -1 1 3) to the Gram matrix of Λ^c_{19}. The resulting Gram matrix is Q'''_{20}. A matrix $T_{20} \in GL(20,\mathbb{Z})$ with $T^{tr}_{20} Q''_{20} T_{20} = Q'''_{20}$ is given below $(Q''_{20} = (\langle \underline{g}_i, \underline{g}_j \rangle)_{1 \le i,j \le 20})$. Hence, the embedding number of Λ^e_{18} into Λ^b_{20} is at least two.

The matrices for the various basis transformations are

T_{19} , T_{20} , and T_{23} satisfying

$(\underline{e}_1,\dots,\underline{e}_{19})\ T_{19} = (\underline{f}_1,\dots,\underline{f}_{19})$,

$T_{20}^{tr}\ Q_{20}''\ T_{20} = Q_{20}'''$,

$(\underline{e}_1,\dots,\underline{e}_{23})\ T_{23} = (\underline{g}_1,\dots,\underline{g}_{23})$.

$$T_{19} =$$

$$\begin{pmatrix}
0 & -2 & 0 & -9 & 6 & 3 & 5 & -4 & 8 & 7 & -3 & 5 & 0 & -4 & -2 & 2 & 2 & -2 & 1 \\
0 & 4 & 0 & 18 & -12 & -5 & -10 & 8 & -17 & -15 & 6 & -10 & 1 & 7 & 4 & -4 & -4 & 3 & 0 \\
0 & 7 & 0 & 30 & -20 & -9 & -16 & 14 & -29 & -26 & 9 & -17 & 1 & 11 & 6 & -7 & -6 & 5 & -1 \\
0 & 6 & 0 & 27 & -18 & -8 & -14 & 13 & -26 & -24 & 8 & -16 & 0 & 11 & 6 & -6 & -5 & 4 & -1 \\
0 & 1 & 0 & 3 & -2 & -1 & -2 & 1 & -3 & -2 & 1 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 \\
-1 & 3 & 0 & 13 & -9 & -4 & -7 & 7 & -13 & -12 & 4 & -8 & 0 & 5 & 3 & -3 & -3 & 2 & 0 \\
0 & 4 & 0 & 22 & -14 & -6 & -11 & 10 & -21 & -19 & 7 & -13 & 0 & 9 & 5 & -5 & -4 & 3 & -1 \\
0 & 2 & 0 & 11 & -7 & -3 & -5 & 5 & -11 & -10 & 4 & -7 & 0 & 5 & 2 & -2 & -2 & 2 & 0 \\
0 & 1 & 0 & 8 & -5 & -2 & -4 & 4 & -8 & -7 & 3 & -5 & 1 & 3 & 2 & -1 & -1 & 1 & 0 \\
0 & 2 & 0 & 8 & -5 & -3 & -4 & 4 & -7 & -7 & 2 & -5 & -1 & 3 & 2 & -2 & -2 & 1 & 0 \\
0 & 0 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 0 & 1 & 1 & -1 & -1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 6 & -4 & -1 & -3 & 3 & -6 & -6 & 2 & -4 & 0 & 3 & 1 & -1 & -1 & 2 & 0 \\
0 & 0 & 0 & 4 & -2 & -1 & -2 & 2 & -4 & -3 & 2 & -2 & 0 & 2 & 2 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 3 & -2 & -1 & -2 & 2 & -3 & -2 & 1 & -2 & 0 & 1 & 1 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 2 & -1 & 0 & -1 & 1 & -2 & -2 & 1 & -1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 2 & -1 & 0 & -1 & 1 & -2 & -2 & 1 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 & -2 & -1 & -1 & 1 & -2 & -2 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}$$

$$T_{20} =$$

$$\begin{pmatrix}
2 & 11 & 0 & 4 & -10 & -11 & -3 & 2 & -2 & -2 & -4 & -2 & -6 & -3 & -1 & 0 & 3 & 0 & -1 & -1 \\
-3 & -17 & 0 & -6 & 15 & 18 & 4 & -4 & 4 & 4 & 6 & 3 & 9 & 4 & 2 & -1 & -5 & -1 & 2 & 2 \\
-4 & -25 & 0 & -9 & 22 & 26 & 6 & -7 & 6 & 7 & 9 & 6 & 13 & 6 & 2 & -1 & -8 & -2 & 3 & 3 \\
-3 & -19 & 0 & -7 & 17 & 19 & 5 & -6 & 5 & 6 & 7 & 5 & 10 & 4 & 0 & 0 & -6 & -2 & 3 & 3 \\
-1 & -5 & 0 & -1 & 4 & 5 & 1 & 0 & 0 & 0 & 2 & 0 & 3 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\
-1 & -9 & 0 & -3 & 8 & 9 & 2 & -3 & 3 & 3 & 3 & 2 & 5 & 2 & 0 & -1 & -3 & -1 & 1 & 2 \\
-2 & -12 & 0 & -4 & 10 & 12 & 3 & -4 & 3 & 4 & 5 & 3 & 6 & 2 & 0 & 0 & -4 & -1 & 2 & 2 \\
-1 & -6 & 0 & -2 & 5 & 6 & 1 & -2 & 2 & 2 & 2 & 2 & 3 & 1 & 0 & 0 & -2 & -1 & 1 & 1 \\
0 & -1 & 0 & -1 & 1 & 2 & 0 & -1 & 1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 1 \\
-1 & -2 & -1 & -1 & 2 & 4 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\
-2 & -8 & 0 & -3 & 7 & 8 & 2 & -2 & 2 & 2 & 3 & 2 & 4 & 1 & 0 & 0 & -2 & 0 & 1 & 1 \\
-1 & -6 & 0 & -2 & 6 & 6 & 1 & -2 & 2 & 2 & 2 & 2 & 3 & 2 & 1 & 0 & -2 & 0 & 0 & 0 \\
0 & -3 & 0 & -1 & 3 & 3 & 1 & -2 & 1 & 1 & 1 & 1 & 2 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & -2 & -2 & -1 & 1 & -1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 4 & 0 & 1 & -3 & -4 & -1 & 0 & 0 & 0 & -2 & 0 & -2 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\
1 & 4 & 0 & 1 & -3 & -4 & -1 & 0 & 0 & 0 & -1 & 0 & -2 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\
1 & 3 & 0 & 1 & -2 & -3 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 1 & -1 & -2 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$T_{23} =$

-45	-130	24	-6	32	40	40	1	-3	-20	39	-26	55	-15	-8	-4	0	-11	2	-16	35	57	-5
95	275	-50	13	-68	-85	-86	-1	7	42	-83	54	-117	31	18	7	0	24	-4	33	-74	-120	10
157	457	-84	21	-113	-141	-141	-2	11	70	-137	90	-194	51	29	13	0	40	-7	55	-123	-200	16
140	407	-75	18	-100	-125	-125	-2	10	62	-122	81	-173	46	25	12	0	36	-6	49	-110	-178	14
17	48	-9	3	-12	-15	-15	0	1	7	-14	9	-20	5	3	1	0	4	0	6	-13	-21	2
71	207	-38	9	-51	-63	-64	-1	5	32	-62	41	-88	23	13	6	0	18	-3	25	-56	-91	7
111	322	-59	14	-79	-99	-99	-2	9	49	-97	64	-137	37	20	9	0	29	-5	38	-87	-141	11
58	169	-31	7	-41	-52	-52	-1	5	26	-51	34	-72	19	10	5	0	15	-3	20	-46	-74	6
39	112	-20	5	-28	-35	-35	-1	3	17	-34	23	-48	13	7	3	0	10	-2	13	-30	-49	4
41	119	-22	5	-29	-36	-36	-1	3	18	-36	24	-51	13	7	4	0	10	-2	15	-32	-52	4
-8	-24	5	0	5	6	6	1	-1	-4	7	-5	10	-3	-1	-1	0	-2	0	-3	7	11	-1
34	101	-19	4	-24	-31	-31	0	3	16	-30	20	-43	12	6	3	0	9	-2	12	-28	-44	4
19	52	-9	2	-13	-16	-16	-1	2	8	-16	11	-23	7	3	1	-1	5	-1	6	-14	-23	2
16	46	-8	2	-12	-14	-14	-1	1	7	-14	9	-20	5	3	1	0	4	-1	6	-12	-20	1
14	42	-8	2	-10	-13	-13	0	1	7	-12	8	-18	5	3	1	0	4	-1	5	-12	-18	2
10	29	-5	1	-7	-9	-9	0	1	4	-9	6	-13	4	2	1	0	3	0	3	-8	-13	1
12	36	-7	2	-9	-11	-11	0	1	5	-11	7	-15	3	2	1	0	3	0	5	-9	-16	1
4	12	-2	0	-2	-4	-4	0	1	2	-4	3	-5	2	1	1	0	1	0	1	-4	-5	1
2	8	-2	0	-2	-2	-2	0	0	1	-2	2	-3	1	1	1	0	1	0	1	-2	-4	0
3	10	-2	0	-2	-3	-3	0	1	2	-3	2	-4	1	0	0	0	1	0	1	-3	-4	1
3	8	-2	2	-2	-3	-3	0	0	1	-2	1	-3	0	0	0	0	1	0	1	-2	-3	0
1	4	-1	0	-1	-1	-1	0	0	1	-1	1	-2	1	1	0	0	0	0	1	-1	-2	0
1	3	-1	1	-1	-1	-1	0	0	0	-1	0	-1	0	0	0	0	0	0	0	-1	-1	0

In Table 3 we list invariants of the lattices described above :

Column 1 : Name of the lattice.

Column 2 : The discriminant, factored into the elementary divisors of the Gram matrix.

Column 3 : $|M| = |T(\underline{a}_1)|_{t(\underline{a}_1,0)} + \ldots + |T(\underline{a}_r)|_{t(\underline{a}_r,0)}$

(According to (3.1),(3.2) we assume that the set M of shortest vectors is partitioned into equivalence classes $T(\underline{a}_i)$ $(1 \le i \le r)$. For each type the subscript $t(\underline{a}_i,0)$ is the number of vectors of M orthogonal to the representative $\underline{a}_i$. For $m=3$ this number uniquely determines the type of $\underline{a}_i$.)

Table 1

Λ_1	$3 = 3$	$2 = 2_0$
Λ_2	$8 = 8$	$4 = 4_0$
Λ_3	$16 = 4^2$	$8 = 8_0$
Λ_4	$32 = 2^2 8$	$12 = 12_0$
Λ_5	$48 = 2^3 6$	$20 = 20_0$
Λ_6	$64 = 2^4 4$	$32 = 32_0$
Λ_7	$64 = 2^6$	$56 = 56_0$
Λ_8	$128 = 2^4 8$	$60 = 24_0+32_4+4_{32}$
Λ_9	$192 = 2^2 4\cdot 12$	$68 = 48_8+8_0+12_{32}$
Λ_{10}^a	$256 = 2^2 4^3$	$80 = 48_{16}+8_0+24_{32}$
Λ_{10}^b	$256 = 4^2 16$	$80 = 24_{16}+32_{12}+24_{36}$
Λ_{11}^a	$256 = 4^4$	$104 = 24_{32}+32_{24}+48_{40}$
Λ_{11}^b	$256 = 2^4 4^2$	$104 = 96_{32}+8_0$
Λ_{12}	$256 = 2^2 4^3$	$136 = 96_{48}+8_{32}+32_{56}$
Λ_{13}^a	$256 = 4^4$	$168 = 104_{64}+64_{72}$
Λ_{13}^b	$256 = 2^4 4^2$	$176 = 128_{72}+48_{64}$
Λ_{14}^a	$256 = 4^2 16$	$200 = 96_{88}+72_{80}+32_{96}$
Λ_{14}^b	$256 = 2^2 4^3$	$208 = 32_{80}+128_{88}+48_{96}$
Λ_{14}^c	$256 = 4^4$	$200 = 96_{80}+8_{96}+96_{88}$
Λ_{14}^d	$256 = 4^4$	$208 = 192_{88}+16_{104}$
Λ_{14}^e	$256 = 2^6 4$	$224 = 224_{96}$
Λ_{15}^a	$192 = 2^2 4\cdot 12$	$272 = 144_{128}+128_{120}$
Λ_{15}^b	$192 = 4^2 12$	$264 = 64_{112}+40_{128}+96_{120}+64_{124}$

Table 1 (continued)

Λ_{15}^{c}	$256 = 4^2 16$	$240 = 64_{112}+128_{104}+16_{120}+32_{124}$
Λ_{15}^{d}	$256 = 4^4$	$248 = 64_{104}+160_{112}+16_{136}+8_{128}$
Λ_{15}^{e}	$256 = 2^4 4^2$	$256 = 64_{128}+192_{112}$
Λ_{15}^{f}	$256 = 2^8$	$280 = 280_{210}$
Λ_{16}^{a}	$128 = 2^4 8$	$380 = 60_{192}+320_{180}$
Λ_{16}^{b}	$128 = 2^2 4\cdot 8$	$368 = 128_{168}+48_{192}+192_{176}$
Λ_{16}^{c}	$192 = 4^2 12$	$312 = 32_{136}+128_{144}+40_{160}+16_{168}+32_{152}+64_{156}$
Λ_{16}^{d}	$192 = 8\cdot 24$	$304 = 192_{144}+16_{152}+96_{156}$
Λ_{16}^{e}	$192 = 2^4 12$	$320 = 128_{144}+192_{160}$
Λ_{16}^{f}	$256 = 2^6 4$	$312 = 224_{144}+56_{160}+32_{164}$
Λ_{17}^{a}	$64 = 2^4 4$	$576 = 512_{280}+64_{320}$
Λ_{17}^{b}	$128 = 4^2 8$	$416 = 64_{192}+128_{200}+64_{224}+32_{232}+128_{208}$
Λ_{17}^{c}	$128 = 2\cdot 8^2$	$400 = 208_{200}+192_{208}$
Λ_{17}^{d}	$128 = 4.32$	$408 = 160_{200}+72_{208}+48_{216}+128_{212}$
Λ_{17}^{e}	$128 = 2^2 4\cdot 8$	$428 = 128_{204}+48_{236}+48_{224}+192_{212}+12_{240}$
Λ_{17}^{f}	$128 = 2^4 8$	$416 = 128_{192}+96_{224}+192_{208}$
Λ_{17}^{g}	$128 = 2^2 4\cdot 8$	$416 = 128_{192}+96_{224}+192_{208}$
Λ_{17}^{h}	$192 = 2^5 6$	$376 = 144_{192}+8_{224}+128_{176}+96_{200}$
Λ_{18}^{a}	$48 = 2^3 6$	$704 = 192_{384}+512_{360}$
Λ_{18}^{b}	$64 = 2^2 4^2$	$640 = 512_{320}+128_{368}$
Λ_{18}^{c}	$64 = 8^2$	$608 = 384_{320}+128_{312}+96_{328}$
Λ_{18}^{d}	$64 = 2^2 4^2$	$608 = 576_{320}+32_{288}$
Λ_{18}^{e}	$128 = 2^4 8$	$484 = 128_{236}+96_{264}+48_{256}+8_{320}+192_{252}+12_{288}$
Λ_{18}^{f}	$128 = 2^4 8$	$472 = 192_{240}+32_{272}+216_{256}+32_{224}$

Table 1 (continued)

Λ_{19}^{a}	$32 = 2^2 8$	$896 = 384_{496}+512_{480}$
Λ_{19}^{b}	$64 = 2^2 4^2$	$680 = 24_{384}+48_{416}+576_{368}+32_{336}$
Λ_{19}^{c}	$64 = 2^2 4^2$	$712 = 192_{416}+8_{512}+512_{368}$
Λ_{19}^{d}	$64 = 2^2 4^2$	$664 = 512_{368}+144_{352}+8_{320}$
Λ_{20}^{a}	$16 = 4^2$	$1280 = 1280_{720}$
Λ_{20}^{b}	$32 = 2^2 8$	$972 = 64_{608}+512_{532}+384_{548}+12_{640}$
Λ_{20}^{c}	$32 = 2^2 8$	$936 = 24_{512}+48_{544}+576_{528}+32_{496}+256_{536}$
Λ_{21}^{a}	$8 = 8$	$1792 = 1792_{1040}$
Λ_{21}^{b}	$12 = 2\cdot 6$	$1492 = 960_{872} + 512_{860} + 20_{896}$
Λ_{22}^{a}	$3 = 3$	$2816 = 2816_{1680}$
Λ_{22}^{b}	$4 = 4$	$2464 = 2464_{1472}$
Λ_{23}	1	$4600 = 4600_{2816}$

Table 2 contains information on the automorphism group of the lattices Λ_i up to dimension i = 18.

Column 1: Name of the lattice.

Column 2: Order of the automorphism group.

Column 3: Length of the orbits on the vectors of minimum length. (If the automorphism group is not transitiv on the vectors of the same type, the corresponding orbit lengths are the put together in parantheses.)

Column 4: If the automorphism group is not solvable, the non abelian composition factors are given in parentheses.

Table 2

Λ_1	2	2 = 2	
Λ_2	2^2	4 = 4	
Λ_3	$2^4 3$	8 = 8	
Λ_4	$2^5 3$	12 = 12	
Λ_5	$2^5 3^2 5$	20 = 20	(A_6)
Λ_6	$2^9 3^2 5$	32 = 32	(A_6)
Λ_7	$2^{10} 3^4 5\cdot 7$	56 = 56	$(Sp_6(2))$
Λ_8	$2^{11} 3^2 5$	60 = 24+32+4	(A_6)
Λ_9	$2^{11} 3^2$	68 = 48+8+12	
Λ_{10}^a	$2^{13} 3^2$	80 = 48+8+24	
Λ_{10}^b	$2^{11} 3$	80 = 24+32+24	
Λ_{11}^a	$2^{13} 3$	104 = 24+32+48	
Λ_{11}^b	$2^{17} 3^3$	104 = 96+8	
Λ_{12}	$2^{16} 3$	136 = 96+8+32	
Λ_{13}^a	$2^{13} 3$	168 = (96+8)+64	
Λ_{13}^b	$2^{19} 3$	176 = 128+48	
Λ_{14}^a	2^{11}	200 = 96+(64+8)+32	
Λ_{14}^b	2^{14}	208 = 32+128+(32+16)	
Λ_{14}^c	$2^{15} 3$	200 = (32+64)+8+96	
Λ_{14}^d	$2^{15} 3^2$	208 = 192+16	
Λ_{14}^e	$2^{20} 3\cdot 7$	224 = 224	$(PSL_3(2))$
Λ_{15}^a	$2^{14} 3$	272 = (48+96)+128	
Λ_{15}^b	$2^{11} 3$	264 = (32+32)+(32+8)+96+64	

Table 2 (continued)

Λ_{15}^{c}	2^{13}	240 = 64+128+16+32	
Λ_{15}^{d}	2^{17}	248 = 64+(128+32)+16+8	
Λ_{15}^{e}	$2^{17}3$	256 = (32+32)+192	
Λ_{15}^{f}	$2^{21}3^{2}5\cdot7$	280 = 280	$(PSL_4(2))$
Λ_{16}^{a}	$2^{15}3^{2}5$	380 = 60+320	(A_6)
Λ_{16}^{b}	$2^{14}3$	368 = 128+48+192	
Λ_{16}^{c}	2^{12}	312 = 32+(32+64+32)+(32+8)+(8+8)+32+64	
Λ_{16}^{d}	$2^{9}3$	304 = 192+16+96	
Λ_{16}^{e}	$2^{14}3$	320 = 128+(96+96)	
Λ_{16}^{f}	$2^{18}3.7$	312 = 224+56+32	$(PSL_3(2))$
Λ_{17}^{a}	$2^{19}3^{2}5$	576 = 512+64	(A_6)
Λ_{17}^{b}	2^{14}	416 = 64+128+64+32+128	
Λ_{17}^{c}	$2^{11}3$	400 = (192+16)+192	
Λ_{17}^{d}	2^{9}	408 = (64+64+32)+(64+8)+(16+32)+128	
Λ_{17}^{e}	$2^{15}3$	428 = 128+48+48+192+12	
Λ_{17}^{f}	$2^{16}3$	416 = 128+96+192	
Λ_{17}^{g}	$2^{16}3$	416 = 128+96+192	
Λ_{17}^{h}	$2^{15}3^{2}$	376 = 144+8+128+96	
Λ_{18}^{a}	$2^{15}3^{3}5$	704 = 192+512	(A_6)
Λ_{18}^{b}	$2^{20}3$	640 = 512+128	
Λ_{18}^{c}	$2^{13}3$	608 = 384+128+96	
Λ_{18}^{d}	$2^{15}3^{2}$	608 = 576+32	
Λ_{18}^{e}	$2^{17}3$	484 = 128+96+48+8+192+12	
Λ_{18}^{f}	$2^{12}3$	472 = 192+32+(24+192)+32	

The lattice Λ_{17}^{a} has a basis $\underline{a}_1,\ldots,\underline{a}_{17}$ with Gram matrix $\tilde{Q}_{17} = (\langle\underline{a}_i,\underline{a}_j\rangle)_{1\le i,j\le 17}$ such that

$K_i = \langle\underline{a}_1,\ldots,\underline{a}_i\rangle_{\mathbb{Z}}$ (i=6,7,...,11) ,

$K_i^{a} = \langle\underline{a}_1,\ldots,\underline{a}_i\rangle_{\mathbb{Z}}$ (i=12,...,16) .

$$\tilde{Q}_{17} = \begin{pmatrix}
3 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 3 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 1 \\
1 & 1 & 3 & 1 & 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & 1 \\
1 & 1 & 1 & 3 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\
1 & -1 & 1 & 1 & 3 & 0 & -1 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & -1 & 0 \\
1 & 1 & 1 & 1 & 0 & 3 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\
-1 & 1 & -1 & 0 & -1 & 1 & 3 & 0 & -1 & -1 & 0 & 0 & 1 & -1 & 1 & 1 & 0 \\
1 & -1 & 0 & 1 & 1 & 1 & 0 & 3 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 \\
0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 1 & 3 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 1 & 3 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 3 & 0 & -1 & 1 & 1 & 0 \\
-1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 3 & 0 & 1 & 1 & -1 \\
0 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 3 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 3 & 1 & 0 \\
0 & 1 & 0 & 1 & -1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 1 & 3 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 3
\end{pmatrix}$$

The lattice Λ_{18}^{a} has a basis $\underline{b}_1,\ldots,\underline{b}_{18}$ with Gram matrix $\tilde{\tilde{Q}}_{18} = (\langle\underline{b}_i,\underline{b}_j\rangle)_{1\le i,j\le 18}$ such that

$K_i^{b} = \langle\underline{b}_1,\ldots,\underline{b}_i\rangle_{\mathbb{Z}}$ (i=12,13,...,16) , $K_{17} = \langle\underline{b}_1,\ldots,\underline{b}_{17}\rangle_{\mathbb{Z}}$.

$$\tilde{\tilde{Q}}_{18} = \begin{pmatrix}
3 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 \\
-1 & 3 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 \\
1 & 1 & 3 & 1 & 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 3 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
1 & -1 & 1 & 1 & 3 & 0 & -1 & 1 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\
1 & 1 & 1 & 1 & 0 & 3 & 1 & 1 & 0 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & 0 & -1 & 1 & 3 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \\
1 & -1 & 0 & 1 & 1 & 1 & 0 & 3 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 3 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 1 & 3 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 1 & 1 & 3 & 0 & -1 & 0 & 0 & 1 & 0 \\
-1 & -1 & -1 & -1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 3 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 3 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 3 & 0 & 0 & -1 \\
-1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 3 & 1 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 3
\end{pmatrix}$$

The matrices for the two basis transformations are T_{17}, T_{18} satisfying

$$(\underline{e}_1, \ldots, \underline{e}_{17})\, T_{17} = (\underline{a}_1, \ldots, \underline{a}_{17}) \quad \text{and}$$

$$(\underline{e}_1, \ldots, \underline{e}_{18})\, T_{18} = (\underline{b}_1, \ldots, \underline{b}_{18}) \quad \text{, respectively.}$$

$$T_{17} = \begin{pmatrix}
0 & 0 & -1 & 1 & -1 & 2 & 0 & 1 & 4 & 3 & 0 & 2 & 1 & -3 & 0 & 0 & -4 \\
1 & 0 & 3 & -1 & 3 & -3 & -1 & -2 & -8 & -4 & 1 & -4 & -2 & 6 & 0 & 0 & 10 \\
1 & 0 & 4 & -3 & 4 & -5 & -1 & -4 & -13 & -7 & 2 & -7 & -4 & 10 & 0 & 0 & 17 \\
1 & 0 & 3 & -3 & 3 & -4 & 0 & -3 & -12 & -7 & 2 & -7 & -3 & 9 & 0 & 0 & 15 \\
0 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 1 & -1 & 0 & 2 \\
1 & 0 & 2 & -1 & 2 & -2 & 0 & -2 & -6 & -3 & 1 & -3 & -2 & 4 & 0 & 0 & 8 \\
1 & 0 & 2 & -2 & 3 & -3 & 0 & -2 & -10 & -6 & 1 & -6 & -2 & 8 & 0 & 0 & 12 \\
1 & 0 & 1 & -1 & 1 & -1 & 0 & -1 & -5 & -2 & 1 & -3 & -1 & 4 & 0 & 0 & 6 \\
0 & 0 & 1 & -1 & 1 & -1 & 0 & -1 & -3 & -2 & 1 & -3 & 0 & 3 & 0 & 0 & 4 \\
1 & 0 & 1 & -1 & 1 & -1 & 0 & -1 & -4 & -2 & 0 & -2 & -2 & 3 & 0 & -1 & 5 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -3 & -1 & 1 & -1 & 0 & 2 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -2 & -2 & 0 & -2 & 0 & 2 & 0 & 0 & 2 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -2 & -1 & 0 & -1 & -1 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2
\end{pmatrix}$$

$$T_{18} = \begin{pmatrix}
-6 & 1 & -1 & -5 & -2 & -2 & -1 & -3 & 1 & 1 & 0 & 1 & 2 & 1 & 1 & 0 & -2 & -2 \\
14 & -2 & 3 & 12 & 5 & 5 & 1 & 6 & -1 & 0 & 0 & -1 & -5 & -2 & -2 & 0 & 4 & 3 \\
22 & -3 & 4 & 19 & 8 & 8 & 2 & 9 & -1 & -1 & 0 & -2 & -7 & -3 & -4 & 0 & 6 & 5 \\
20 & -3 & 4 & 17 & 8 & 7 & 1 & 8 & -1 & -1 & 0 & -2 & -6 & -2 & -3 & 0 & 5 & 5 \\
2 & 0 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 1 & 1 \\
10 & -1 & 2 & 9 & 4 & 4 & 1 & 4 & -1 & 0 & 0 & -1 & -3 & -1 & -2 & 0 & 2 & 2 \\
17 & -3 & 4 & 14 & 7 & 6 & 0 & 7 & 0 & 0 & 0 & -1 & -5 & -2 & -2 & -1 & 5 & 4 \\
9 & -2 & 2 & 7 & 4 & 3 & 0 & 3 & 0 & 0 & 0 & 0 & -2 & -2 & -2 & 0 & 2 & 2 \\
6 & -1 & 1 & 5 & 2 & 2 & 0 & 2 & 0 & 0 & 1 & 0 & -2 & -1 & 0 & 0 & 2 & 1 \\
6 & -1 & 2 & 5 & 3 & 2 & 0 & 2 & 0 & 0 & 0 & -1 & -1 & 0 & -2 & 0 & 1 & 2 \\
-1 & 0 & -1 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
5 & -1 & 1 & 4 & 2 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 \\
3 & 0 & 1 & 3 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 2 & 1 \\
2 & 0 & 1 & 2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 1 \\
2 & 0 & 1 & 2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

In Table 3 we list invariants of the lattices described above :

Column 1 : Name of the lattice.

Column 2 : The discriminant, factored into the elementary divisors of the Gram matrix.

Column 3 : $|M| = |T(\underline{a}_1)|_{t(\underline{a}_1,0)} + \ldots + |T(\underline{a}_r)|_{t(\underline{a}_r,0)}$

(According to (3.1),(3.2) we assume that the set M of shortest vectors is partitioned into equivalence classes $T(\underline{a}_i)$ $(1 \leq i \leq r)$. For each type the subscript $t(\underline{a}_i,0)$ is the number of vectors of M orthogonal to the representative $\underline{a}_i$. For $m=3$ this number uniquely determines the type of $\underline{a}_i$.)

Table 3

K_6	$96 = 2^2 24$	$24 = 12_0+8_4+4_8$
K_7	$144 = 12^2$	$32 = 8_0+24_8$
K_8	$216 = 3^2 24$	$40 = 4_0+12_8+24_{12}$
$\tilde{K}_8$	$192 = 4^2 12$	$44 = 8_0+12_{16}+24_8$
K_9	$243 = 3^3 9$	$56 = 54_{16}+2_0$
K_{10}	$243 = 3^5$	$80 = 80_{24}$
K_{11}	$324 = 3^3 12$	$92 = 60_{32}+20_{24}+12_{40}$
K_{12}^a	$324 = 3 \cdot 6 \cdot 18$	$116 = 60_{48}+54_{40}+2_{24}$
K_{12}^b	$324 = 3^2 6^2$	$116 = 60_{48}+20_{24}+36_{40}$
K_{13}^a	$288 = 2 \cdot 6 \cdot 24$	$152 = 120_{64}+24_{56}+8_{48}$
K_{13}^b	$324 = 3^2 36$	$144 = 36_{60}+48_{64}+18_{44}+2_{28}+24_{56}+12_{52}+4_{68}$
K_{14}^a	$192 = 2^2 4 \cdot 12$	$224 = 224_{96}$
K_{14}^b	$243 = 3^3 9$	$200 = 162_{84}+36_{96}+2_{36}$
K_{15}^a	$128 = 2^4 8$	$320 = 320_{144}$
K_{15}^b	$216 = 3^2 24$	$248 = 144_{112}+84_{120}+18_{104}+2_{72}$
K_{16}^a	$64 = 2^6$	$512 = 512_{240}$
K_{16}^b	$144 = 12^2$	$344 = 264_{168}+72_{160}+8_{144}$
K_{17}	$96 = 2^2 24$	$464 = 144_{240}+288_{232}+32_{216}$

Table 4 contains information on the automorphism group of the lattices Λ_i up to dimension i = 18.

Column 1: Name of the lattice.

Column 2: Order of the automorphism group.

Column 3: Length of the orbits on the vectors of minimum length. (If the automorphism group is not transitiv on the vectors of the same type, the corresponding orbit lengths are the put together in parantheses.)

Column 4: If the automorphism group is not solvable, the non abelian composition factors are given in parentheses.

Table 4

K_6	$2^6 3$	$24 = 12+8+4$	
K_7	$2^7 3$	$32 = 8+24$	
K_8	$2^6 3$	$40 = 4+12+24$	
$\tilde{K}_8$	$2^8 3$	$44 = 24_8+6_{16}+8_0$	
K_9	$2^5 3^4$	$56 = 54+2$	
K_{10}	$2^7 3^4 5$	$80 = 80$	$(PSU_4(2))$
K_{11}	$2^6 3^2 5$	$92 = 60+20+12$	(A_6)
K_{12}^a	$2^5 3^3$	$116 = (24+36)+54+2$	
K_{12}^b	$2^7 3^3 5$	$116 = 60+20+36$	(A_6)
K_{13}^a	$2^8 3$	$152 = (96+24)+24+8$	
K_{13}^b	$2^6 3^2$	$144 = 36+(24+24)+18+2+24+12+4$	
K_{14}^a	$2^{12} 3^2$	$224 = (128+96)$	
K_{14}^b	$2^6 3^5$	$200 = 162+36+2$	
K_{15}^a	$2^{14} 3^2 5$	$320 = 320$	(A_6)
K_{15}^b	$2^7 3^2$	$248 = (72+72)+(72+12)+18+2$	
K_{16}^a	$2^{18} 3^4 5 \cdot 7$	$512 = 512_{240}$	$(Sp_6(2))$
K_{16}^b	$2^{10} 3^2$	$344 = (192+48+24)+72+8$	
K_{17}	$2^{11} 3^2$	$464 = 144+288+32$	