MATHEMATICS OF COMPUTATION
VOLUME 45, NUMBER 172
OCTOBER 1985, PAGES 417-437

The Numerical Solution of Weakly Singular Volterra Integral Equations By Collocation on Graded Meshes

By Hermann Brunner

Abstract. Since the solution of a second-kind Volterra integral equation with weakly singular kernel has, in general, unbounded derivatives at the left endpoint of the interval of integration, its numerical solution by polynomial spline collocation on uniform meshes will lead to poor convergence rates. In this paper we investigate the convergence rates with respect to graded meshes, and we discuss the problem of how to select the quadrature formulas to obtain the fully discretized collocation equation.

1. Introduction. In this paper we present an analysis of certain numerical methods for solving the (nonlinear) Volterra integral equation

$$(1.1)\quad y(t) = g(t) + \int_0^t (t-s)^{-\alpha} \cdot k(t, s, y(s))\, ds, \qquad t \in I := [0, T],\ T < \infty,$$

where $0 < \alpha < 1$, and where g and k denote given smooth functions. In practical applications one very frequently encounters the linear counterpart of (1.1),

$$(1.2)\quad y(t) = g(t) + \int_0^t (t-s)^{-\alpha} \cdot K(t, s) y(s)\, ds, \qquad t \in I\ (0 < \alpha < 1);$$

in the subsequent analysis we shall, for ease of exposition, usually utilize the linear version of (1.1) to display the principal ideas.

The numerical methods to be analyzed will be *collocation methods* in the polynomial spline space,

$$(1.3)\qquad S_{m-1}^{(-1)}(Z_N) := \{u \colon u|_{\sigma_n} =: u_n \in \pi_{m-1}, 0 \leqslant n \leqslant N-1\},$$

associated with a given partition (or: mesh) Π_N of the interval I,

$$\Pi_N \colon 0 = t_0^{(N)} < t_1^{(N)} < \cdots < t_N^{(N)} = T$$

(the index indicating the dependence of the mesh points on N will, for ease of notation, subsequently be suppressed). Here, π_{m-1} denotes, for given $m \geqslant 1$, the space of (real) polynomials of degree not exceeding $m-1$, and we have set $\sigma_0 := [t_0, t_1]$, $\sigma_n := (t_n, t_{n+1}]$ $(1 \leqslant n \leqslant N-1)$; the set $Z_N := \{t_n \colon 1 \leqslant n \leqslant N-1\}$ (i.e., the interior mesh points) will be referred to as the knots of these polynomial splines. In addition, we define

$$(1.4)\quad h := \max\{h_n \colon 0 \leqslant n \leqslant N-1\}, \qquad h' := \min\{h_n \colon 0 \leqslant n \leqslant N-1\},$$

Received February 3, 1984; revised November 6, 1984.
1980 *Mathematics Subject Classification*. Primary 65R20; Secondary 45D05.
Key words and phrases. Volterra integral equations, weakly singular kernels, polynomial spline collocation, graded meshes.

where $h_n := t_{n+1} - t_n$; the quantity h is often called the diameter of the mesh Π_N. (Note that, according to the above remark on our notation, both h and h' will depend on N.)

In order to describe these collocation methods we rewrite (1.1), for $t \in \sigma_n$, in "one-step form",

$$y(t) = F_n(y; t) + \int_{t_n}^{t} (t-s)^{-\alpha} \cdot k(t, s, y(s))\, ds, \tag{1.5}$$

where

$$F_n(y; t) := g(t) + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (t-s)^{-\alpha} \cdot k(t, s, y(s))\, ds \tag{1.6}$$

$$(0 \leqslant n \leqslant N-1).$$

For given parameters $\{c_j\}$ with $0 \leqslant c_1 < \cdots < c_m \leqslant 1$ we introduce the sets

$$X_n := \{t_{nj} := t_n + c_j h_n \colon 1 \leqslant j \leqslant m\} \qquad (0 \leqslant n \leqslant N-1), \tag{1.7}$$

and we define

$$X(N) := \bigcup_{n=0}^{N-1} X_n;$$

the set $X(N) \subset I$ will be referred to as the set of *collocation points*, while the c_j's will be called *collocation parameters*. A numerical approximation to the exact solution y of (1.1) (or (1.2)) is an element of $S_{m-1}^{(-1)}(Z_N)$ satisfying the given integral equation on $X(N)$; i.e., by (1.5), this approximation u is computed recursively from

$$\begin{aligned} u_n(t_{nj}) &= F_n(u; t_{nj}) \\ &\quad + h_n^{1-\alpha} \cdot \int_0^{c_j} (c_j - v)^{-\alpha} \cdot k(t_{nj}, t_n + vh_n, u_n(t_n + vh_n))\, dv \end{aligned} \tag{1.8}$$

$$(1 \leqslant j \leqslant m),$$

where

$$F_n(u; t_{nj}) := g(t_{nj}) + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (t_{nj} - s)^{-\alpha} \cdot k(t_{nj}, s, u_i(s))\, ds \tag{1.9}$$

$$(0 \leqslant n \leqslant N-1).$$

It follows by a standard contraction mapping argument that, for any continuous $k(t, s, y)$ with bounded partial derivative $k_y(t, s, y)$, and for any mesh Π_N whose mesh diameter h tends to zero as N tends to infinity, (1.8) will define a unique approximation $u \in S_{m-1}^{(-1)}(Z)$ for all sufficiently large N; once the values $\{u_n(t_{nj})\colon 1 \leqslant j \leqslant m\}$ have been found we have

$$u_n(t_n + vh_n) = \sum_{j=1}^{m} L_j(v) u_n(t_{nj}), \qquad t_n + vh_n \in \sigma_n\, (0 \leqslant n \leqslant N-1), \tag{1.10}$$

where L_j denotes the jth Lagrange fundamental polynomial for the m collocation parameters $\{c_j\}$; i.e.

$$L_j(v) := \prod_{\substack{k=1 \\ k \neq j}}^{m} (v - c_k)/(c_j - c_k) \qquad (1 \leqslant j \leqslant m). \tag{1.11}$$

We note in passing that the particular choice: $c_1 = 0$ and $c_m = 1$, implies that the approximating element u will be continuous on the entire interval I; that is, u is then an element of the smoother polynomial spline space

$$S_{m-1}^{(0)}(Z_N) := S_{m-1}^{(-1)}(Z_N) \cap C(I).$$

In the following, we shall be interested in studying the attainable order of convergence of u on I, as $N \to \infty$. It is well-known that, were the exact solution y of (1.1) (or (1.2)) in $C^m(I)$, then we would obtain, for a uniform mesh (where $h_n = h = TN^{-1}$),

$$\|y - u\|_\infty = \mathcal{O}(N^{-m}). \tag{1.12}$$

Unfortunately, smooth g and k (or K) in (1.1) (or in (1.2)) lead, for $0 < \alpha < 1$, to an exact solution y which behaves like $y(t) = \mathcal{O}(t^{1-\alpha})$ near $t = 0$; it has thus unbounded derivatives at $t = 0$ (compare [16], [12], [14], [3]). As a consequence, the collocation approximation $u \in S_{m-1}^{(-1)}(Z_N)$ given by (1.8), with the underlying mesh being the uniform one, satisfies only

$$\|y - u\|_\infty = \mathcal{O}(N^{-(1-\alpha)}), \tag{1.13}$$

and this order is best possible for any $m \geqslant 1$. (Compare also Section 3 below.)

In view of results from classical approximation theory (see, e.g., [22, pp. 409–425]) this disappointing result is no surprise. However, it has been known for some time that by using polynomial spline functions of degree $m - 1$ on certain nonuniform meshes tailored to the behavior of the function $f(t) = t^{1-\alpha}$ (so-called graded meshes; cf. Section 4 below) one can restore the convergence behavior shown in (1.12) (compare [17], [2], [6], [26], and [21, pp. 268–296]; related results on the use of graded meshes in numerical quadrature for integrals containing weakly singular functions in their integrands may be found, e.g., in [19] and in [13]).

This idea has recently been employed to devise high-order methods for the numerical solution of Fredholm integral equations of the second kind with weakly singular kernels: see [7] and [20] for studies of product integration methods on graded meshes; [9] and [10] for Galerkin methods; [23] and [25] for collocation methods (compare also [24] for a comprehensive survey). A survey of collocation methods for Fredholm and Volterra integral equations of the second kind with weakly singular kernels, as well as additional references, may also be found in [4].

As far as Volterra integral equations of the forms (1.1) and (1.2) are concerned, [5] presents a study of product integration techniques (extending the functional-analytic techniques used in, e.g., [7], [9], [20]). In order to construct high-order methods on uniform meshes it is necessary to abandon polynomial spline spaces in favor of special nonpolynomial spline spaces reflecting the behavior of the exact solution of (1.1) or (1.2) near $t = 0$. This approach has been investigated in [18] (for $\alpha = 1/2$) and in [3].

In the present paper we carry out an analysis of the convergence properties of collocation approximations in $S_{m-1}^{(-1)}(Z_N)$ to the solution of the Volterra integral equations (1.1), (1.2), both for quasi-uniform sequences of meshes and for graded meshes. Moreover, we extend this analysis to the fully discretized version of the collocation equation (1.8) in which the integrals have been approximated by appropriate quadrature processes (note that the above-mentioned analyses for Fredholm integral equations are all based on the assumption that the integrals be evaluated exactly).

2. The Attainable Order of Convergence. In this section we state the results on the attainable order of convergence of the collocation approximation $u \in S_{m-1}^{(-1)}(Z_N)$ with respect to the two types of mesh sequences mentioned above, assuming that the integrals occurring in (1.8) and (1.9) are known exactly. The fully discretized collocation equation will be investigated in Section 5. We shall formulate these results for the linear integral equation (1.2) so as not to be burdened with too many technical assumptions; when giving the proofs (in Sections 3 and 4) we shall indicate how these results can be extended to the nonlinear case (1.1).

A sequence of meshes for the interval I is called *quasi-uniform* if there exists a finite constant γ such that, for all $N \in \mathbf{N}$,

$$h/h' \leqslant \gamma \tag{2.1}$$

holds (recall the notation introduced in (1.4)). It is easily seen that such a mesh sequence has the property

$$h_n \leqslant h \leqslant \gamma \cdot TN^{-1}, \qquad 0 \leqslant n \leqslant N-1 \ \ (N \in \mathbf{N}); \tag{2.2}$$

hence, $h = \mathcal{O}(N^{-1})$ for any compact interval I. This holds, of course, trivially for uniform meshes, where we have $\gamma = 1$ and $h_n = TN^{-1}$ for all n.

THEOREM 2.1. *Let the functions g and K in* (1.2) *satisfy $g \in C^m(I)$ and $K \in C^m(S)$, with $m \geqslant 1$, and assume that neither function vanishes identically. If $u \in S_{m-1}^{(-1)}(Z_N)$ is the collocation approximation defined by* (1.8), *and if y denotes the exact solution of* (1.2), *then*

$$\|y - u\|_\infty = \mathcal{O}(N^{-(1-\alpha)}) \tag{2.3}$$

for any quasi-uniform mesh sequence. The exponent $1-\alpha$ in (2.3) *is best possible for all $m \geqslant 1$ and for all collocation parameters $\{c_j\}$ with $0 \leqslant c_1 < \cdots < c_m \leqslant 1$.*

Consider now *graded meshes* of the form

$$t_n := \left(\frac{n}{N}\right)^r \cdot T, \qquad 0 \leqslant n \leqslant N-1 \ (N \geqslant 2), \tag{2.4}$$

where the *grading exponent $r \in \mathbf{R}$* will always be assumed to satisfy $r \geqslant 1$. (We again suppress the index showing the dependence of t_n on N.) For any such mesh we have $0 < h_0 = h' < h_1 < \cdots < h_{N-1} = h$, and, in analogy to (2.2),

$$h_n \leqslant h \leqslant r \cdot TN^{-1}, \qquad 0 \leqslant n \leqslant N-1 \ \ (N \in \mathbf{N}). \tag{2.5}$$

Thus the mesh diameters of a sequence of graded meshes of the form (2.4) behave like $h = \mathcal{O}(N^{-1})$ on compact intervals.

THEOREM 2.2. *Let the functions g and K in* (1.2) *satisfy the conditions stated in Theorem* 2.1. *If $u \in S_{m-1}^{(-1)}(Z_N)$ is the collocation approximation defined by* (1.8), *and if y denotes the exact solution of* (1.2), *then*

$$\|y - u\|_\infty = \mathcal{O}(N^{-m}), \tag{2.6}$$

provided we employ the sequence of graded meshes (2.4) *corresponding the the grading exponent*

$$r = m/(1-\alpha). \tag{2.7}$$

This holds for all collocation parameters $\{c_j\}$ with $0 \leqslant c_1 < \cdots < c_m \leqslant 1$.

Note that the choice (2.7) for the grading exponent leads to optimal (global) convergence, in the sense that the exponent m in (2.6) cannot be replaced by $m+1$. This agrees, of course, with the well-known result in approximation theory which states that $\mathcal{O}(N^{-m})$-convergence is best possible when approximating a function $f \in C^m(I)$ in $S_{m-1}^{(-1)}(Z_N)$ or in $S_{m-1}^{(0)}(Z_N)$.

3. Proof of Theorem 2.1: Convergence on Quasi-Uniform Meshes. For $g \in C^m(I)$ and $K \in C^m(S)$ the (unique) solution of (1.2) is in $C[0,T] \cap C^m(0,T]$; more precisely, it has the form

$$y(t) = g(t) + \sum_{k=1}^{\infty} \psi_k(t) \cdot t^{k(1-\alpha)}, \qquad t \in I, \tag{3.1}$$

where $\psi_k \in C^m(I)$ $(k \geqslant 1)$, and where the series converges absolutely and uniformly on I (compare [3]; see also [16] and [14]). If α is rational, $\alpha = p/q$ (with p and q coprime), then (3.1) may be written as

$$y(t) = v_0(t) + \sum_{s=1}^{q-1} v_s(t) \cdot t^{s(1-\alpha)}, \qquad t \in I, \tag{3.2}$$

with $v_s \in C^m(I)$ $(0 \leqslant s \leqslant q-1)$. (See also [12] for the case $\alpha = 1/2$.) For the sake of simplicity of notation (and, not least, in view of practical applications where one usually encounters the values $\alpha = 1/2$, $\alpha = 1/3$, and $\alpha = 2/3$) we shall give the proofs of Theorem 2.1 and Theorem 2.2 for the case of rational α; the generalization of the ideas involved in the subsequent arguments to irrational α is straightforward.

On the initial interval $\sigma_0 = [t_0, t_1]$ (where $t_0 = 0$) the exact solution (3.2) is not continuously differentiable (unless $y(t) \equiv 0$; this case has been excluded by assuming $g(t) \not\equiv 0$ and $K(t,s) \not\equiv 0$). However, since $v_s \in C^m(I)$, we may write

$$v_s(t_0 + vh_0) = \sum_{l=1}^{m} c_{0l}^{(s)} v^{l-1} + h_0^m R_{0s}(v), \qquad v \in [0,1],$$

where we have set

$$c_{0l}^{(s)} := v_s^{(l-1)}(t_0) h_0^{l-1}/(l-1)!, \tag{3.3a}$$

and

$$R_{0s}(v) := v_s^{(m)}(\xi_{0s}) \cdot v^m/m! \qquad (t_0 < \xi_{0s} < t_0 + vh_0). \tag{3.3b}$$

Thus, by (3.2) (setting $(t_0 + vh_0)^{s(1-\alpha)} = h_0^{s(1-\alpha)} \cdot [1 + (v^{s(1-\alpha)} - 1)]$), we obtain

$$y(t_0 + vh_0) = \sum_{l=1}^{m} c_{0l} v^{l-1} + h_0^{1-\alpha} \cdot C_0(v) + h_0^m \cdot R_0(v), \qquad v \in [0,1], \tag{3.4}$$

with

$$\begin{aligned} c_{0l} &:= \sum_{s=0}^{q-1} h_0^{s(1-\alpha)} \cdot c_{0l}^{(s)}, \\ C_0(v) &:= \sum_{s=1}^{q-1} h_0^{(s-1)(1-\alpha)} \cdot (v^{s(1-\alpha)} - 1) \cdot \sum_{l=1}^{m} c_{0l}^{(s)} v^{l-1}, \end{aligned} \tag{3.5}$$

and

$$R_0(v) := \sum_{s=0}^{q-1} h_0^{s(1-\alpha)} \cdot R_{0s}(v) \cdot v^{s(1-\alpha)}. \tag{3.6}$$

For $1 \leqslant n \leqslant N-1$ we have, since $y \in C^m[t_1, T]$ $(t_1 > 0)$,

$$y(t_n + vh_n) = \sum_{l=1}^{m} c_{nl} v^{l-1} + h_n^m \cdot R_n(v), \qquad t_n + vh_n \in \sigma_n, \tag{3.7}$$

with

$$c_{nl} := y^{(l-1)}(t_n) \cdot h_n^{l-1}/(l-1)!, \tag{3.8}$$

and

$$R_n(v) := y^{(m)}(t_n + \theta_n v h_n) \cdot v^m/m! \qquad (0 < \theta_n < 1). \tag{3.9}$$

Suppose now that the restriction of the approximation $u \in S_{m-1}^{(-1)}(Z_N)$ to the subinterval σ_n is given by

$$u_n(t_n + vh_n) = \sum_{l=1}^{m} \alpha_{nl} v^{l-1}.$$

Thus, the error $e := y - u$ (with $e_n := y - u_n$ denoting its restriction to σ_n) assumes the form

$$e_n(t_n + vh_n) = \begin{cases} \sum_{l=1}^{m} \beta_{0l} v^{l-1} + h_0^{1-\alpha} C_0(v) + h_0^m R_0(v) & \text{if } n = 0; \\ \sum_{l=1}^{m} \beta_{nl} v^{l-1} + h_n^m R_n(v) & \text{if } 1 \leqslant n \leqslant N-1, \end{cases} \tag{3.10}$$

where we have defined $\beta_{nl} := c_{nl} - \alpha_{nl}$ $(1 \leqslant l \leqslant m; 0 \leqslant n \leqslant N-1)$.

Subtracting the collocation equation (1.8) (with $k(t, s, y) = K(t, s)y$) from the integral equation (1.2) (with $t = t_{nj}$), we obtain

$$\begin{aligned} e_n(t_{nj}) &= h_n^{1-\alpha} \cdot \int_0^{c_j} (c_j - v)^{-\alpha} \cdot K_{nj}(t_n + vh_n) e_n(t_n + vh_n)\, dv \\ &\quad + \sum_{i=0}^{n-1} h_i^{1-\alpha} \int_0^1 \left(\frac{t_{nj} - t_i}{h_i} - v \right)^{-\alpha} \cdot K_{nj}(t_i + vh_i) e_i(t_i + vh_i)\, dv \end{aligned} \tag{3.11}$$

$$(1 \leqslant j \leqslant m; 0 \leqslant n \leqslant N-1).$$

Here, we have set $K_{nj}(\cdot) := K(t_{nj}, \cdot)$. The expressions for the errors e_i given in (3.10) can now be used in (3.11) to derive a recurrence relation for the components of the vectors $\beta_n := (\beta_{n1}, \ldots, \beta_{nm})^T \in \mathbf{R}^m$ $(0 \leqslant n \leqslant N-1)$; it reads

$$\begin{aligned} &\sum_{l=1}^{m} \beta_{nl} \left\{ c_j^{l-1} - h_n^{1-\alpha} \int_0^{c_j} (c_j - v)^{-\alpha} \cdot K_{nj}(t_n + vh_n) v^{l-1}\, dv \right\} \\ &= \sum_{i=0}^{n-1} h_i^{1-\alpha} \cdot \sum_{l=1}^{m} \beta_{il} \cdot \int_0^1 \left(\frac{t_{nj} - t_i}{h_i} - v \right)^{-\alpha} K_{nj}(t_i + vh_i) v^{l-1}\, dv + q_{nj} \end{aligned} \tag{3.12}$$

$$(1 \leqslant j \leqslant m; 0 \leqslant n \leqslant N-1),$$

where the remainder terms q_{nj} are defined by

$$q_{nj} := -h_n^m R_n(c_j) + h_n^{1-\alpha} \cdot \int_0^{c_j} (c_j - v)^{-\alpha} \cdot K_{nj}(t_n + vh_n)(h_n^m R_n(v))\, dv$$

$$(3.13) \qquad + \sum_{i=1}^{n-1} h_i^{1-\alpha} \cdot \int_0^1 \left(\frac{t_{nj} - t_i}{h_i} - v \right)^{-\alpha} \cdot K_{nj}(t_i + vh_i)(h_i^m R_i(v))\, dv$$

$$+ h_0^{1-\alpha} \cdot \int_0^1 \left(\frac{t_{nj} - t_0}{h_0} \right)^{-\alpha} \cdot K_{nj}(t_0 + vh_0)\{ h_0^{1-\alpha} C_0(v) + h_0^m R_0(v) \}\, dv.$$

For the initial interval σ_0 we obtain, in particular,

$$\sum_{l=1}^{m} \beta_{0l} \left\{ c_j^{l-1} - h_0^{1-\alpha} \cdot \int_0^{c_j} (c_j - v)^{-\alpha} \cdot K_{0j}(t_0 + vh_0) v^{l-1}\, dv \right\}$$

$$(3.14) \qquad = h_0^{1-\alpha} \cdot \Big\{ -C_0(c_j) - h_0^{m+\alpha-1} \cdot R_0(c_j)$$

$$+ h_0^{1-\alpha} \int_0^{c_j} (c_j - v)^{-\alpha} \cdot K_{0j}(t_0 + vh_0)(C_0(v) + h_0^{m+\alpha-1} \cdot R_0(v))\, dv \Big\}$$

$$(1 \leqslant j \leqslant m).$$

We shall now show that, for quasi-uniform mesh sequences, all vectors β_n have l_1-norms satisfying

$$(3.15) \quad \|\beta_n\|_1 = \mathcal{O}(N^{-(1-\alpha)}) \qquad (0 \leqslant n \leqslant N-1;\ N \to \infty, \text{with } Nh \leqslant \gamma T).$$

This result will then be used in (3.10) to establish the assertion (2.3) of Theorem 2.1.

We begin by observing that, since the kernel $K(t, s)$ is bounded on S and since we have $h_n = \mathcal{O}(N^{-1})$ for $0 \leqslant n \leqslant N - 1$ (recall (2.2)), the matrices $V_m - h_n^{1-\alpha} \cdot C_{nn}$, with

$$(3.16) \quad V_m := (c_j^{l-1}), \quad \text{and} \quad C_{nn} := \left(\int_0^{c_j} (c_j - v)^{-\alpha} \cdot K_{nj}(t_n + vh_n) v^{l-1}\, dv \right)$$

$$(1 \leqslant j, l \leqslant m),$$

occurring in (3.12) and (3.14) possess uniformly bounded inverses for all sufficiently large N (note that V_m is a Vandermonde matrix corresponding to the collocation parameters $\{c_j\}$ satisfying $0 \leqslant c_1 < \cdots < c_m \leqslant 1$). Hence (3.12) and (3.14) define a unique sequence of vectors β_n for all sufficiently large values of N, and there exists a finite constant C_0' such that

$$(3.17) \qquad \left\| (V_m - h_n^{1-\alpha} \cdot C_{nn})^{-1} \right\|_1 \leqslant C_0', \qquad 0 \leqslant n \leqslant N - 1.$$

In order to show that the sequence $\{\|\beta_n\|_1\}$ is governed by a generalized Gronwall inequality, we require the following result.

LEMMA 3.1. *Consider a quasi-uniform sequence of meshes for I. Then, for* $0 \leqslant i \leqslant n - 1$ $(n \leqslant N - 1)$, *and for all* $\{c_j\}$ *with* $0 \leqslant c_1 < \cdots < c_m \leqslant 1$, *we have,*

$$(3.18) \quad \int_0^1 \left(\frac{t_{nj} - t_i}{h_i} - v \right)^{-\alpha} v^{l-1}\, dv < \frac{\gamma^\alpha (1 + \gamma)^\alpha}{1 - \alpha} \cdot (n - i)^{-\alpha} \qquad (1 \leqslant j, l \leqslant m).$$

Proof of Lemma 3.1. For $i = n - 1$, we have

$$\int_0^1 \left(1 + c_j \frac{h_n}{h_{n-1}} - v \right)^{-\alpha} v^{l-1}\, dv \leqslant \int_0^1 (1 - v)^{-\alpha}\, dv = 1/(1 - \alpha) < \frac{\gamma^\alpha (1 + \gamma)^\alpha}{1 - \alpha},$$

since $\gamma \geqslant 1$.

Suppose, then, that $i \leqslant n-2$. Since $c_j \in [0,1]$, we obtain

$$I_{ni}(\alpha) := \int_0^1 \left(\frac{t_{nj}-t_i}{h_i} - v\right)^{-\alpha} v^{l-1}\,dv \leqslant \int_0^1 \left(\frac{t_n-t_i}{h_i} - v\right)^{-\alpha} dv$$

$$= \frac{1}{1-\alpha}\left[\left(\frac{t_n-t_i}{h_i}\right)^{1-\alpha} - \left(\frac{t_n-t_i}{h_i} - 1\right)^{1-\alpha}\right]$$

$$= \frac{1}{1-\alpha}\left(\frac{t_n-t_i}{h_i}\right)^{1-\alpha} \cdot \left[1 - \left(1 - \left(\frac{t_n-t_i}{h_i}\right)^{-1}\right)\right]^{1-\alpha}.$$

Application of the Mean-Value Theorem yields

$$\left(1 - \left(\frac{t_n-t_i}{h_i}\right)^{-1}\right)^{1-\alpha} = 1 - (1-\alpha)\cdot\left(\frac{t_n-t_i}{h_i}\right)^{-1}\cdot\left(1 - \theta_{ni}\left(\frac{t_n-t_i}{h_i}\right)^{-1}\right)^{-\alpha}$$

(with $0 < \theta_{ni} < 1$), and we thus find

$$I_{ni}(\alpha) \leqslant \left(\frac{t_n-t_i}{h_i}\right)^{-\alpha}\cdot\left(1 - \theta_{ni}\cdot\left(\frac{t_n-t_i}{h_i}\right)^{-1}\right)^{-\alpha}. \tag{3.19}$$

So far, we have not specified the type of mesh sequence containing the points $\{t_n\}$. Suppose now that the mesh sequence is quasi-uniform. Hence, by (2.1),

$$\frac{t_n-t_i}{h_i} \geqslant \frac{(n-i)h'}{h} \geqslant \gamma^{-1}\cdot(n-i).$$

Moreover, since $i \leqslant n-2$,

$$1 - \theta_{ni}\cdot\left(\frac{t_n-t_i}{h_i}\right)^{-1} = 1 - \theta_{ni}\cdot\left(\frac{h_i + \cdots + h_{n-1}}{h_i}\right)^{-1}$$

$$\geqslant 1 - \left(\frac{h_i + h_{i+1}}{h_i}\right)^{-1} = 1 - (1 + h_{i+1}/h_i)^{-1}$$

$$\geqslant 1 - (1 + h'/h)^{-1} \geqslant 1 - (1 + 1/\gamma)^{-1} = (1+\gamma)^{-1}.$$

Using these results in (3.19) we obtain, for $0 < \alpha < 1$,

$$I_{ni}(\alpha) \leqslant \gamma^{\alpha}(1+\gamma)^{\alpha}\cdot(n-i)^{-\alpha} < \frac{\gamma^{\alpha}(1+\gamma)^{\alpha}}{1-\alpha}\cdot(n-i)^{-\alpha}.$$ □

LEMMA 3.2. *Let the assumptions of Lemma* 3.1 *hold. Then*:

$$\sum_{i=0}^{n-1} h_i^{1-\alpha}\cdot\int_0^1 \left(\frac{t_{nj}-t_i}{h_i} - v\right)^{-\alpha} dv \leqslant T^{1-\alpha}/(1-\alpha), \qquad 1 \leqslant n \leqslant N. \tag{3.20}$$

Proof of Lemma 3.2. Using the initial argument of the previous proof, we find

$$\sum_{i=0}^{n-1} h_i^{1-\alpha}\cdot\int_0^1 \left(\frac{t_{nj}-t_i}{h_i} - v\right)^{-\alpha} dv \leqslant \sum_{i=0}^{n-1} h_i^{1-\alpha}\cdot\int_0^1 \left(\frac{t_n-t_i}{h_i} - v\right)^{-\alpha} dv$$

$$= \frac{1}{1-\alpha}\cdot\sum_{i=0}^{n-1} h_i^{1-\alpha}\left\{\left(\frac{t_n-t_i}{h_i}\right)^{1-\alpha} - \left(\frac{t_n-t_{i+1}}{h_i}\right)^{1-\alpha}\right\}$$

$$= \frac{1}{1-\alpha}\cdot\sum_{i=0}^{n-1}\left\{(t_n-t_i)^{1-\alpha} - (t_n-t_{i+1})^{1-\alpha}\right\} = t_n^{1-\alpha}/(1-\alpha)$$

$$\leqslant T^{1-\alpha}/(1-\alpha) \quad \text{for } n \leqslant N.$$ □

Note that (3.20) will also be valid for graded meshes; this fact will be used in Section 4.

We now return to (3.12): for $0 \leqslant i \leqslant n-1$, define the matrices C_{ni} by

$$C_{ni} := \left(\int_0^1 \left(\frac{t_{nj} - t_i}{h_i} - v \right)^{-\alpha} \cdot K_{nj}(t_i + vh_i) v^{l-1}\, dv \right) \qquad (1 \leqslant j, l \leqslant m),$$

and introduce the vectors $q_n := (q_{n1}, \ldots, q_{nm})^T$, with components defined in (3.13). Thus, (3.12) can be rewritten as

$$\beta_n = \left(V_m - h_n^{1-\alpha} C_{nn}\right)^{-1} \cdot \left\{ \sum_{i=0}^{n-1} h_i^{1-\alpha} \cdot C_{ni}\beta_i + q_n \right\}, \tag{3.21}$$

provided N is sufficiently large. If $K_0 := \max\{|K(t,s)|: (t,s) \in S\}$ then, by Lemma 3.1, we find

$$\|C_{ni}\|_1 \leqslant C(\alpha) \cdot (n-i)^{-\alpha}, \qquad 0 \leqslant i \leqslant n-1, \tag{3.22}$$

where $C(\alpha) := mK_0 \cdot \gamma^\alpha (1+\gamma)^\alpha/(1-\alpha)$. This can be used, together with (3.17), to obtain

$$\|\beta_n\|_1 \leqslant C_0 h^{1-\alpha} \cdot \sum_{i=0}^{n-1} (n-i)^{-\alpha} \cdot \|\beta_i\|_1 + C_0' \cdot \|q_n\|_1 \qquad (0 \leqslant n \leqslant N-1), \tag{3.23}$$

with $C_0 := C_0' \cdot C(\alpha)$. This represents a generalized discrete Gronwall inequality (compare [15], [8], [1]), and it follows that

$$\|\beta_n\|_1 = \mathcal{O}\left(\|q_n\|_1\right), \qquad 0 \leqslant n \leqslant N-1, \tag{3.24}$$

since C_0' (given in (3.17)) is a finite constant, and since $Nh \leqslant \gamma T$.

It is clear from (3.13) that the order of $\|q_n\|_1$ will essentially be governed by that of the terms $h_i^m R_i(\cdot)$, with $R_i(v)$ defined in (3.9) and (3.6); for $i \geqslant 1$ these terms involve the mth derivative of the solution y (if $i = 0$ then, by (3.6) and (3.3b), $h_0^m R_0(v) = \mathcal{O}(N^{-m})$). It follows from (3.2) and from the Leibniz product rule that this derivative has the form

$$y^{(m)}(t) = v_0^{(m)}(t) + \sum_{s=1}^{q-1} \sum_{k=0}^{m} \binom{m}{k} \binom{s(1-\alpha)}{k} k! \cdot v_s^{(m-k)}(t) \cdot t^{s(1-\alpha)-k}, \qquad t > 0. \tag{3.25}$$

Thus, upon setting

$$\gamma_{mk}^{(s)} := \binom{m}{k} \binom{s(1-\alpha)}{k} \cdot k!, \qquad M_{s\nu} := \max\left\{ \left| v_s^{(\nu)}(t) \right| : t \in I \right\},$$

we are led to

$$h_i^m \cdot |R_i(v)| \leqslant h_i^m \cdot \left\{ M_{0m} + \sum_{s=1}^{q-1} \sum_{k=0}^{m} \left|\gamma_{mk}^{(s)}\right| \cdot M_{s,m-k} (t_i + \theta_i v h_i)^{s(1-\alpha)-k} \right\}. \tag{3.26}$$

This, in turn, reveals that the order of $h_i^m R_i(v)$ will depend on the orders of the products $h_i^m \cdot t_i^{s(1-\alpha)-k}$. To be precise, we state

LEMMA 3.3. *Consider any quasi-uniform sequence of meshes for I, and assume that $1 \leqslant k \leqslant m$. Then, for $s \geqslant 1$,*

$$h_i^m \cdot t_i^{s(1-\alpha)-k} = \begin{cases} \mathcal{O}(N^{-m}), & \text{if } s(1-\alpha) - k \geqslant 0; \\ \mathcal{O}(N^{-(1-\alpha)}), & \text{if } s(1-\alpha) - k < 0. \end{cases} \tag{3.27}$$

Proof of Lemma 3.3. Assume first that $s(1-\alpha)-k \geqslant 0$. The first part of (3.27) then follows trivially since $t_i^{s(1-\alpha)-k} \leqslant T^{s(1-\alpha)-k}$ and, by (2.2), $h_i = \mathcal{O}(N^{-1})$ for $1 \leqslant i \leqslant N-1$.

Now let $s(1-\alpha)-k < 0$. In this case we have, for $i \geqslant 1$,

$$t_i \geqslant t_1 = h_0 \geqslant h' \geqslant h/\gamma \geqslant \gamma^{-1} T N^{-1},$$

and hence, by (2.2),

$$\begin{aligned} h_i^m \cdot t_i^{s(1-\alpha)-k} &\leqslant (\gamma T)^m N^{-m} \cdot (\gamma^{-1} T)^{s(1-\alpha)-k} \cdot N^{-(s(1-\alpha)-k)} \\ &= (\gamma T)^m \cdot (\gamma^{-1} T)^{s(1-\alpha)-k} \cdot N^{-(m+s(1-\alpha)-k)} \\ &\leqslant (\gamma T)^m \cdot (\gamma^{-1} T)^{s(1-\alpha)-k} \cdot N^{-(1-\alpha)}, \qquad 1 \leqslant i \leqslant N-1, \end{aligned}$$

since $m + s(1-\alpha) - k \geqslant s(1-\alpha) \geqslant 1 - \alpha > 0$ for all k with $1 \leqslant k \leqslant m$, $s \geqslant 1$, and $\alpha \in (0,1)$. □

Consider again (3.26): for $\alpha \in (0,1)$ there is at least one pair (s,k), with $1 \leqslant k \leqslant m$, $s \geqslant 1$, for which $s(1-\alpha)-k < 0$ (take $(s,k) = (1,1)$). Consequently, (3.26) yields

$$h_i^m |R_i(v)| = \mathcal{O}(N^{-(1-\alpha)}), \qquad v \in [0,1] \ (1 \leqslant i \leqslant N-1), \tag{3.28}$$

where the exponent cannot be replaced by some $\beta > 1-\alpha$.

If we now use the results (3.26), (3.27), (3.20) in (3.13) we verify readily that

$$|q_{nj}| = \mathcal{O}(N^{-(1-\alpha)}), \qquad 1 \leqslant j \leqslant m \ (0 \leqslant n \leqslant N-1),$$

and hence

$$\|q_n\|_1 = \mathcal{O}(N^{-(1-\alpha)}), \qquad 0 \leqslant n \leqslant N-1. \tag{3.29}$$

To bring the proof of Theorem 2.1 to its conclusion we return to (3.10): since (3.24) and (3.29) imply $\|\beta_n\|_1 = \mathcal{O}(N^{-(1-\alpha)})$ for all n we find

$$|e_n(t_n + vh_n)| \leqslant \|\beta_n\|_1 + \mathcal{O}(N^{-(1-\alpha)}) = \mathcal{O}(N^{-(1-\alpha)}),$$

$t_n + vh_n \in \sigma_n$, $0 \leqslant n \leqslant N-1$ (as $N \to \infty$, $Nh \leqslant \gamma T$). This is equivalent to (2.3). □

We conclude this section with two remarks.

(i) As has been mentioned above, the proof is easily extended to the case of *irrational* α: this follows from the fact that the infinite series in (3.1) converges absolutely and uniformly on I, and by Lemma 3.3 which holds for all $s \geqslant 1$.

(ii) If the given integral equation is *nonlinear*, i.e. (1.1), then we can use a result due to Lubich [14] which states that if $g(t)$ is of the form $g(t) = G(t, t^{1-\alpha})$ near $t = 0$, and if G and the kernel k are real analytic functions in a neighborhood of the origin (excluding the trivial cases $g \equiv 0$, $k \equiv 0$), then the exact solution of (1.1) near $t = 0$ is given by

$$y(t) = Y(t, t^{1-\alpha}), \tag{3.30}$$

where Y is a real analytic function in a neighborhood of $(0,0)$. It is then easily seen that by expressing Y as a power series, the solution y near $t = 0$ can be written in a form analogous to (3.1). In the corresponding error analysis the role of $K_{nj}(t_i + vh_i)$ will then be taken by the partial derivative $\partial k(t_{nj}, t_i + vh_i, y)/\partial y$, evaluated at some suitable value of y (stemming from the application of the Mean-Value Theorem in the linearization of the error equation); in order that the analogue of (3.17) hold, $\partial k/\partial y$ has to be bounded.

4. Proof of Theorem 2.2: Convergence on Graded Meshes. The proof of Theorem 2.2 proceeds in complete analogy to the one of Theorem 2.1 given in the previous section, except that now we shall obtain a different estimate for $\|q_n\|_1$, and hence for $\|\beta_n\|_1$ (cf. (3.29) and (3.24)).

Let us begin by stating two simple properties of graded meshes of the form (2.4) with $r \geqslant 1$; namely,

$$t_n = n^r \cdot t_1, \qquad 1 \leqslant n \leqslant N, \tag{4.1a}$$

with

$$t_1 = h_0 = TN^{-r}; \tag{4.1b}$$

and

$$h/h' = N^r \cdot (1 - (1 - N^{-1})^r). \tag{4.2}$$

(This last result shows, incidentally, that a sequence of graded meshes with $r > 1$ is not quasi-uniform, since $h/h' \to \infty$ as $N \to \infty$.)

For graded meshes we obtain the following analogue of Lemma 3.1:

LEMMA 4.1. *Consider any graded mesh of the form* (2.4) *and with grading exponent* $r \geqslant 1$. *Then, for* $0 \leqslant i \leqslant n-1 \leqslant N-1$ *and for all collocation parameters* $\{c_j\}$ *satisfying* $0 \leqslant c_1 < \cdots < c_m \leqslant 1$,

$$\int_0^1 \left(\frac{t_{nj} - t_i}{h_i} - v\right)^{-\alpha} \cdot v^{l-1}\,dv \leqslant \frac{2^\alpha}{1-\alpha} \cdot (n-i)^{-\alpha} \qquad (1 \leqslant j, l \leqslant m). \tag{4.3}$$

Proof of Lemma 4.1. The first half of the proof of Lemma 3.1 carries over without any change: there, we have shown that, for $i \leqslant n-2$,

$$I_{ni}(\alpha) := \int_0^1 \left(\frac{t_{nj} - t_i}{h_i} - v\right)^{-\alpha} \cdot v^{l-1}\,dv \leqslant \left(\frac{t_n - t_i}{h_i}\right)^{-\alpha} \cdot \left(1 - \theta_{ni} \cdot \left(\frac{t_n - t_i}{h_i}\right)^{-1}\right)^{-\alpha},$$

with $0 < \theta_{ni} < 1$. For a graded mesh (2.4) with $r \geqslant 1$ we obtain

$$\frac{t_n - t_i}{h_i} = \frac{h_{n-1} + \cdots + h_i}{h_i} \geqslant \frac{(n-i)\cdot h_i}{h_i} = n - i,$$

since $0 < h_0 < \cdots < h_{N-1}\,(= h)$. Moreover, since $i \leqslant n-2$, we have

$$1 - \theta_{ni} \cdot \left(\frac{t_n - t_i}{h_i}\right)^{-1} \geqslant 1 - \left(\frac{t_n - t_i}{h_i}\right)^{-1} \geqslant 1 - \left(\frac{h_i + h_{i+1}}{h_i}\right)^{-1}$$
$$= 1 - (1 + h_{i+1}/h_i)^{-1} \geqslant 1 - \frac{1}{1+1} = 1/2,$$

and this yields

$$\left(1 - \theta_{ni} \cdot \left(\frac{t_n - t_i}{h_i}\right)^{-1}\right)^{-\alpha} \leqslant 2^\alpha < 2^\alpha/(1-\alpha).$$

Hence,

$$I_{ni}(\alpha) \leqslant \frac{2^\alpha}{1-\alpha} \cdot (n-i)^{-\alpha}, \qquad 0 \leqslant i \leqslant n-1. \qquad \square$$

Since the mesh diameter h of a graded mesh satisfies $h = \mathcal{O}(N^{-1})$ (recall (2.5)) we may use again the contraction mapping argument of Section 3 (cf. (3.16) and (3.17)) to show that, for all sufficiently large N, (3.12) defines a unique sequence of vectors $\{\beta_n := (\beta_{n1}, \ldots, \beta_{nm})^T: 0 \leqslant n \leqslant N-1\}$, for which the generalized discrete Gronwall inequality (3.23) holds. Note that here we have made use of Lemma 3.2 which is valid both for quasi-uniform and for graded mesh sequences.

As an immediate consequence, the l_1-norms of these vectors β_n satisfy again (3.24). However, the estimate for $h_i^m R_i(v)$ (which will eventually determine the order of $\|\beta_n\|_1$) turns out to be rather different than that for quasi-uniform mesh sequences given in (3.28). This is due to the following results.

LEMMA 4.2. *Consider a graded mesh of the form* (2.4), *and assume that the grading exponent r is given by*

$$r = m/(1-\alpha). \tag{4.4}$$

Then, for $1 \leqslant k \leqslant \nu \leqslant m$ *and for* $s \geqslant 1$,

$$h_i^\nu \cdot t_i^{s(1-\alpha)-k} \leqslant c \cdot N^{-\nu}, \qquad 1 \leqslant i \leqslant N-1, \tag{4.5}$$

where $c := r^\nu \cdot 2^{\nu(r-1)} \cdot T^{\nu-k+s(1-\alpha)}$.

Proof of Lemma 4.2. Since $r \geqslant 1$ we find, using (4.1) and the Mean-Value Theorem,

$$h_i = t_{i+1} - t_i = \left((i+1)^r - i^r\right) \cdot t_1 = i^r \cdot \left((1+i^{-1})^r - 1\right) \cdot TN^{-r}$$
$$= r \cdot i^{r-1} \cdot \left(1 + \theta_i \cdot i^{-1}\right)^{r-1} \cdot TN^{-r}, \quad \text{with } 0 < \theta_i < 1,$$

and hence $h_i \leqslant r \cdot 2^{r-1} \cdot T \cdot i^{r-1} \cdot N^{-r}$. This yields, again employing (4.1),

$$h_i^\nu \cdot t_i^{s(1-\alpha)-k} \leqslant \left(r \cdot 2^{r-1} \cdot T\right)^\nu \cdot i^{\nu(r-1)} \cdot N^{-\nu r} \cdot \left(i^r \cdot TN^{-r}\right)^{s(1-\alpha)-k}$$
$$= c \cdot i^{\nu(r-1)+rs(1-\alpha)-rk} \cdot N^{-\nu r - rs(1-\alpha)+rk}$$
$$= c \cdot i^{r(\nu-k)+rs(1-\alpha)-\nu} \cdot N^{-r(\nu-k)-rs(1-\alpha)},$$

with the constant c as defined in Lemma 4.2.

For $r = m/(1-\alpha)$ this reduces to

$$h_i^\nu \cdot t_i^{s(1-\alpha)-k} \leqslant c \cdot \left(\frac{i}{N}\right)^{r(\nu-k)} \cdot i^{ms-\nu} \cdot N^{-ms},$$

where the exponent of i satisfies $ms - \nu \geqslant ms - m = m(s-1) \geqslant 0$, since $s \geqslant 1$. It thus follows that, for all $i \leqslant N$, and with $1 \leqslant k \leqslant \nu \leqslant m$,

$$h_i^\nu \cdot t_i^{s(1-\alpha)-k} \leqslant c \cdot N^{ms-\nu} \cdot N^{-ms} = c \cdot N^{-\nu}. \qquad \square$$

If we now use the result of Lemma 4.2, with $\nu = m$, in (3.26), we find with no further difficulties the estimate

$$h_i^m \cdot |R_i(v)| = \mathcal{O}(N^{-m}), \qquad v \in [0,1] \ (1 \leqslant i \leqslant N-1), \tag{4.6}$$

where the exponent m is best possible. By (3.13) this then leads to

$$\|q_n\|_1 = \mathcal{O}(N^{-m}), \tag{4.7}$$

since, by (4.1b), $h_0 = TN^{-r}$; hence $h_0^{1-\alpha} = T^{1-\alpha} \cdot N^{-r(1-\alpha)} = T^{1-\alpha} \cdot N^{-m}$. By (3.23) and (3.24) we have thus shown that

$$\|\beta_n\|_1 = \mathcal{O}(N^{-m}), \qquad 0 \leqslant n \leqslant N-1 \ (N \to \infty, Nh \leqslant rT), \tag{4.8}$$

provided the grading exponent r is as in (4.4). Using once more the expression (3.10) for the error function $e_n(t_n + vh_n)$, together with the fact that $h_0^{1-\alpha} = \mathcal{O}(N^{-m})$ for the above grading exponent, we obtain assertion (2.6) of Theorem 2.2. □

The above proof is easily modified to deal with the case where, instead of (4.4), we have

$$r = \mu/(1-\alpha), \qquad 1 \leqslant \mu < m. \tag{4.9}$$

We now find, for $1 \leqslant k \leqslant \nu \leqslant m$,

$$h_i^\nu \cdot t_i^{s(1-\alpha)-k} \leqslant c \cdot \left(\frac{i}{N}\right)^{r(\nu-k)} \cdot i^{rs(1-\alpha)-\nu} \cdot N^{-rs(1-\alpha)}$$
$$\leqslant c \cdot i^{\mu s-\nu} N^{-\mu s}, \qquad 1 \leqslant i \leqslant N.$$

If $\mu s - \nu \geqslant 0$, then $h_i^\nu \cdot t_i^{s(1-\alpha)-k} = \mathcal{O}(N^{-\nu})$. However, since $\mu < m$, we shall also have $\mu s - \nu < 0$ for some values of (s, ν) (e.g., for $(s, \nu) = (1, m)$), in which case the above estimate will no longer be valid. Instead, writing $\mu = m - (m - \mu)$, and observing that $ms - \nu \geqslant 0$ and $m - \mu > 0$, we obtain

$$h_i^\nu \cdot t_i^{s(1-\alpha)-k} \leqslant c \cdot i^{(m-(m-\mu))s-\nu} \cdot N^{-(m-(m-\mu))s}$$
$$\leqslant c \cdot N^{ms-\nu} \cdot i^{-(m-\mu)s} \cdot N^{-ms+(m-\mu)s}$$
$$\leqslant c \cdot N^{-(\nu-(m-\mu))s}, \qquad 1 \leqslant i \leqslant N \ (s \geqslant 1).$$

For the value of ν relevant in our analysis, $\nu = m$, this becomes

$$h_i^m \cdot t_i^{s(1-\alpha)-k} \leqslant c \cdot N^{-\mu s}, \qquad s \geqslant 1. \tag{4.10}$$

Hence, if the grading exponent r in (2.4) is given by (4.9), then there results the estimate $\|q_n\|_1 = \mathcal{O}(N^{-\mu})$ (note that now $h_0^{1-\alpha} = T^{1-\alpha} \cdot N^{-\mu}$) and, by (3.24), $\|\beta_n\|_1 = \mathcal{O}(N^{-\mu})$ $(0 \leqslant n \leqslant N-1)$. By (3.10) we then readily establish the following result.

THEOREM 4.1. *Let the functions g and K in* (1.2) *satisfy the smoothness hypotheses stated in Theorem* 2.1, *and let* $u \in S_{m-1}^{(-1)}(Z_N)$ *denote the collocation approximation defined by* (1.8), *with collocation parameters* $\{c_j\}$ *satisfying* $0 \leqslant c_1 < \cdots < c_m \leqslant 1$. *Then, for the sequence of graded meshes* (2.4) *corresponding to the grading exponent* $r = \mu/(1-\alpha)$ $(1 \leqslant \mu \leqslant m)$, *the collocation error behaves like*

$$\|y - u\|_\infty = \mathcal{O}(N^{-\mu}). \tag{4.11}$$

In particular, the choice $\mu = 1$ (i.e., $r = 1/(1-\alpha)$) will yield collocation approximations which, on I, converge linearly to the solution y of (1.2), independent of how one selects m.

The proofs of the above results are again easily extended to linear integral equations (1.2) with irrational α, and to nonlinear integral equations (1.1). We refer to the remarks made at the end of Section 3.

5. Discretization of the Collocation Equation. Until now it has been assumed that the integrals

$$\Phi_{ni}^{(j)}[u_i] := \begin{cases} \displaystyle\int_0^1 \left(\frac{t_{nj} - t_i}{h_i} - v\right)^{-\alpha} \cdot k\big(t_{nj}, t_i + vh_i, u_i(t_i + vh_i)\big)\, dv, & 0 \leqslant i \leqslant n-1, \\ \displaystyle\int_0^{c_j} (c_j - v)^{-\alpha} \cdot k\big(t_{nj}, t_n + vh_n, u_n(t_n + vh_n)\big)\, dv, & i = n \end{cases} \tag{5.1}$$

$$(1 \leqslant j \leqslant m)$$

occurring in the collocation equation (1.8) are known exactly; i.e., that the collocation approximation $u \in S_{m-1}^{(-1)}(Z_N)$ is obtained from what we shall refer to as the *exact collocation equation*

$$u_n(t_{nj}) = g(t_{nj}) + h_n^{1-\alpha} \cdot \Phi_{nn}^{(j)}[u_n] + \sum_{i=0}^{n-1} h_i^{1-\alpha} \cdot \Phi_{ni}^{(j)}[u_i] \tag{5.2}$$

$$(1 \leqslant j \leqslant m; 0 \leqslant n \leqslant N-1).$$

In practical applications this will rarely be possible, making a further discretization step necessary which will involve numerical quadrature. Suppose, then, that the integrals in (5.1) are approximated by

$$\hat{\Phi}_{ni}^{(j)}[u_i] := \begin{cases} \sum_{l=1}^{\mu_1} w_{jl}^{(n,i)}(\alpha) \cdot k\big(t_{nj}, t_i + d_l h_i, u_i(t_i + d_l h_i)\big), \\ \qquad\qquad 0 \leqslant i \leqslant n-1, \\ \sum_{l=1}^{\mu_0} w_{jl}(\alpha) \cdot k\big(t_{nj}, t_n + d_{jl} h_n, u_n(t_n + d_{jl} h_n)\big), \quad i = n \end{cases} \tag{5.3}$$

$$(1 \leqslant j \leqslant m);$$

for the case where $c_1 = 0$ we set $\hat{\Phi}_{nn}^{(1)}[u_n] := 0$ $(= \Phi_{nn}^{(1)}[u_n])$. It will be assumed that the quadrature abscissas in (5.3) are characterized by the parameters

$$0 \leqslant d_1 < \cdots < d_{\mu_1} \leqslant 1, \tag{5.4a}$$

and

$$0 \leqslant d_{j1} < \cdots < d_{j\mu_0} \leqslant c_j \qquad (1 \leqslant j \leqslant m), \tag{5.4b}$$

with $\mu_0 \geqslant 1$, $\mu_1 \geqslant 1$ (and, usually, $\mu_0 \leqslant m$, $\mu_1 \leqslant m$). (Note that due to the choice (5.4b) the quadrature formulas $\hat{\Phi}_{nn}^{(j)}[u_n]$ will only involve kernel values $k(t, s, \cdot)$ lying in the domain of k; in general, it may not be possible to extend $k(t, s, \cdot)$ smoothly to points (t, s) with $s > t$.) Moreover, we shall assume that the quadrature weights in (5.3) are given by

$$w_{jl}^{(n,i)}(\alpha) := \int_0^1 \left(\frac{t_{nj} - t_i}{h_i} - v\right)^{-\alpha} \cdot \lambda_l(v)\, dv \qquad (1 \leqslant l \leqslant \mu_1; 1 \leqslant j \leqslant m), \tag{5.5a}$$

and by

$$w_{jl}(\alpha) := \int_0^{c_j} (c_j - v)^{-\alpha} \cdot \lambda_{jl}(v)\, dv \qquad (1 \leqslant l \leqslant \mu_0; 1 \leqslant j \leqslant m), \tag{5.5b}$$

where

$$\lambda_l(v) := \prod_{\substack{k=1 \\ k \neq l}}^{\mu_1} (v - d_k)/(d_l - d_k) \quad \text{and} \quad \lambda_{jl}(v) := \prod_{\substack{k=1 \\ k \neq l}}^{\mu_0} (v - d_{jk})/(d_{jl} - d_{jk})$$

represent, respectively, the Lagrange fundamental polynomials for the points given in (5.4). In other words, we consider the discretization of the exact collocation equation (5.2) by quadrature formulas based on *product integration* (compare also [19] and the references listed there).

The *fully discretized collocation equation* is obtained from (5.2) by replacing the exact integrals (5.1) by the corresponding approximations (5.3). In general, one will now generate an approximation $\hat{u} \in S_{m-1}^{(-1)}(Z_N)$ which will be different from the one defined by the exact collocation (5.2); i.e., $\hat{u}$ will be given by

$$\hat{u}_n(t_{nj}) = g(t_{nj}) + h_n^{1-\alpha} \cdot \hat{\Phi}_{nn}^{(j)}[\hat{u}_n] + \sum_{i=0}^{n-1} h_i^{1-\alpha} \cdot \hat{\Phi}_{ni}^{(j)}[\hat{u}_i] \tag{5.6}$$

$$(1 \leqslant j \leqslant m;\ 0 \leqslant n \leqslant N-1),$$

where, in analogy to (1.10), we write

$$\hat{u}_n(t_n + vh_n) = \sum_{j=1}^{m} L_j(v) \cdot \hat{u}_n(t_{nj}), \qquad t_n + vh_n \in \sigma_n. \tag{5.7}$$

Setting $\hat{e} := y - \hat{u}$, $e := y - u$, and $\varepsilon := u - \hat{u}$, it follows from $\hat{e} = (y-u) + (u - \hat{u})$ that

$$\|\hat{e}\|_\infty \leqslant \|e\|_\infty + \|\varepsilon\|_\infty. \tag{5.8}$$

Global convergence results for $\hat{u}$ will thus be obtained by estimating the order of the perturbation ε due to the full discretization of (5.2), and by using the results on the behavior of e derived in the previous sections. For simplicity, we shall state the results again for the linear equation (1.2); according to the remark at the end of Section 3, their extension to nonlinear equations is straightforward.

THEOREM 5.1. *Let g and K in* (1.2) *be m times continuously differentiable on their respective domains. Assume that* u, $\hat{u} \in S_{m-1}^{(-1)}(Z_N)$ *denote the solution of the exact collocation equation* (5.2) *and that of its fully discretized version* (5.6), *where the quadrature formulas* (5.3), *satisfying* (5.4) *and* (5.5), *have been used. Then the perturbation* $\varepsilon := u - \hat{u}$ *behaves like*

$$\|\varepsilon\|_\infty = \mathcal{O}(N^{-\mu}), \tag{5.9}$$

where $\mu := \min(\mu_0 + 1 - \alpha, \mu_1)$, *and this holds for quasi-uniform sequences of meshes as well as for graded mesh sequences* (2.4) *with* $r \geqslant 1$.

Proof. Let

$$E_{ni}^{(j)}[u_i] := \Phi_{ni}^{(j)}[u_i] - \hat{\Phi}_{ni}^{(j)}[u_i]. \tag{5.10}$$

Hence, subtracting (5.6) from (5.2) and setting $k(t,s,y) = K(t,s) \cdot y$, we obtain

$$\begin{aligned} \varepsilon_n(t_{nj}) &= h_n^{1-\alpha} \cdot \hat{\Phi}_{nn}^{(j)}[\varepsilon_n] + \sum_{i=0}^{n-1} h_i^{1-\alpha} \cdot \hat{\Phi}_{ni}^{(j)}[\varepsilon_i] \\ &\quad + \sum_{i=0}^{n} h_i^{1-\alpha} \cdot E_{ni}^{(j)}[u_i] \qquad (1 \leqslant j \leqslant m; 0 \leqslant n \leqslant N-1), \end{aligned} \tag{5.11}$$

where $\varepsilon_n(t)$ denotes the restriction of $\varepsilon(t)$ to the subinterval σ_n. Since $\varepsilon_n \in \pi_{m-1}$ we may write

$$\varepsilon_n(t_n + vh_n) = \sum_{l=1}^{m} L_l(v) \cdot \varepsilon_n(t_n + c_l h_n), \qquad t_n + vh_n \in \sigma_n, \tag{5.12}$$

with $L_l(v)$ representing the lth Lagrange fundamental polynomial associated with the m collocation parameters. The terms $\hat{\Phi}_{ni}^{(j)}[\varepsilon_i]$ in (5.11) are thus of the form

$$\hat{\Phi}_{ni}^{(j)}[\varepsilon_i] = \begin{cases} \sum_{l=1}^{m}\left(\sum_{s=1}^{\mu_1} w_{js}^{(n,i)}(\alpha)\cdot K_{nj}(t_i + d_s h_i)\cdot L_l(d_s)\right)\cdot \varepsilon_i(t_i + c_l h_i), & i < n, \\ \sum_{l=1}^{m}\left(\sum_{s=1}^{\mu_0} w_{js}(\alpha)\cdot K_{nj}(t_n + d_{js} h_n)\cdot L_l(d_{js})\right)\cdot \varepsilon_n(t_n + c_l h_n), & i = n \end{cases}$$

$$(1 \leqslant j \leqslant m).$$

Let $Q_{ni}(\alpha)$ $(0 \leqslant i \leqslant n \leqslant N-1)$ denote the square matrix of order m whose elements are

$$(5.13) \quad q_{jl}^{(n,i)}(\alpha) := \begin{cases} \sum_{s=1}^{\mu_1} w_{js}^{(n,i)}(\alpha)\cdot K_{nj}(t_i + d_s h_i)\cdot L_l(d_s), & 0 \leqslant i \leqslant n-1, \\ \sum_{s=1}^{\mu_0} w_{js}(\alpha)\cdot K_{nj}(t_n + d_{js} h_n) L_l(d_{js}), & i = n \end{cases}$$

$$(1 \leqslant j, l \leqslant m),$$

and define the vectors

$$r_{ni} := \left(E_{ni}^{(1)}[u_i], \ldots, E_{ni}^{(m)}[u_i]\right)^T,$$

$$\eta_i := \left(\varepsilon_i(t_i + c_1 h_i), \ldots, \varepsilon_i(t_i + c_m h_i)\right)^T.$$

With this notation, Eq. (5.11) can be expressed in the form

$$(5.14) \quad \left(I_m - h_n^{1-\alpha}\cdot Q_{nn}(\alpha)\right)\cdot \eta_n = \sum_{i=0}^{n-1} h_i^{1-\alpha}\cdot Q_{ni}(\alpha)\eta_i + \sum_{i=0}^{n} h_i^{1-\alpha}\cdot r_{ni}$$

$$(0 \leqslant n \leqslant N-1),$$

where I_m is the identity matrix of order m. Consider the matrix multiplying η_n: since the elements of $Q_{nn}(\alpha)$ are bounded (this follows from the boundedness of the kernel $K(t,s)$ and from that of the quadrature weights (5.5b)), and since $h_n = \mathcal{O}(N^{-1})$ $(n \leqslant N-1)$ both for quasi-uniform and for graded mesh sequences (recall (2.2) and (2.5)), there exists a finite constant Q_0' such that, for all sufficiently large N,

$$(5.15) \quad \left\|\left(I_m - h_n^{1-\alpha}\cdot Q_{nn}(\alpha)\right)^{-1}\right\|_1 \leqslant Q_0', \quad 0 \leqslant n \leqslant N-1.$$

In order to show that the l_1-norms of the vectors η_n are governed, in analogy to (3.23), by a generalized discrete Gronwall inequality we require the following

LEMMA 5.1. *The quadrature weights* $w_{jl}^{(n,i)}(\alpha)$ $(i < n)$ *defined by* (5.5a) *satisfy*

$$(5.16) \quad \left|w_{jl}^{(n,i)}(\alpha)\right| \leqslant w(\alpha)\cdot(n-i)^{-\alpha} \quad (1 \leqslant j \leqslant m; 1 \leqslant l \leqslant \mu_1),$$

where the constant $w(\alpha)$ *is given by*

$$w(\alpha) := \begin{cases} \dfrac{\gamma^\alpha(1+\gamma)^\alpha}{1-\alpha}\cdot\Lambda_1 & \text{for quasi-uniform meshes}, \\ \dfrac{2^\alpha}{1-\alpha}\cdot\Lambda_1 & \text{for graded meshes (2.4)}; \end{cases}$$

here, $\Lambda_1 := \max\{\sum_{l=1}^{\mu_1}|\lambda_l(v)|: v \in [0,1]\}$ *denotes the Lebesgue constant associated with the quadrature parameters* $\{d_1, \ldots, d_{\mu_1}\}$.

The proof of this assertion is an immediate consequence of Lemma 3.1 (for quasi-uniform mesh sequences) and of Lemma 4.1 (for graded meshes). □

The above lemma allows us to derive bounds for the norms $\|Q_{ni}(\alpha)\|_1$: using (5.13) we find

$$\|Q_{ni}(\alpha)\|_1 = \max\left\{ \sum_{j=1}^{m} \left|q_{jl}^{(n,i)}(\alpha)\right| : 1 \leqslant l \leqslant m \right\}$$
$$\leqslant Q(\alpha)\cdot(n-i)^{-\alpha}, \qquad 0 \leqslant i \leqslant n-1 \ (n \leqslant N-1),$$

with the constant $Q(\alpha)$ depending on the bound for $K(t, s)$ and on the Lebesgue constant Λ_1. Applying the above results in (5.14) we obtain

$$\|\eta_n\|_1 \leqslant Q_0 \cdot h^{1-\alpha} \cdot \sum_{i=0}^{n-1} (n-i)^{-\alpha} \cdot \|\eta_i\|_1 + Q_0' \cdot \sum_{i=0}^{n} h_i^{1-\alpha} \cdot \|r_{ni}\|_1 \tag{5.17}$$
$$(0 \leqslant n \leqslant N-1),$$

with $Q_0 := Q_0' \cdot Q(\alpha)$. This is the desired generalized discrete Gronwall inequality; in analogy to (3.24), the order of the quantities $\|\eta_n\|_1$ will be given by the order of the terms $\sum_{i=0}^{n} h_i^{1-\alpha} \cdot \|r_{ni}\|_1$.

LEMMA 5.2. *Let the assumptions of Theorem* 5.1 *hold. Then we have*

$$\sum_{i=0}^{n} h_i^{1-\alpha} \cdot \|r_{ni}\|_1 = \mathcal{O}(N^{-\mu}), \qquad 0 \leqslant n \leqslant N-1, \tag{5.18}$$

with $\mu := \min(\mu_0 + 1 - \alpha, \mu_1)$, *independent of whether we consider quasi-uniform or graded mesh sequences.*

Proof of Lemma 5.2. Recall that the components of r_{ni} are the quadrature errors introduced in (5.10). According to the hypotheses imposed on the quadrature formulas (5.3) these quadrature errors are bounded; specifically, we have

$$\left|E_{ni}^{(j)}[u_i]\right| \leqslant \begin{cases} \gamma_1 h_i^{\mu_1} \cdot \int_0^1 \left(\frac{t_{nj}-t_i}{h_i} - v\right)^{-\alpha} dv & \text{if } i \leqslant n-1, \\ \gamma_0 h_i^{\mu_0} \cdot \int_0^{c_j} (c_j - v)^{-\alpha} dv & \text{if } i = n \quad (1 \leqslant j \leqslant m), \end{cases} \tag{5.19}$$

with γ_0 and γ_1 denoting suitable constants. To show this, let $i < n$,

$$\phi_{nj}(t_i + vh_i) := K_{nj}(t_i + vh_i)u_i(t_i + vh_i),$$

and denote by $\psi_{nj}(t_i + vh_i)$ the interpolating polynomial (of degree μ_1) for ϕ_{nj} with respect to the points $\{t_i + d_s h_i : 1 \leqslant s \leqslant \mu_1\}$. Since ϕ_{nj} has continuous derivatives of order m on σ_i, the interpolation error has the form

$$\phi_{nj}(t_i + vh_i) - \psi_{nj}(t_i + vh_i) = \phi_{nj}^{(\mu_1)}(\xi_i) \cdot h_i^{\mu_1} \cdot \prod_{s=1}^{\mu_1} (v - d_s)/\mu_1!,$$

with $\xi_i \in \sigma_i$, and for all $\mu_1 \leqslant m$. An analogous expression holds when $i = n$, with μ_1 replaced by μ_0. According to (5.5), $E_{ni}^{(j)}[u_i]$ is equal to the weighted integral of the above interpolation error, with weight functions as in (5.5); from this, (5.19) follows immediately.

We thus obtain, setting first $i \leqslant n-1$ and using $t_{nj} - t_i \geqslant t_n - t_i$,

$$\|r_{ni}\|_1 = \sum_{j=1}^{m} |E_{ni}^{(j)}[u_i]| \leqslant m \cdot \gamma_1 h_i^{\mu_1} \cdot \int_0^1 \left(\frac{t_n - t_i}{h_i} - v \right)^{-\alpha} dv$$

$$= \frac{m \cdot \gamma_1 h_i^{\mu_1}}{1-\alpha} \cdot h_i^{\alpha-1} \cdot \left\{ (t_n - t_i)^{1-\alpha} - (t_n - t_{i+1})^{1-\alpha} \right\};$$

if $i = n$ we find

$$\|r_{nn}\|_1 \leqslant \frac{m \cdot \gamma_0 h_n^{\mu_0}}{1-\alpha}.$$

It now follows that

$$\sum_{i=0}^{n} h_i^{1-\alpha} \cdot \|r_{ni}\|_1 = h_n^{1-\alpha} \cdot \|r_{nn}\|_1 + \sum_{i=0}^{n-1} h_i^{1-\alpha} \|r_{ni}\|_1$$

$$\leqslant \gamma(\alpha) \left\{ h^{\mu_0+1-\alpha} + h^{\mu_1} \sum_{i=0}^{n-1} \left\{ (t_n - t_i)^{1-\alpha} - (t_n - t_{i+1})^{1-\alpha} \right\} \right\}$$

$$\leqslant \gamma(\alpha) \left\{ h^{\mu_0+1-\alpha} + h^{\mu_1} t_n^{1-\alpha} \right\} \leqslant \gamma(\alpha) \cdot h^{\mu} \cdot \left\{ h^{\mu_0+1-\alpha-\mu} + T^{1-\alpha} \cdot h^{\mu_1-\mu} \right\},$$

with $\gamma(\alpha) := \max(m \cdot \gamma_0/(1-\alpha), m \cdot \gamma_1/(1-\alpha))$, and with μ defined as in Lemma 5.2. Since the factor multiplying h^{μ} is uniformly bounded, we have established (5.18). □

We now return to the Gronwall inequality (5.17): since the mesh diameter h satisfies $h = \mathcal{O}(N^{-1})$ for both types of mesh sequences considered here, (5.17) implies $\|\eta_n\|_1 = \mathcal{O}(N^{-\mu})$, on the basis of the above result. On the other hand, (5.12) leads to

$$|\varepsilon_n(t_n + vh_n)| \leqslant \|L_l\|_\infty \cdot \|\eta_n\|_1 \leqslant \Lambda \cdot \|\eta_n\|_1, \qquad t_n + vh_n \in \sigma_n \ (0 \leqslant n \leqslant N-1),$$

and the result of Theorem 5.1 follows, since the Lebesgue constant Λ corresponding to the (fixed number) m of collocation parameters $\{c_j\}$ is bounded. □

We are now in a position to derive our results on the attainable order of convergence of the approximation $\hat{u} \in S_{m-1}^{(-1)}(Z_N)$ defined by the fully discretized collocation equation (5.6); since quasi-uniform mesh sequences are of no interest in practical applications (recall Theorem 2.1), we shall state only the result for graded meshes. The proof of the following theorem is, of course, a direct consequence of (5.8), Theorem 2.2, and Theorem 5.1.

THEOREM 5.2. *Let g and K in* (1.2) *be m times continuously differentiable on their respective domains I and S, and let $\hat{u} \in S_{m-1}^{(-1)}(Z_N)$ denote the solution of the fully discretized collocation equation* (5.6). *Moreover, assume that the quadrature approximations* (5.3) *used in* (5.6) *correspond to*

$$\mu_0 = \mu_1 = m, \quad d_l = c_l, \quad d_{jl} = c_j c_l \qquad (1 \leqslant j, l \leqslant m).$$

Then for the graded mesh (2.4) *with grading exponent $r = m/(1-\alpha)$ we have*

$$\|y - \hat{u}\|_\infty = \mathcal{O}(N^{-m}). \tag{5.20}$$

It is clear that the above result can be generalized to cover the cases where the quadrature parameters (5.4) are not related to the collocation parameters $\{c_j\}$, and where the grading exponent is given the value $r = \mu/(1-\alpha)$, $1 \leqslant \mu \leqslant m$ (compare Theorem 4.1). By (5.8), Theorem 5.1, and Theorem 4.1, the corresponding results are obvious, and we therefore refrain from stating them explicitly.

6. An Example. Consider the fully discretized collocation equation (5.6) where the quadratures $\hat{\Phi}_{ni}^{(j)}[\hat{u}_i]$ are characterized by $\mu_0 = \mu_1 = m$, $d_l = c_l$, $d_{jl} = c_j c_l$ $(1 \leqslant j, l \leqslant m)$. Setting

$$\hat{Y}_{is} := \hat{u}_i(t_i + c_s h_i) \qquad (1 \leqslant s \leqslant m),$$

and using (5.7), the quadrature approximations in (5.6) assume the form

$$\text{(6.1)} \quad \hat{\Phi}_{ni}^{(j)}[\hat{u}_i] = \begin{cases} \sum_{l=1}^{m} w_{jl}^{(n,i)}(\alpha) \cdot k\big(t_{nj}, t_i + c_l h_i, \hat{Y}_{il}\big) & \text{if } 0 \leqslant i \leqslant n-1, \\ \sum_{l=1}^{m} w_{jl}(\alpha) \cdot k\left(t_{nj}, t_n + c_j c_l h_n, \sum_{s=1}^{m} L_s(c_j c_l) \cdot \hat{Y}_{ns}\right) & \text{if } i = n, \end{cases}$$

with $t_{nj} := t_n + c_j h_n$ $(1 \leqslant j \leqslant m)$. In the expressions (5.5) for the quadrature weights we now have $\lambda_l(v) = L_l(v)$, and hence the above weights become

$$\text{(6.2a)} \qquad w_{jl}^{(n,i)}(\alpha) = \int_0^1 \left(\frac{t_{nj} - t_i}{h_i} - v\right)^{-\alpha} \cdot L_l(v)\, dv,$$

and

$$w_{jl}(\alpha) = \int_0^{c_j} (c_j - v)^{-\alpha} \cdot \prod_{\substack{k=1 \\ k \neq l}}^{m} (v - c_j c_k)/\big(c_j(c_l - c_k)\big)\, dv \qquad (1 \leqslant j, l \leqslant m).$$

This last expression can be simplified by an obvious substitution; we find

$$\text{(6.2b)} \qquad w_{jl}(\alpha) = c_j^{1-\alpha} \cdot \int_0^1 (1-v)^{-\alpha} \cdot L_l(v)\, dv.$$

Thus, according to (5.6) and (6.1), the fully discretized collocation equation,

$$\text{(6.3)} \qquad \hat{Y}_{nj} = \hat{F}_n(\hat{u}; t_{nj}) + h_n^{1-\alpha} \cdot \hat{\Phi}_{nn}^{(j)}[\hat{u}_n] \qquad (1 \leqslant j \leqslant m),$$

with

$$\text{(6.4)} \qquad \hat{F}_n(\hat{u}; t_{nj}) := g(t_{nj}) + \sum_{i=0}^{n-1} h_i^{1-\alpha} \cdot \hat{\Phi}_{ni}^{(j)}[\hat{u}_i] \qquad (0 \leqslant n \leqslant N-1),$$

consititutes, for each n, a system of m nonlinear algebraic equations for $\{\hat{Y}_{n1}, \dots, \hat{Y}_{nm}\}$; once these values have been determined, the approximation $\hat{u}$ on σ_n is given by

$$\text{(6.5)} \qquad \hat{u}_n(t_n + v h_n) = \sum_{j=1}^{m} L_j(v) \cdot \hat{Y}_{nj}.$$

For $m = 2$ (i.e., $\hat{u} \in S_1^{(-1)}(Z_N)$), the quadrature weights (6.2) are:

$$w_{j1}(\alpha) = \frac{c_j^{1-\alpha} \cdot ((2-\alpha)c_2 - 1)}{(1-\alpha)(2-\alpha)(c_2 - c_1)}, \qquad w_{j2}(\alpha) = \frac{c_j^{1-\alpha} \cdot (1 - (2-\alpha)c_1)}{(1-\alpha)(2-\alpha)(c_2 - c_1)};$$

$$w_{j1}^{(n,i)}(\alpha) = \frac{1}{(1-\alpha)(2-\alpha)(c_2 - c_1)} \cdot \left\{ \left(\frac{t_{nj} - t_i}{h_i} \right)^{1-\alpha} \cdot \left((2-\alpha)c_2 - \frac{t_{nj} - t_i}{h_i} \right) - \left(\frac{t_{nj} - t_{i+1}}{h_i} \right)^{1-\alpha} \cdot \left((2-\alpha)(c_2 - 1) - \frac{t_{nj} - t_{i+1}}{h_i} \right) \right\},$$

$$w_{j2}^{(n,i)}(\alpha) = \frac{1}{(1-\alpha)(2-\alpha)(c_2 - c_1)} \cdot \left\{ \left(\frac{t_{nj} - t_i}{h_i} \right)^{1-\alpha} \cdot \left(\frac{t_{nj} - t_i}{h_i} - (2-\alpha)c_1 \right) - \left(\frac{t_{nj} - t_{i+1}}{h_i} \right)^{1-\alpha} \cdot \left(\frac{t_{nj} - t_{i+1}}{h_i} - (2-\alpha)(c_1 - 1) \right) \right\}$$

$$(j = 1, 2; 0 \leqslant i \leqslant n-1).$$

The corresponding fully discretized collocation equation reads

$$\hat{Y}_{nj} = \hat{F}_n(\hat{u}; t_{nj}) + h_n^{1-\alpha} \cdot \left\{ w_{j1}(\alpha) \cdot k\left(t_{nj}, t_n + c_j c_1 h_n, L_1(c_j c_1)\hat{Y}_{n1} + L_2(c_j c_1)\hat{Y}_{n2}\right) + w_{j2}(\alpha) \cdot k\left(t_{nj}, t_n + c_j c_2 h_n, L_1(c_j c_2)\hat{Y}_{n1} + L_2(c_j c_2)\hat{Y}_{n2}\right) \right\}$$

$$(j = 1, 2),$$

with

$$\hat{F}_n(\hat{u}; t_{nj}) = g(t_{nj}) + \sum_{i=0}^{n-1} h_i^{1-\alpha} \cdot \left\{ w_{j1}^{(n,i)}(\alpha) \cdot k(t_{nj}, t_i + c_1 h_i, \hat{Y}_{i1}) + w_{j2}^{(n,i)}(\alpha) \cdot k(t_{nj}, t_i + c_2 h_i, \hat{Y}_{i2}) \right\}$$

$$(0 \leqslant n \leqslant N-1).$$

For the graded mesh $t_n = (n/N)^r \cdot T\ (0 \leqslant n \leqslant N)$, we obtain

$$\|y - \hat{u}\|_\infty = \begin{cases} \mathcal{O}(N^{-2}), & \text{if } r = 2/(1-\alpha), \\ \mathcal{O}(N^{-1}), & \text{if } r = 1/(1-\alpha), \\ \mathcal{O}(N^{-(1-\alpha)}), & \text{if } r = 1 \text{ (uniform mesh)}. \end{cases}$$

Acknowledgment. The author gratefully acknowledges the generous hospitality extended to him by CWI (formerly Mathematisch Centrum) at Amsterdam during a recent visit where part of this work was carried out.

Institut de Mathématiques
Université de Fribourg
CH-1700 Fribourg, Switzerland

1. P. R. BEESACK, "More generalised discrete Gronwall inequalities," Preprint, 1983.

2. C. DE BOOR, "Good approximation by splines with variable knots," in *Spline Functions and Approximation Theory* (A. Meir and A. Sharma, eds.), Birkhäuser-Verlag, Basel, 1973, pp. 57–72.

3. H. BRUNNER, "Nonpolynomial spline collocation for Volterra equations with weakly singular kernels," *SIAM J. Numer. Anal.*, v. 20, 1983, pp. 1106–1119.

4. H. BRUNNER, "The numerical solution of integral equations with weakly singular kernels," in *Numerical Analysis, Dundee* 1983 (D. F. Griffiths, ed.), Lecture Notes in Math., Vol. 1066, Springer-Verlag, Berlin, 1984, pp. 50–71.

5. H. BRUNNER & I. G. GRAHAM, "Product integration for weakly singular Volterra integral equations." (To appear.)

6. H. G. BURCHARD, "On the degree of convergence of piecewise polynomial approximation on optimal meshes. II," *Trans. Amer. Math. Soc.*, v. 234, 1977, pp. 531–559.

7. G. A. CHANDLER, *Superconvergence of Numerical Methods to Second Kind Integral Equations*, Ph. D. Thesis, Australian National University, Canberra, 1979.

8. J. DIXON & S. MCKEE, *Singular Gronwall Inequalities*, Numerical Analysis Report NA/83/44, Hertford College, University of Oxford, 1983.

9. I. G. GRAHAM, *The Numerical Solution of Fredholm Integral Equations of the Second Kind*, Ph. D. Thesis, University of New South Wales, Kensington, 1980.

10. I. G. GRAHAM, "Galerkin methods for second kind integral equations with singularities", *Math. Comp.*, v. 39, 1982, pp. 519–533.

11. I. G. GRAHAM, *Estimates for the Modulus of Smoothness*, Research Report No. 22, Dept. of Mathematics, University of Melbourne, 1982.

12. F. DE HOOG & R. WEISS, "On the solution of a Volterra integral equation with a weakly singular kernel," *SIAM J. Math. Anal.*, v. 4, 1973, pp. 561–573.

13. R. B. KEARFOTT, "A sinc approximation for the indefinite integral," *Math. Comp.*, v. 41, 1983, pp. 559–572.

14. CH. LUBICH, "Runge-Kutta theory for Volterra and Abel integral equations of the second kind," *Math. Comp.*, v. 41, 1983, pp. 87–102.

15. S. MCKEE, "Generalised discrete Gronwall lemmas," *Z. Angew. Math. Mech.*, v. 62, 1982, pp. 429–434.

16. R. K. MILLER and A. FELDSTEIN, "Smoothness of solutions of Volterra integral equations with weakly singular kernels," *SIAM J. Math. Anal.*, v. 2, 1971, pp. 242–258.

17. J. R. RICE, "On the degree of convergence of nonlinear spline approximation," in *Approximation with Special Emphasis on Spline Functions* (I. J. Schoenberg, ed.), Academic Press, New York, 1969, pp. 349–365.

18. H. J. J. TE RIELE, "Collocation methods for weakly singular second-kind Volterra integral equations with non-smooth solution," *IMA J. Numer. Anal.*, v. 2, 1982, pp. 437–449.

19. C. SCHNEIDER, "Produktintegration mit nicht-äquidistanten Stützstellen," *Numer. Math.*, v. 35, 1980, pp. 35–43.

20. C. SCHNEIDER, "Product integration for weakly singular integral equations," *Math. Comp.*, v. 36, 1981, pp. 207–213.

21. L. L. SCHUMAKER, *Spline Functions: Basic Theory*, Wiley, New York, 1981.

22. A. F. TIMAN, *Theory of Approximation of Functions of a Real Variable*, Pergamon Press, Oxford, 1963.

23. P. R. UBA, "The method of piecewise-linear collocation on a nonuniform grid for the solution of integral equations with a singularity," *Uchen. Zap. Tartu. Gos. Univ.*, v. 580, 1981, pp. 52–57. (Russian)

24. G. VAINIKKO, A. PEDAS & P. UBA, *Methods for Solving Weakly Singular Integral Equations*, Tartu. Gos. Univ., Tartu, 1984. (Russian)

25. G. VAINIKKO & P. UBA, "A piecewise polynomial approximation to the solution of an integral equation with weakly singular kernel," *J. Austral. Math. Soc. Ser. B.*, v. 22, 1981, pp. 431–438.

26. R. DEVORE & K. SCHERER, "Variable knot, variable degree spline approximation to x^β," in *Quantitative Approximation* (R. DeVore and K. Scherer, eds), Academic Press, New York, 1980, pp. 121–131.