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Evaluation of Howland-Type Integrals Involving tanh and coth Functions

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Abstract. Four Howland-type integrals involving tanh and coth functions in their integrands are evaluated in this paper. The results are tabulated to 20D.

This paper presents tables of 20D values of the following four integrals of Howland type involving tanh and coth functions:

$$(1)\qquad \begin{aligned} III_k &= \frac{2^k}{k!}\int_0^\infty \frac{x^k \tanh x\,dx}{\sinh 2x \pm 2x}, && (k \geqslant 0),\\ III_k^* &= \frac{2^k}{k!}\int_0^\infty \frac{x^k \tanh x\,dx}{\sinh 2x \pm 2x}, && (k \geqslant 2),\\ IV_k &= \frac{2^k}{k!}\int_0^\infty \frac{x^k \coth x\,dx}{\sinh 2x \pm 2x}, && (k \geqslant 2),\\ IV_k^* &= \frac{2^k}{k!}\int_0^\infty \frac{x^k \coth x\,dx}{\sinh 2x \pm 2x}, && (k \geqslant 4), \end{aligned}$$

which are encountered in mathematical elasticity. These integrals can be evaluated in terms of the following integrals and series:

$$(2)\qquad \begin{aligned} I_k &= \frac{2^k}{k!}\int_0^\infty \frac{x^k\,dx}{\sinh 2x \pm 2x}, && (k \geqslant 1),\\ I_k^* &= \frac{2^k}{k!}\int_0^\infty \frac{x^k\,dx}{\sinh 2x \pm 2x}, && (k \geqslant 3),\\ II_k &= \frac{2^k}{k!}\int_0^\infty \frac{x^k e^{-2x}\,dx}{\sinh 2x \pm 2x}, && (k \geqslant 1),\\ II_k^* &= \frac{2^k}{k!}\int_0^\infty \frac{x^k e^{-2x}\,dx}{\sinh 2x \pm 2x}, && (k \geqslant 3),\\ S_k &= \sum_{n=1}^\infty \frac{1}{n^k}, && (k \geqslant 2),\\ U_k &= \sum_{n=1}^\infty \frac{1}{(2n-1)^k}, && (k \geqslant 2),\\ s_k &= \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^k}, && (k \geqslant 1), \end{aligned}$$

where the four integrals are ordinary Howland integrals. Their relations are

$$(3)\qquad \begin{aligned} III_k &= I_k + (I_{k-1} - II_{k-1} - s_k)/k, && (k \geqslant 2),\\ III_k^* &= I_k^* - (I_{k-1}^* - II_{k-1}^* - s_k)/k, && (k \geqslant 4),\\ IV_k &= I_k - (I_{k-1} + II_{k-1} - S_k)/k, && (k \geqslant 2),\\ IV_k^* &= I_k^* + (I_{k-1}^* + II_{k-1}^* - S_k)/k, && (k \geqslant 4). \end{aligned}$$

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The integrals III_0, III_1, III_2^* and III_3^* are not covered by (3), but they can be separately evaluated as

$$
\begin{aligned}
III_0 &= \sum_{n=1}^{\infty} \frac{1}{2n}(U_{2n} - I_{2n-1}),\\
III_1 &= \sum_{n=1}^{\infty} (U_{2n+1} - I_{2n}),\\
III_2^* &= \sum_{n=1}^{\infty} \frac{2n+1}{2}(I_{2n+1}^* - U_{2n+2}),\\
III_3^* &= \sum_{n=1}^{\infty} \frac{(2n+1)(2n+2)}{6}(I_{2n+2}^* - U_{2n+3}).
\end{aligned} \tag{4}
$$

The following numerical tables are available: (i) 32D values of S_k, U_k and s_k [1], (ii) 25D values of I_k and I_k^* [3], [4], and (iii) 20D values of II_k and II_k^* [5]. With these tables of high precision, the integrals in (3) can be computed to 20D and those in (4) to 21D. The following formulas may be used for checking purposes:

$$
\begin{aligned}
&\sum_{k=1}^{\infty} \frac{1}{2}(III_{2k+1}^* - III_{2k+1}) = III_1,\\
&\sum_{k=2}^{\infty} \frac{k}{2}(III_{2k}^* - III_{2k}) = III_2,\\
&\sum_{k=2}^{\infty} \frac{k}{2}(IV_{2k}^* - IV_{2k}) = IV_2,\\
&\sum_{k=2}^{\infty} \frac{k(2k+1)}{6}(IV_{2k+1}^* - IV_{2k+1}) = IV_3.
\end{aligned} \tag{5}
$$

Or

$$
\begin{aligned}
&2\,III_1 + \sum_{k=2}^{\infty} (-1)^k k(1 - III_k) = \frac{3}{4} + I_1 - II_1,\\
&2\,III_3^* - \sum_{k=4}^{\infty} (-1)^k \binom{k}{3}(III_k^* - 1) = \frac{15}{16} + I_3^* - II_3^*,\\
&\sum_{k=2}^{\infty} (-1)^k k(1 - IV_k) = \frac{3}{4} - I_1 - II_1,\\
&\sum_{k=4}^{\infty} (-1)^k \binom{k}{3}(IV_k^* - 1) = I_3^* + II_3^* - \frac{15}{16}.
\end{aligned} \tag{6}
$$

It is noted that the evaluation of the four integrals in question was considered before by the author. The reader may consult the author's previous paper [2] for derivations of the relations in (3) and the first two relations in (4). The other two relations in (4) are new. Also, the second set of checking formulas as in (6) is new. The results are shown in the accompanying tables to 20D for k up to 60. By developing the integrands of the integrals into series of $\exp(-2x)$, we find

$$
\begin{aligned}
\frac{III_k}{IV_k^*} &= 1 \mp \frac{k+3}{2^{k+1}} + \frac{4k^2 + 24k + 47}{3^{k+3}} \mp \cdots,\\
\frac{IV_k}{III_k^*} &= 1 \mp \frac{k-1}{2^{k+1}} + \frac{4k^2 + 23}{3^{k+3}} \mp \cdots.
\end{aligned} \tag{7}
$$

TABLE 1
Values of III_k and III_k^*

k	III_k	III_k^*
0	0.29661 99339 44595 97112	-
1	0.47442 96568 44400 38418	-
2	0.63084 14007 56394 71239	2.13561 79285 77824 94912
3	0.75383 61380 88364 70967	1.41506 33609 72854 32434
4	0.84283 92075 97609 05654	1.19555 30856 62227 22928
5	0.90323 68041 38941 72530	1.10049 26035 57012 36691
6	0.94219 18007 41668 47549	1.05357 96662 57560 48355
7	0.96631 46744 35813 14361	1.02903 43071 42502 10827
8	0.98077 05523 77242 24405	1.01583 42008 66031 23633
9	0.98920 65773 86084 21391	1.00864 67795 84489 60628
10	0.99402 49922 06105 00479	1.00471 53780 14052 97031
11	0.99672 96737 31447 25230	1.00256 42740 49514 14013
12	0.99822 66146 53647 40993	1.00138 96013 41781 18900
13	0.99904 56852 39138 46194	1.00075 01799 98261 39463
14	0.99948 96912 34774 36906	1.00040 34288 02376 36775
15	0.99972 85510 46781 31800	1.00021 61385 29685 15785
16	0.99985 62445 06325 07098	1.00011 53807 26827 69980
17	0.99992 41536 98507 71646	1.00006 13846 38756 86013
18	0.99996 01111 69596 89982	1.00003 25543 06883 72782
19	0.99997 90799 94077 83888	1.00001 72138 31286 34127
20	0.99998 90549 89633 34142	1.00000 90774 16519 06784
21	0.99999 42860 51251 56941	1.00000 47747 76000 10268
22	0.99999 70227 03086 80599	1.00000 25057 29053 29209
23	0.99999 84513 39381 89658	1.00000 13121 49875 40531
24	0.99999 91957 18898 51663	1.00000 06857 60310 13140
25	0.99999 95829 04425 04331	1.00000 03577 38642 20147
26	0.99999 97839 82165 46273	1.00000 01863 04385 46698
27	0.99999 98882 58781 31913	1.00000 00968 71860 10514
28	0.99999 99422 64228 25313	1.00000 00502 96543 86347
29	0.99999 99701 99890 33980	1.00000 00260 78862 81209
30	0.99999 99846 33962 08112	1.00000 00135 04829 85857
31	0.99999 99920 84035 76641	1.00000 00069 85151 40249
32	0.99999 99959 25561 81941	1.00000 00036 08957 36474
33	0.99999 99979 04558 79829	1.00000 00018 62674 34056
34	0.99999 99989 23170 45323	1.00000 00009 60436 72970
35	0.99999 99994 47031 50379	1.00000 00004 94768 76335
36	0.99999 99997 16239 15663	1.00000 00002 54659 80169
37	0.99999 99998 54481 37493	1.00000 00001 30967 68945
38	0.99999 99999 25421 61908	1.00000 00000 67302 76695
39	0.99999 99999 61801 28710	1.00000 00000 34560 85449
40	0.99999 99999 80445 88647	1.00000 00000 17735 16625
41	0.99999 99999 89995 56615	1.00000 00000 09094 95387
42	0.99999 99999 94884 09505	1.00000 00000 04661 16274
43	0.99999 99999 97385 20369	1.00000 00000 02387 42443
44	0.99999 99999 98664 17999	1.00000 00000 01222 13380
45	0.99999 99999 99317 87909	1.00000 00000 00625 27771
46	0.99999 99999 99651 83410	1.00000 00000 00319 74427
47	0.99999 99999 99822 36433	1.00000 00000 00163 42484
48	0.99999 99999 99909 40581	1.00000 00000 00083 48878
49	0.99999 99999 99953 81472	1.00000 00000 00042 63257
50	0.99999 99999 99976 46327	1.00000 00000 00021 76037
51	0.99999 99999 99988 00959	1.00000 00000 00011 10223
52	0.99999 99999 99993 89377	1.00000 00000 00005 66214
53	0.99999 99999 99996 89138	1.00000 00000 00002 88658
54	0.99999 99999 99998 41793	1.00000 00000 00001 47105
55	0.99999 99999 99999 19509	1.00000 00000 00000 74940
56	0.99999 99999 99999 59061	1.00000 00000 00000 38164
57	0.99999 99999 99999 79183	1.00000 00000 00000 19429
58	0.99999 99999 99999 89418	1.00000 00000 00000 09888
59	0.99999 99999 99999 94622	1.00000 00000 00000 05031
60	0.99999 99999 99999 97268	1.00000 00000 00000 02559

TABLE 2

Values of IV_k and IV_k^*

k	IV_k	IV_k^*
2	1.09596 74127 81010 85878	-
3	0.94313 77055 08848 67791	-
4	0.93632 30754 57122 67709	1.70756 94031 43097 55236
5	0.95164 35866 53520 53556	1.24000 07445 07150 12412
6	0.96751 54936 93683 27020	1.10538 54294 02360 29909
7	0.97953 75278 14738 14097	1.05082 23685 77313 70040
8	0.98763 45988 74063 43243	1.02560 82341 55080 92253
9	0.99274 59473 93955 13798	1.01319 92806 69181 95449
10	0.99583 85646 54282 37673	1.00688 59990 48109 07772
11	0.99765 38966 76473 56783	1.00361 50333 53774 75744
12	0.99869 54997 30458 46138	1.00190 34159 95424 91800
13	0.99928 27114 09536 83414	1.00100 31566 44122 71579
14	0.99960 91749 46440 43910	1.00052 85675 09881 64186
15	0.99978 86529 35058 24499	1.00027 82413 68014 66299
16	0.99988 64275 65132 83220	1.00014 62701 91356 62952
17	0.99993 92932 65386 26683	1.00007 67729 99299 90470
18	0.99996 76982 83520 95590	1.00004 02288 37576 83404
19	0.99998 28797 61580 86957	1.00002 10442 43279 59493
20	0.99999 09570 86200 06940	1.00001 09902 32217 83127
21	0.99999 52378 84050 98006	1.00000 57303 50517 17140
22	0.99999 74988 96428 71618	1.00000 29832 25992 41811
23	0.99999 86895 33451 09555	1.00000 15507 97293 91173
24	0.99999 93148 50075 46958	1.00000 08050 48882 93785
25	0.99999 96424 81947 61774	1.00000 04173 70724 82992
26	0.99999 98137 75087 63794	1.00000 02161 16192 95492
27	0.99999 99031 56689 86685	1.00000 01117 76296 71740
28	0.99999 99497 13685 05045	1.00000 00577 48254 35475
29	0.99999 99739 24792 88887	1.00000 00298 04542 48757
30	0.99999 99864 96473 60984	1.00000 00153 67609 06208
31	0.99999 99930 15312 34737	1.00000 00079 16520 08614
32	0.99999 99963 91207 29193	1.00000 00040 74634 49746
33	0.99999 99981 37384 00941	1.00000 00020 95510 42487
34	0.99999 99990 39583 91064	1.00000 00010 76853 91810
35	0.99999 99995 05238 52539	1.00000 00005 52977 06414
36	0.99999 99997 45342 76804	1.00000 00002 83763 85135
37	0.99999 99998 69033 21516	1.00000 00001 45519 67971
38	0.99999 99999 32697 55104	1.00000 00000 74578 75023
39	0.99999 99999 65439 25714	1.00000 00000 38198 84207
40	0.99999 99999 82264 87288	1.00000 00000 19554 15865
41	0.99999 99999 90905 05983	1.00000 00000 10004 44959
42	0.99999 99999 95338 84205	1.00000 00000 05115 91044
43	0.99999 99999 97612 57724	1.00000 00000 02614 79822
44	0.99999 99999 98777 86679	1.00000 00000 01335 82068
45	0.99999 99999 99374 72249	1.00000 00000 00682 12114
46	0.99999 99999 99680 25580	1.00000 00000 00348 16598
47	0.99999 99999 99836 57518	1.00000 00000 00177 63570
48	0.99999 99999 99916 51123	1.00000 00000 00090 59420
49	0.99999 99999 99957 36744	1.00000 00000 00046 18528
50	0.99999 99999 99978 23963	1.00000 00000 00023 53673
51	0.99999 99999 99988 89777	1.00000 00000 00011 99041
52	0.99999 99999 99994 33786	1.00000 00000 00006 10623
53	0.99999 99999 99997 11342	1.00000 00000 00003 10862
54	0.99999 99999 99998 52895	1.00000 00000 00001 58207
55	0.99999 99999 99999 25060	1.00000 00000 00000 80491
56	0.99999 99999 99999 61836	1.00000 00000 00000 40939
57	0.99999 99999 99999 80571	1.00000 00000 00000 20817
58	0.99999 99999 99999 90112	1.00000 00000 00000 10582
59	0.99999 99999 99999 94969	1.00000 00000 00000 05378
60	0.99999 99999 99999 97441	1.00000 00000 00000 02732

These series are useful in evaluation when k is large. The first two terms give 20D values when $k \geqslant 50$, 15D values when $k \geqslant 39$, or 10D values when $k \geqslant 28$. If the first three terms are used, they give 20D, 15D, or 10D values when $k \geqslant 41$, 28, or 22, respectively.

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