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Breeding Amicable Numbers in Abundance

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Abstract. We give some new methods for the constructive search for amicable number pairs. Our numerical experiments using these methods produced a total of 3501 new amicable pairs of a very special form. They provide some experimental evidence for the infinity of such pairs.

1. Amicable Numbers of Euler's First Form.

1.1. *Historical Remarks on Thabit ibn Kurrah's Formula.* Two natural numbers A, B are called *amicable* if each of them is the sum of all proper divisors of the other one. Equivalently, this means that

$$\sigma(A) = A + B = \sigma(B), \tag{1}$$

where $\sigma(m)$ denotes the sum of all divisors of m. The famous rule of Thabit ibn Kurrah (†Bagdad 801 A.D.) states that

$$A = 2^n \cdot r_1 \cdot r_2, \quad \text{and} \quad B = 2^n \cdot s \tag{2}$$

are *amicable* numbers, in the case that

$$\begin{aligned} r_1 &= 3 \cdot 2^{n-1} - 1, \quad r_2 = 3 \cdot 2^n - 1, \quad \text{and} \\ s &= (r_1 + 1)(r_2 + 1) - 1 = 9 \cdot 2^{2n-1} - 1 \end{aligned} \tag{3}$$

are *prime* numbers. For example, $n = 2$ gives the amicable pair $A = 2^2 \cdot 5 \cdot 11$, $B = 2^2 \cdot 71$ attributed to the legendary Pythagoras (500 B.C.) by Iamblichos (300 A.D.). Two further examples are obtained for $n = 4$, resp. $n = 7$, as discovered in the early 14th century by Ibn al-Bannā' in Marakesh, and also by Kamaladdin Fārisī in Bagdad, according to [12], [1], [11], resp. in the 17th century, by Muhammad Bāqir Yazdī in Iran, see [1], [11]. Note that until recently, these first examples of amicable numbers, and also Thabit's rule, had been attributed to Fermat (1636), and Descartes (1638). So much for our little updating of the early history of the subject (cf., [2], [5]).

1.2. *A New Analogue of Thabit's Formula.* Let us now state an analogue of Thabit's theorem (1.1) which seems to be new:

THEOREM. *Let n be a positive integer, and choose β, $0 < \beta < n$, such that with $g = 2^{n-\beta} + 1$, the number*

$$r_1 = 2^\beta \cdot g - 1 \tag{4}$$

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is prime. Now choose α, $0 < \alpha < n$, *such that*

$$p = 2^\alpha + (2^{n+1} - 1)g, \tag{5}$$

$$r_2 = 2^{n-\alpha}gp - 1, \quad \textit{and} \tag{6}$$

$$s = (r_1 + 1)(r_2 + 1) - 1 = 2^{n-\alpha+\beta} \cdot g^2 p - 1 \tag{7}$$

are also prime. Then

$$A = 2^n p r_1 r_2 \quad \textit{and} \quad B = 2^n ps \tag{8}$$

are amicable numbers.

In fact, by the assumptions of the theorem, the reader may verify (1) by straightforward calculations. For example, $n = 2$ and $\alpha = \beta = 1$ give the amicable pair $A = 2^2 \cdot 23 \cdot 5 \cdot 137$ and $B = 2^2 \cdot 23 \cdot 827$, which was discovered by L. Euler without using such explicit formulae. To avoid confusion, let us mention at this point that another analogue of Thabit's rule was introduced in [3], and further investigated in [13], [5]. Note that the "Thabit rules" of [3], to which we shall come back briefly in Subsection 2.5 of the present paper, are of a quite different nature than the above theorem.

1.3. *Euler's Generalization of Thabit's Formula.* Thabit's explicit formulae (3) have been generalized by L. Euler to exhaust *all* amicable pairs of the form (2) with primes r_1, r_2, s as follows: If for $g = 2^{n-\beta} + 1$ with some β, $0 < \beta < n$, the numbers

$$\begin{aligned} r_1 &= 2^\beta \cdot g - 1, \quad r_2 = 2^n \cdot g - 1, \quad \text{and} \\ s &= (r_1 + 1)(r_2 + 1) - 1 = 2^{n+\beta} \cdot g^2 - 1 \end{aligned} \tag{3'}$$

are prime, then the numbers (2) are amicable, and this gives all amicable pairs of the form (2) (with primes r_1, r_2, s). For example, $\beta = n - 1$ gives Thabit's rule. Euler's generalization gives further amicable numbers for $n = 8$, $\beta = 1$, resp. $n = 40$, $\beta = 29$, as discovered by Legendre and Chebyshev, resp. by te Riele [13].

1.4. *A New Analogue of Euler's Formula.* Let us now state an analogue of Euler's exhaustive formulae (1.3), dealing with amicable numbers of the form (8), which seems to be new.

THEOREM. *Let* n, γ *be positive integers with* $0 < \gamma < n$. *Put either*

$$\text{(i)} \quad C = 2^n + 2^\gamma, \qquad \textit{or} \qquad \text{(ii)} \quad C = 2^n(2^{n+1} - 1) + 2^\gamma.$$

Take any factorization of $C = fD$ *into two positive factors* f, D. *Whenever the four numbers* p, r_1, r_2, s *given below in* (9)–(11) *are primes, then*

$$A = 2^n \cdot p \cdot r_1 r_2 \quad \textit{and} \quad B = 2^n \cdot p \cdot s \tag{8}$$

are amicable; and all amicable pairs (8) *with* p, r_1, r_2, s *odd primes, are obtained in this way. Here*,

$$p = D + 2^{n+1} - 1, \tag{9}$$

$$\begin{aligned} r_1 &= pf - 1, \qquad r_2 = (r_1 + 1) \cdot 2^{n-\gamma} - 1 \quad \textit{in case (i), resp.} \\ r_1 &= 2^n + f - 1, \qquad r_2 = p(2^n + f) \cdot 2^{n-\gamma} - 1 \quad \textit{in case (ii), and} \end{aligned} \tag{10}$$

$$s = (r_1 + 1)(r_2 + 1) - 1. \tag{11}$$

For example, $n = 8$, $\gamma = 7$, and $f = 2^3 \cdot 31$ give E. J. Lee's amicable pair [9]

$$A = 2^8 \cdot 1039 \cdot 503 \cdot 1\,047\,311, \qquad B = 2^8 \cdot 1039 \cdot 527\,845\,247.$$

If we choose for the factor f a power of two in case (ii), then we obtain Theorem 1.2 as a special case of the present Theorem 1.4. On the other hand, Theorem 1.4 will be seen to be itself a special case of an even more general result (Theorem 1.6 below), and this is the way to prove it.

Remark. As pointed out by the referee, we may allow in case (ii) also $n < \gamma < 2n$, provided that the number f has sufficiently many factors 2 to make $f \cdot 2^{n-\gamma}$ integral. For example, $n = 7$, $\gamma = 9$, and $f = 2^4 \cdot 7 \cdot 37$ give P. Poulet's amicable pair

$$A = 2^7 \cdot 263 \cdot 4271 \cdot 280\,883, \qquad B = 2^7 \cdot 263 \cdot 1\,199\,936\,447.$$

1.5. *Euler's Search Procedure for Amicable Pairs of the Simplest Form.* Let us now consider amicable numbers of the form

$$A = a \cdot r_1 r_2, \qquad B = a \cdot s \tag{12}$$

with primes r_1, r_2, s not dividing a. Here we allow for the common divisor a now an arbitrary number, in place of the 2-power 2^n in 1.3. As noted by Euler, for purposes of algebraic discussions, this is the simplest form the prime decomposition of an amicable pair may have, and therefore it is sometimes referred to as "*Euler's first form*" [7]. Although Euler was able to find an additional 13 of them using several clever methods, such pairs later turned out to be relatively rare. On the other hand, they also turned out to be particularly useful, as inputs for certain methods to construct further amicable numbers of different forms, as will be discussed later (2.5).

So for various good reasons, amicable pairs of Euler's first form (12) have been investigated much more extensively, and more systematically, than those of any other form. For a list of investigators and references, we refer to Table 1 below.

TABLE 1

Documentation of the 98 *amicable pairs of Euler's first type presently known to us.*

year	authors	number of new pairs	reference
≈ -500	Pythagoras	1	[11]
≈ 1300	al-Bannā' (Fermat 1636)	1	[12]
≈ 1600	Yazdi (Descartes 1638)	1	[1], [11]
≈ 1750	Euler	13	[10]
1830	Legendre/Chebychev	1	[10]
1884	Seelhoff	1	[10]
1921	Mason	1	[10]
1929	Poulet/Gerardin	4	[10]
1946	Escott	8	[10]
1957	Garcia	8	[10]
1968	Lee	3	[9] or [10]
1974	te Riele	1	[13]
1978	Costello	7	[7]
1979	Borho, Hoffmann Nebgen, Reckow	18	[5]
1984	Borho/Hoffmann	30	Table 2 of this note

In order to find all amicable pairs of the form (12) for a specified numerical value of a, one may proceed as follows: Take any factorization of a^2 into two factors, $a^2 = d_1 d_2$; whenever the three numbers

$$(13) \qquad r_i = (d_i + \sigma(a) - a)/(2a - \sigma(a)) \quad \text{for } i = 1, 2, \quad \text{and}$$

$$(14) \qquad s = (r_1 + 1)(r_2 + 1) - 1$$

are different prime numbers not dividing a, then $A = ar_1r_2$, $B = as$ is an amicable pair; and all such pairs are obtained in this way. For more details of this method, which essentially goes back to Euler, we refer to our paper [5], where we used such an "*Euler search*" to compute the complete list of (exactly 60, as it turns out) amicable pairs of type (12) with $a \leqslant 10^7$. Note that the above statements, applied to the case of a 2-power $a = 2^n$, readily yield Euler's generalization (1.3) of Thabit's explicit formulae.

1.6. *A New Analogue of "Euler's Search"*. Let us now state another method of search for amicable numbers of Euler's first type, which is a more sophisticated analogue of the "Euler search" (1.5), and seems to be new. The idea is to assume that the common factor a contains a simple prime factor p, which is another unknown, to be determined along with r_1, r_2, and s, after specification of the cofactor $b = a/p$.

THEOREM. *Given a natural number b, take first any factorization $b^2 = d_1 d_2$ of b^2 into two factors d_1, d_2, and take secondly any factorization $C = fD$ of the number C below into two factors f, D; here C is either*

$$(15) \qquad \begin{array}{ll} \text{(i)} & C := b + d_1, \\ \textit{or}\ \text{(ii)} & C := b\sigma(b) + d_1 \cdot (2b - \sigma(b)). \end{array}$$

If the four numbers p, r_1, r_2, s, where

$$(16) \qquad p := (D + \sigma(b))/(2b - \sigma(b)),$$

$$(17\text{i}) \qquad \textit{and} \quad \begin{array}{l} r_1 := pf - 1, \\ r_2 := b(r_1 + 1)/d_1 - 1 \end{array} \qquad \textit{in case (i)},$$

$$(17\text{ii}) \qquad \textit{resp.} \quad \begin{array}{l} r_1 := (\sigma(b) - b + f)/(2b - \sigma(b)), \\ r_2 := bp(r_1 + 1)/d_1 - 1 \end{array} \qquad \textit{in case (ii)},$$

$$(18) \qquad \textit{and} \quad s := (r_1 + 1)(r_2 + 1) - 1,$$

are all prime, pairwise different, and prime to b, then

$$(19) \qquad A = bpr_1r_2, \qquad B = bps$$

are amicable, and all such amicable pairs are obtained by the above formulae.

This theorem can be proved by applying 1.5 to the case where $a = bp$ with p prime, and not dividing b. We leave the verification of the details to the reader. Putting $b = 2^n$ we obtain Theorem 1.4 as an immediate corollary of the more general theorem above.

By an extensive computer search based on this theorem, we found that in the range $b \leqslant 10^6$, and with the largest prime $s \leqslant 10^{14}$, there are exactly 86 amicable pairs of the form (19) considered in the theorem, 56 of which had been known before. The 30 new pairs are listed in Table 2.

TABLE 2

The 30 *new amicable pairs of Euler's first form* $A = ar_1r_2$, $B = as$, *found by means of Theorem* 1.6.

no.	$a = b \cdot p$	$r_1 \cdot r_2$	s
1	$3^4 \cdot 5 \cdot 11^3 \cdot 83$	$331 \cdot 659$	2 19119
2	$3^3 \cdot 5 \cdot 11 \cdot 103 \cdot 109$	$41 \cdot 4\,71533$	198 04427
3	$3^4 \cdot 5^2 \cdot 13 \cdot 769$	$389 \cdot 1\,24577$	485 85419
4	$3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 103 \cdot 109$	$41 \cdot 4\,71533$	198 04427
5	$3^2 \cdot 5^3 \cdot 11 \cdot 16349$	$29 \cdot 10\,79033$	323 71019
6	$3^5 \cdot 5 \cdot 13 \cdot 37 \cdot 2663$	$89 \cdot 47933$	43 14059
7	$3^3 \cdot 5^2 \cdot 19 \cdot 37 \cdot 4079$	$73 \cdot 73421$	54 33227
8	$3^6 \cdot 7 \cdot 11 \cdot 17 \cdot 101$	$857 \cdot 1\,65437$	1419 45803
9	$2^6 \cdot 211 \cdot 4219$	$173 \cdot 14\,68211$	2554 68887
10	$2^6 \cdot 587 \cdot 5869$	$83 \cdot 9\,85991$	828 23327
11	$3^3 \cdot 5 \cdot 13 \cdot 23 \cdot 1103$	$2417 \cdot 2\,05157$	4960 72043
12	$2^3 \cdot 17^2 \cdot 307 \cdot 1877$	$1733 \cdot 11261$	195 28307
13	$3 \cdot 5^2 \cdot 7 \cdot 23^2 \cdot 83$	$1049 \cdot 18\,44093$	19362 98699
14	$3^4 \cdot 7^2 \cdot 11 \cdot 19 \cdot 461$	$5531 \cdot 38723$	2142 21167
15	$3^3 \cdot 5^2 \cdot 19 \cdot 31 \cdot 521$	$15629 \cdot 2\,96969$	46416 41099
16	$2^7 \cdot 349 \cdot 27919$	$491 \cdot 68\,68073$	33790 92407
17	$3^2 \cdot 5 \cdot 7 \cdot 113 \cdot 1321$	$3\,79679 \cdot 6\,34079$	24 07474 94399
18	$3^3 \cdot 5 \cdot 11^2 \cdot 43 \cdot 5689$	$8513 \cdot 11\,26421$	95903 56907
19	$3^2 \cdot 5^2 \cdot 11 \cdot 43 \cdot 8599$	$2579 \cdot 221\,85419$	5 72383 83599
20	$2^8 \cdot 1259 \cdot 3\,37411$	$431 \cdot 182\,20193$	78711 23807
21	$2^8 \cdot 599 \cdot 3709$	$28751 \cdot 533\,20583$	153 30734 31167
22	$2^7 \cdot 347 \cdot 971$	$72869 \cdot 2830\,27079$	2062 41833 19599
23	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 521 \cdot 2083$	$2459 \cdot 3945\,61859$	97 06221 75599
24	$3 \cdot 5 \cdot 7 \cdot 13 \cdot 67 \cdot 1499$	$22511 \cdot 4386\,91343$	987 58195 36127
25	$3^2 \cdot 5 \cdot 7 \cdot 107 \cdot 3851$	$67409 \cdot 2595\,95909$	1749 93602 93099
26	$2^7 \cdot 263 \cdot 75743$	$4733 \cdot 3585\,67361$	169 74578 91707
27	$2^9 \cdot 1039 \cdot 2\,07797$	$49871 \cdot 24\,93563$	12 43590 23807
28	$3^3 \cdot 5 \cdot 11^2 \cdot 43 \cdot 4289$	$2\,79413 \cdot 278\,69921$	778 72463 85707
29	$2^4 \cdot 37 \cdot 227 \cdot 79549$	$743 \cdot 1\,34348\,71511$	999 55444 04927
30	$3^3 \cdot 5 \cdot 11^2 \cdot 43 \cdot 31583$	$459 \cdot 3\,67493\,47139$	9040 33939 64399

2. Advice for Constructing "Breeders", and How to "Breed" Amicable Numbers from Them.

2.1. *Lee's BDE Method* [9]. Given natural numbers a_1, a_2 one may determine all amicable pairs of the form

$$A = a_1 q, \qquad B = a_2 s_1 s_2, \tag{20}$$

where q, resp. s_1, s_2 ($s_1 \neq s_2$), are primes not dividing a_1, resp. a_2, by solving a bilinear Diophantine equation (BDE) on s_1, s_2 as follows: Take any factorization of the number

$$(F + D)F + DG = d_1 d_2 \tag{21}$$

into two different natural factors d_1, d_2, where

$$\begin{aligned} F &:= (\sigma(a_1) - a_1)\sigma(a_2), \qquad G := a_1\sigma(a_1), \\ D &:= a_2\sigma(a_1) + a_1\sigma(a_2) - \sigma(a_2)\sigma(a_1). \end{aligned} \tag{22}$$

If then, for $i = 1, 2$,

$$s_i = (d_i + F)/D \tag{23}$$

are integer, prime, and prime to a_2, and if also

$$(24)\qquad q = \sigma(a_2)\sigma(a_1)^{-1}(s_1 + 1)(s_2 + 1) - 1$$

is prime and prime to a_1, then we have an amicable pair (20), and this procedure gives all such pairs. The idea of this method goes essentially back to Euler, who formulated—and extensively used—it in several special cases, as did many authors later on. In the present general form, the method seems to be explicitly formulated for the first time in E. J. Lee's paper [9]. For example, the "Euler search" procedure stated in 1.5 is obtained again as the special case $a_1 = a_2 = a$.

2.2. *te Riele's Trick*: *Daughter Pairs From Mother Pairs*. Most of the amicable numbers currently known have been found by use of some version of the BDE method. A successful search requires two essential ingredients: 1. clever choices for the input numbers a_1, a_2, and 2. clever handling of a large amount of primality testing. Before powerful primality tests and adequate computing facilities became available, the second point—for a long time—put a severe restriction on the numerical use of the BDE method, because this method tends to lead too soon to too large primes. Since this restriction has been sufficiently removed, the investigators could focus attention on the first point, on the clever choice of inputs. Recently, te Riele [14] has discovered the remarkable efficiency of the following trick: Take the inputs a_1, a_2 from the lists of *already known* amicable pairs (A_1, A_2) by splitting both numbers A_i into $A_i = a_i v_i$ $(i = 1, 2)$, where v_i is either 1, or a large simple prime factor of A_i. From a list of 1592 known ("mother") pairs, te Riele computed in [14] in this manner 2324 new ("daughter" and "granddaughter") pairs. A substantial portion of these were produced by the special case of his trick (cf. [15, Lemma 1]), where the mother pair (A_1, A_2) is of Euler's first form (ar_1r_2, as), and $v_1 = 1$, $v_2 = s$. For example, after sending te Riele our 18 new pairs of Euler's first form found in [5], he almost immediately returned to us a list of 455 new amicable pairs derived from these 18 "mother pairs" as "daughter pairs" in this way. Let us restrict our attention here only to this special case, which is particularly nice and particularly productive for two reasons: First, this particular choice of inputs (a_1, a_2) allows us to eliminate the divisions in Eqs. (23) of the BDE-method, so that the values for s_1, s_2 are automatically integer, and second, it guarantees a particularly highly factorizable number in Eq. (21), so that an abundance of values for s_1, s_2, q enter the primality testing, and it becomes likely that at least a few of them successfully pass it.

2.3. *Amicable Breeders*. To improve te Riele's trick further, we take as inputs (a_1, a_2) for the BDE method not only data from amicable ("mother") pairs, because these are still relatively rare, but we allow more general inputs (a_1, a_2), called "breeder" pairs, which are not that rare.

Definition. A pair of positive integers a_1, a_2 is called a "breeder", if the equations

$$(25)\qquad a_1 + a_2 x = \sigma(a_1) = \sigma(a_2)(x + 1)$$

have a positive integer solution x.

The idea is that "breeders" may be used to "breed" amicable "daughter pairs" in the same way as te Riele uses his data from amicable pairs, the only difference being that there might not be any "mother pair" now. Taking a breeder (a_1, a_2) as input in the BDE-method 2.1 will produce "daughter pairs" of the form $(a_1 q,\ a_2 s_1 s_2)$ as discussed there (for example). Only if the solution x of (25) happens to be a prime not dividing a_2, then (a_1, a_2x) is an amicable pair, and is a "mother pair", in te Riele's terms, of our daughter pairs. So we may say that the "mother pair", if it exists, is breeded from our "breeder" in 0th generation.

Our point is that "breeders" may not only serve as clever *inputs* for the BDE method, but that they may also—on the other hand—be produced as *outputs* of the BDE method. Therefore, from one breeder whole generations of other breeders may be "bred" by the BDE method, and this process may then be used to "breed" an abundance of new amicable pairs.

In fact, to construct breeders (b_1, b_2) by the BDE method, one simply proceeds as follows: Starting from any input (a_1, a_2) as explained in 2.1, just check that (23) gives two different primes s_1, s_2 prime to a_2 (but ignore (24) now). Then $b_1 = a_2s_1s_2$, $b_2 = a_1$ is already a breeder. Alternatively, one may check whether (24) gives a prime q not dividing a_1, and (23) gives at least one prime, s_1 say, not dividing a_2; if so, then $b_1 = a_1q$, $b_2 = a_2s_1$ will be a breeder. Many variations are possible.

2.4. *More General Breeders.* More generally, we may define a "*breeder of type* (i, j)" to be a pair of positive integers (b_1, b_2) such that the equations

$$(26)\quad b_1x_1x_2 \cdots x_i + b_2y_1y_2 \cdots y_j = \sigma(b_1)\prod_{\nu=1}^{i}(x_\nu + 1) = \sigma(b_2)\prod_{\mu=1}^{j}(y_\mu + 1)$$

have a solution in positive integers $x_1, \ldots, x_i$, $y_1, \ldots, y_j$. Note that "breeders" in the sense of 2.3 are "breeders of type $(0,1)$" in the present sense. Although we shall discuss in the sequel only these in more detail, let us point out here that breeders of type $(1,1)$, e.g., are of similar interest.* Note that "breeders of type $(0,0)$" are just amicable pairs (and that type $(1,0)$ is of course equivalent to type $(0,1)$). Equation (26) says that if the positive integers $x_1, \ldots, x_i$, $y_1, \ldots, y_j$ happen to be all prime, pairwise different, and not divisors of b_1, resp. b_2, then $(b_1x_1 \cdots x_i, b_2y_1 \cdots y_j)$ will be an amicable pair. In the special case where $b_1 = b_2 =: a$, such an amicable pair is called *regular*, *of Euler type* (i, j). In this terminology, the amicable pairs considered in 1.5 are the regular amicable pairs of Euler type $(2,1)$. They are called "of Euler's first form" because $(2,1)$ are the smallest possible values for (i, j) in this situation, as noted by Euler.

2.5. *More Special Breeders, and Thabit Rules* [3]. A *breeder* in the sense of 2.3 is called *special*, if it is of the form (au, a) with u prime to a. For example, an amicable pair of Euler's first form 1.5 (12) gives rise to a special breeder, with $u = r_1r_2$ the product of two primes. However, our search procedures (1.5 and 1.6 modified by omitting the primality check for s, cf. 2.3) produced roughly seven times more special breeders than amicable pairs of Euler's first type.

Let us point out that already in the paper [3], special breeders in the above sense (resp. in particular amicable pairs of Euler's first form) were used to "breed" new amicable pairs by a method quite different from te Riele's trick (2.2), by constructing so-called "Thabit rules" (cf. loc. cit., Theorems 2 and 4, resp. in particular Theorem 3).

*We are presently performing some numerical experiments with type $(1,1)$ breeders, and we intend to report the results in a sequel to the present paper.

THEOREM [3]. *Given a special breeder* (au, a), *assume that*

$$t := u + \sigma(u) \tag{27}$$

is a prime not dividing a. Then the following ("Thabit") rule holds for $n = 1, 2, 3, \ldots$

$$A = aut^n[t^n(u+1) - 1], \qquad B = at^n[t^n(u+1)(t-u) - 1] \tag{28}$$

are amicable, whenever the numbers in square brackets are primes prime to a.

So each special breeder gives rise to a "Thabit rule" whenever (27) gives a prime t (it essentially never happens that t then divides a). In addition to the 67 Thabit rules previously known (see [3], [13], [7], [5]), we found 34 new ones in this way, as a by-product of our search procedure reported in 1.6 (which was exhaustive in the range $b \leqslant 10^6$, $s \leqslant 10^{14}$, notation 1.6). Since some readers may wish to use them in a search for very large amicable pairs, we list these new rules in Table 3 below.

TABLE 3

The 34 *new Thabit rules* (*notation Theorem* 2.5) *found by means of Theorem* 1.6 (*cf*. 2.5).

no.	$a = b \cdot p$	$u = r_1 \cdot r_2$	t
1	$3^3 \cdot 5^2 \cdot 31^2 \cdot 19$	$557 \cdot 10601$	118 20673
2	$3^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 43$	$1091 \cdot 20123$	439 29601
3	$3^5 \cdot 7 \cdot 13 \cdot 17 \cdot 41$	$3137 \cdot 65\,61557$	4 11737 73313
4	$2^8 \cdot 65951$	$257 \cdot 680\,61431$	3 50516 37223
5	$3^4 \cdot 5^2 \cdot 13 \cdot 769$	$389 \cdot 1\,24577$	970 45873
6	$3^3 \cdot 5^4 \cdot 19 \cdot 71$	$41 \cdot 19949$	16 55809
7	$2^4 \cdot 47 \cdot 181 \cdot 193$	$28949 \cdot 12\,76049$	7 38819 90001
8	$3^4 \cdot 5 \cdot 11 \cdot 79 \cdot 103$	$3089 \cdot 7109$	439 29601
9	$3^3 \cdot 5 \cdot 13 \cdot 23 \cdot 919$	$18379 \cdot 7\,16819$	2 63495 68001
10	$3^2 \cdot 5 \cdot 7 \cdot 113 \cdot 1399$	$12203 \cdot 1\,51091$	36876 90241
11	$2^2 \cdot 11 \cdot 29 \cdot 109 \cdot 367$	$653 \cdot 69\,60521$	90974 01601
12	$3^3 \cdot 5 \cdot 13 \cdot 23 \cdot 1609$	$1013 \cdot 5\,43841$	11023 66721
13	$2^6 \cdot 179 \cdot 7517$	$233 \cdot 70\,35911$	32857 70671
14	$3^6 \cdot 7 \cdot 11 \cdot 17 \cdot 101$	$857 \cdot 1\,65437$	2837 25313
15	$3^4 \cdot 5 \cdot 11 \cdot 47 \cdot 751$	$5639 \cdot 42\,35639$	4 77737 77921
16	$3^3 \cdot 5^2 \cdot 19 \cdot 31 \cdot 521$	$15629 \cdot 2\,96969$	92829 69601
17	$3 \cdot 5 \cdot 7 \cdot 13 \cdot 67 \cdot 3109$	$1741 \cdot 6217$	216 55553
18	$3^2 \cdot 5^2 \cdot 13 \cdot 101 \cdot 1009$	$19 \cdot 12107$	4 72193
19	$3^4 \cdot 7^2 \cdot 11 \cdot 19 \cdot 461$	$5531 \cdot 38723$	4283 98081
20	$3^5 \cdot 5 \cdot 13 \cdot 41 \cdot 641$	$71 \cdot 31\,53719$	4509 81889
21	$3^3 \cdot 5^2 \cdot 17 \cdot 37 \cdot 1091$	$1997 \cdot 50\,07689$	2 00057 19553
22	$2^6 \cdot 1459 \cdot 5281$	$71 \cdot 2773\,79243$	3 96652 31821
23	$2^6 \cdot 139 \cdot 1\,04527$	$751 \cdot 786\,04303$	11 81422 68161
24	$2^3 \cdot 53 \cdot 317 \cdot 7607$	$11 \cdot 91283$	20 99521
25	$2^8 \cdot 547 \cdot 10939$	$13463 \cdot 2945\,65391$	793 17622 96921
26	$3^5 \cdot 5 \cdot 13 \cdot 37 \cdot 2663$	$89 \cdot 47933$	85 80097
27	$3 \cdot 5 \cdot 7 \cdot 11 \cdot 433 \cdot 3463$	$1\,03889 \cdot 11\,42789$	23 74476 59521
28	$2^5 \cdot 97 \cdot 193 \cdot 3089$	$35\,77061 \cdot 71\,54123$	5118 14794 76191
29	$2^8 \cdot 523 \cdot 23719$	$1\,92463 \cdot 87\,28591$	335 98705 40321
30	$2^7 \cdot 257 \cdot 1\,02797$	$24671 \cdot 12\,33563$	6 08677 23781
31	$2^5 \cdot 89 \cdot 223 \cdot 13883$	$17393 \cdot 2414\,80901$	840 03961 20481
32	$3^2 \cdot 5 \cdot 7 \cdot 149 \cdot 3\,61927$	$173 \cdot 93833\,19401$	325 60118 32321
33	$2^8 \cdot 1259 \cdot 3\,37411$	$431 \cdot 182\,20193$	1 57240 26991
34	$2^5 \cdot 97 \cdot 193 \cdot 3\,01079$	$1559 \cdot 4696\,83239$	146 49420 24001

2.6. *Breeding Amicable Pairs From Special Breeders.* As a very special case of the general strategy explained in 2.3, let us now describe more explicitly, and in more detail, how special breeders may be used to "breed" new amicable pairs.

THEOREM. *Let (au, a) be a special breeder. Take any factorization of*

$$C := \sigma(u)(u + \sigma(u) - 1) \tag{29}$$

into two different factors D_1, D_2 ($C = D_1 D_2$). Then if the numbers

$$s_i = D_i + \sigma(u) - 1 \quad \text{for } i = 1, 2, \tag{30}$$

and also

$$q = u + s_1 + s_2, \tag{31}$$

are primes not dividing a, then

$$(auq, as_1 s_2) \tag{32}$$

is an amicable pair.

In fact, going into the BDE-method in 2.1 with $a_1 = a$, $a_2 = a$, one may derive formulas (29)–(31) above readily from the general formulae (21)–(24) in 2.1, using the "breeder condition" (25). In the special case where $s := \sigma(u) - 1$ happens to be a prime not dividing a, we will be in a situation with a "mother pair" (au, as), and then the above recipe is equivalent to the one stated by te Riele in [15, Lemma 1], by which he produced about half of his new amicable pairs (cf., 2.2). It turned out in our numerical experiments, that arbitrary "special breeders", used as inputs into this recipe, usually "breed" new amicable pairs at a similarly high rate of fertility.

2.7. *Numerical Experiments on Breeding.* Let us conclude with a brief summary of our numerical experiments about special breeders (au, a) with $u = r_1 r_2$ the product of two different primes r_1, r_2. Note that such breeders will breed regular amicable pairs of Euler type $(3, 2)$ (notation 2.4). By our previous remarks (cf. 2.5), it is clear how such special breeders may be found by (a trivial modification of) our Euler search procedure (1.5), resp. by the new procedure described in 1.6, along with the search for amicable pairs of Euler's first type (1.6) and for Thabit rules (2.5). In the range $a \leqslant 10^7$, $r_1 \leqslant r_2 \leqslant 10^{12}$ there turned out to be exactly 141 special breeders $(ar_1 r_2, a)$, *in addition* to those associated with a "mother pair" (and hence covered by te Riele's investigations). From these special breeders, we derived 1669 amicable pairs by the above recipe (Theorem 2.6), 1604 of which were new. Our new search procedure in 1.6 generated more than 300 further new special breeders of the same type $(ar_1 r_2, a)$, of which only those 153 with $r_1 r_2 \leqslant 10^{12}$ were actually employed for breeding, according to the above recipe. This resulted in a list of another 1867 new amicable pairs. The smallest member of the list, for example, is the pair:

$$2 \cdot 5 \cdot 19^2 \cdot 37 \cdot 127 \cdot 29 \cdot 4217 \cdot 1889453,$$
$$2 \cdot 5 \cdot 19^2 \cdot 37 \cdot 127 \cdot 147629 \cdot 1619531.$$

From a single special breeder, we have thus bred about a dozen new amicable pairs, in the average. Our "champion breeder" was $(a \cdot 1019 \cdot 3918\ 82979,\ a)$, where $a = 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 73 \cdot 277$, which bred 110 new amicable pairs. For more details about our numerical results, and in particular for the full list of our $30 + 1604 + 1867 = 3501$ new amicable pairs found here, we refer to a forthcoming joint report of H. J. J. te Riele and the authors [6].

What, in conclusion, do the results of our numerical experiments about breeding amicable numbers suggest? We obtain about three and a half thousand amicable pairs of a very special type (Euler's type (3, 2)) by a very special recipe. In combination with some analysis of their distribution, we think that this gives some rather convincing numerical evidence to expect that there is an infinity of amicable pairs, even of this particular special type.

2.8. *Why "Special" Breeders*? As a final point, let us briefly come back to the breeder condition (25) in 2.3. Under a certain mild regularity assumption, we shall now show that such breeders are necessarily "special" in the sense of 2.5.

PROPOSITION. *Assume that* (a_1, a_2) *is a breeder in the sense of* 2.3, *and that it "breeds" at least one regular amicable pair* $A = a_1q$, $B = a_2s_1s_2$ *as in* (20) *by the process described in* 2.1. *Then* (a_1, a_2) *is necessarily a special breeder in the sense of* 2.5.

Proof. From the *regularity* of the pair A, B it follows that we must have $a_1 = au$, $a_2 = av$, with a, u, v pairwise relatively prime to each other. From the breeder condition (25), we have—using the notations as in 2.1:

$$F := (\sigma(a_1) - a_1)\sigma(a_2) = a_2\sigma(a_2)x = a_2[\sigma(a_1) - \sigma(a_2)],$$

which gives

$$D := a_2\sigma(a_1) - F = a_2\sigma(a_2),$$

or also

$$F = Dx.$$

Putting this into Eqs. (23), we obtain for $i = 1, 2$ that

$$s_i = d_i/D + x$$

must be integer, and hence d_1d_2 is divisible by D^2. Now putting $F = Dx$ into Eq. (21) gives

$$d_1d_2 = x(x+1)D^2 + DG,$$

hence we must have

$$d_1d_2 \equiv 0 \equiv DG \bmod D^2.$$

Recalling Eq. (22), this is equivalent to

$$a_1\sigma(a_1) \equiv 0 \bmod a_2\sigma(a_2).$$

Recalling now $a_1 = au$, $a_2 = av$ from above, this implies

$$u\sigma(a_1) \equiv 0 \bmod v.$$

But since u, v are relatively prime, v must divide $\sigma(a_1)$, which divides $\sigma(a_1q) = \sigma(A) = \sigma(B) = A + B = A + avs_1s_2$. It follows that v must also divide $A = a_1q = auq$. By our assumptions, this implies $v = 1$, and so proves the proposition. Q.E.D.

Remark. The proposition says that "nonspecial" breeders are usually useless for our purpose. Let us mention here another, more obvious, reason which causes a breeder to be useless for our purpose: More generally, let (a_1, a_2) even be an arbitrary type (i, j) breeder in the sense of Subsection 2.4, and let c_k be the greatest common divisor of a_k and $\sigma(a_k)$. We observe that *then* c_1 *must divide* a_2, *and* c_2

must divide a_1, *or else the breeder would become useless*. In fact, any amicable pair A_1, A_2 "breeded" from a_1, a_2 should have the form $A_k = a_k u_k$ $(k = 1, 2)$, with $a_1 a_2$ relatively prime to u_1, u_2, since the "unknown" prime factors of $u_1 u_2$ should be different from the "known" prime factors of $a_1 a_2$. Then Eqs. (1) imply immediately that c_1 divides $A_2 = a_2 u_2$, and hence a_2. Similarly, c_2 divides a_1. Needless to say, in numerical experiments one should make sure not to use "useless" breeders.

2.9. *A Criterion for Type* $(1,1)$ *Breeders*. By definition 2.4, a pair of natural numbers a_1, a_2 is a type $(1,1)$ breeder, if the equations

$$a_1 x + a_2 y = \sigma(a_1)(x+1) = \sigma(a_2)(y+1) \tag{33}$$

have a solution in positive integers x, y. Let us rewrite the two equations (33) as

$$a_1 x - \tau(a_2) y = \sigma(a_2), \qquad -\tau(a_1) x + a_2 y = \sigma(a_1), \tag{34}$$

where τ denotes the function $\tau(n) := \sigma(n) - n$. This is an *inhomogeneous* system of linear equations with determinant

$$D := a_1 a_2 - \tau(a_1)\tau(a_2) = a_1 \sigma(a_2) + a_2 \sigma(a_1) - \sigma(a_1)\sigma(a_2). \tag{35}$$

It is therefore solvable if and only if $D \neq 0$, the solutions then being given by

$$Dy = F + G, \qquad Dx = F' + G', \tag{36}$$

where F, G are as in 2.1, and F', G' are defined analogously by interchanging a_1 and a_2; or in detail:

$$\begin{aligned} F &:= \tau(a_1)\sigma(a_2), & G &:= a_1 \sigma(a_1), \\ F' &:= \tau(a_2)\sigma(a_1), & G' &:= a_2 \sigma(a_2). \end{aligned} \tag{37}$$

Obviously, the solutions x, y are *positive* if and only if D is positive. Let us now put

$$H := F + D = a_2 \sigma(a_1), \qquad H' := F' + D = a_1 \sigma(a_2), \tag{38}$$

and rewrite (36) in the equivalent form

$$\begin{aligned} D(y+1) &= H + G = (a_1 + a_2)\sigma(a_1), \\ D(x+1) &= H' + G' = (a_1 + a_2)\sigma(a_2). \end{aligned} \tag{39}$$

Then it becomes obvious that the solutions x, y are integers if and only if

$$(a_1 + a_2)\sigma(a_i) \equiv 0 \bmod D \tag{40}$$

for $i = 1$ and 2. This may be written as a single congruence for the greatest common divisor:

$$(a_1 + a_2)(\sigma(a_1), \sigma(a_2)) \equiv 0 \bmod D. \tag{41}$$

In conclusion, we have proved the following

LEMMA. *A necessary and sufficient condition for two natural numbers* a_1, a_2 *to be a type* $(1,1)$ *breeder is that the determinant* $D := a_1 a_2 - \tau(a_1)\tau(a_2)$ *be positive, and divides* $(a_1 + a_2)(\sigma(a_1), \sigma(a_2))$.

2.10. *Breeders Which Are of Types* $(0,1)$ *and* $(1,1)$ *Simultaneously*. Let us now take for a_1, a_2 a breeder in the sense of 2.3, that is to say, a type $(1,0)$ breeder in the terminology of 2.4. In this case, it follows from the breeder condition (25) that the determinant D is

$$D = a_1 \sigma(a_1) > 0. \tag{42}$$

Let us now assume that a_1, a_2 are simultaneously also a $(1,1)$ breeder. Then condition (40) with $i = 1$, in combination with Eq. (42), implies that necessarily a_1 must divide a_2. So let us write $a_1 = a$, $a_2 = a \cdot u$. If we restrict attention to *regular* amicable pairs, we may assume, in addition, that a and u are relatively prime. So in conclusion, we find that (a_1, a_2) must be even a "*special* breeder" in the sense of 2.5.

On the other hand, if $(a, a \cdot u)$ is such a special breeder, then $D = a\sigma(a)$, and $\sigma(a)$ divides $\sigma(a_2) = \sigma(au) = \sigma(a)\sigma(u)$, so that conditions (40) are satisfied for both $i = 1, 2$. Now Lemma 2.9 implies that $(a, a \cdot u)$ is also a $(1,1)$ breeder. In conclusion, we have shown in particular the following

PROPOSITION. *Any special breeder (as defined in* 2.5) *is a breeder of types* $(0,1)$ *and* $(1,1)$ *simultaneously.*

Remark. Note that whenever the positive integer solutions x, y of (33) are *prime numbers*, not dividing a_1, a_2, then the given type $(1,1)$ breeder belongs to an amicable pair (a "mother pair", as te Riele would say), of the form $(a_1 x, a_2 y)$.

Let us point out here, however, that such is *never* the case for the special breeders $(a_1, a_2) = (a, a \cdot u)$ as considered in the above proposition: In fact, from Eqs. (39) it follows in this case that

$$D(y+1) = a(u+1)\sigma(a),$$

and since $D = a\sigma(a)$ by (42), we have $y = u$, and so y is *never* prime to a_2 in this case. Thus, such $(1,1)$ breeders *never* belong to a "mother pair". But let us point out that they nevertheless can be very "fertile" breeders, when used as inputs for the BDE-method (2.1). Numerical experiments shall be reported elsewhere.

Note Added in Proof (August 31, 1985). Numerical experiments with type $(1,1)$ breeders, carried out by Stefan Battiato and the first author, generated (in first and second generation) about a thousand amicable pairs so far, 600 of which were new. The most frequent type of these new pairs are $(3,3)$, $(4,2)$, $(4,3)$ or $(3,2)$ (about 80%), but there are also some of types $(4,4)$, $(5,2)$, $(5,3)$, and about 10% are of "exotic" type. We are extending these breeding experiments to higher generations. All of these new amicable pairs will be published in [6] as well.

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