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A Note on Class-Number One in Certain Real Quadratic and Pure Cubic Fields

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Abstract. Let p be any odd prime and let $h(p)$ be the class number of the real quadratic field $\mathcal{Q}(\sqrt{p})$. The results of a computer run to determine the density of the field $\mathcal{Q}(\sqrt{p})$ with $h(p) = 1$ and $p < 10^8$ are presented. Similar results are given for pure cubic fields $\mathcal{Q}(\sqrt[3]{p})$ with $p < 10^6$.

1. Introduction. Let p be any odd prime and let $h = h(p)$ be the class number of the quadratic field $\mathcal{Q}(\sqrt{p})$. It is well known that $h(p)$ is odd, but the problem of how frequently $h(p) = 1$, although it goes back to Gauss, is still unsolved.

If we let $\pi(a, b; x)$ denote the number of primes of the form $a + bk$ less than or equal to x and $f(a, b; x)$ denote the number of these primes p for which $h(p) = 1$, we find (see Lakein [5]) from a large table of Kuroda [4], that

$$r(1,4;x) = f(1,4;x)/\pi(1,4;x) = .7765$$

for $x = 2776817$; that is, over 77% of all the primes ($\equiv 1 \pmod 4$) up to 2776817 have $h(p) = 1$. Indeed, according to the recent heuristic results of Cohen and Lenstra (see Cohen [1]), we would expect that $h(p) = 1$ with probability .75446.

In order to test this heuristic, we developed and ran a computer program which determined whether or not $h(p) = 1$ for all primes $p < 10^8$. In the next section of this note we give the results of this computer run. In the following section we present some data for certain pure cubic fields $\mathcal{Q}(\sqrt[3]{p})$ with $p < 10^6$.

2. The Quadratic Case. In order to find $h(p)$, we made use of the well-known formula

$$2hR = \sqrt{\Delta}\, L(1, \chi),$$

where Δ is the discriminant of $\mathcal{Q}(\sqrt{p})$, R is the regulator, and $L(1, \chi)$ is the value of the Dirichlet L-function

$$\sum_{n=1}^{\infty} \left(\frac{\Delta}{n}\right) \frac{1}{n^s}$$

for $s = 1$. To evaluate $L(1, \chi)$ we employed a routine similar to the SPEEDY routine mentioned in Shanks [8]. Most of the time needed to find $h(p)$ was taken up computing R. This was done by using the techniques developed by Lenstra [6] and Schoof [7] (see, also, Williams [10]). The implementation of these ideas permitted us

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TABLE 1

x	$a=-1, b=4$			$a=1, b=4$		
	$\pi(a,b;x)$	$f(a,b;x)$	$r(a,b;x)$	$\pi(a,b;x)$	$f(a,b;x)$	$r(a,b;x)$
5000000	174319	134661	.7724975	174193	134862	.7742102
10000000	332398	255697	.7692495	332180	256345	.7717051
15000000	485429	372854	.7680917	485274	373925	.7705441
20000000	635436	487699	.7675029	635170	488752	.7694821
25000000	783173	600560	.7668293	783059	602016	.7688003
30000000	929079	712172	.7665355	928779	713887	.7686296
35000000	1073601	822569	.7661775	1073173	824136	.7679433
40000000	1216966	932017	.7658529	1216687	933970	.7676337
45000000	1359235	1040345	.7653900	1358924	1042888	.7674366
50000000	1500681	1148210	.7651259	1500452	1151039	.7671282
55000000	1641343	1255778	.7650917	1640856	1258288	.7668485
60000000	1781444	1362483	.7648194	1780670	1365129	.7666378
65000000	1920648	1468646	.7646617	1919905	1471506	.7664472
70000000	2059345	1574494	.7645605	2058718	1577494	.7662506
75000000	2197469	1679748	.7644012	2196834	1682861	.7660392
80000000	2335008	1784833	.7643798	2334373	1787874	.7658904
85000000	2472052	1889752	.7644467	2471678	1892924	.7658457
90000000	2608560	1993991	.7644029	2608393	1997296	.7657189
95000000	2745067	2098012	.7642844	2744681	2101465	.7656500
100000000	2880950	2201430	.7641333	2880504	2205112	.7655299

to evaluate R much more rapidly than was done in Williams and Broere [11]. Indeed, without this innovation we would not have been able to complete our calculations because of time constraints. If we define

$$r(a,b;x) = f(a,b;x)/\pi(a,b;x),$$

the results of running our program are summarized in Table 1.

Notice that the value of $r(a, b; x)$ in both cases is tending to decrease more slowly as x increases. These results are certainly consistent with the heuristic we get from [1].

3. The Pure Cubic Case. Let $H(p)$ denote the class number of $\mathcal{Q}(\sqrt[3]{p})$. In order for $H(p) = 1$, we must have $p = 3$ or $p \equiv -1 \pmod 3$ (Honda [3]); also, it has been noted by Eisenbeis, Frey, and Ommerborn [2] that $H(p)$ tends to be 1 more frequently for $p \equiv -1 \pmod 9$, an observation that was tested empirically by Williams and Shanks [13]. Thus, in the cubic case we performed our computations on the primes in each of the residue classes $-1, 2, 5 \pmod 9$.

Let $F(a, b; x)$ be the number of primes p of the form $a + bk$ less than or equal to x for which $H(p) = 1$, and put $R(a, b; x) = F(a, b; x)/\pi(a, b; x)$. The results of our computer runs for the pure cubic case are given in Tables 2 and 3. These tables were computed by making use of the algorithms given in Williams, Dueck and

TABLE 2

x	$a = -1,\ b = 9$		
	$\pi(a, b; x)$	$F(a, b; x)$	$R(a, b; x)$
200000	2993	1827	.6104
250000	3671	2240	.6102
300000	4337	2627	.6057
350000	4992	3041	.6092
400000	5650	3437	.6083
450000	6287	3820	.6076
500000	6924	4199	.6064
550000	7550	4568	.6050
600000	8174	4940	.6044
650000	8802	5332	.6058
700000	9416	5701	.6055
750000	10033	6065	.6045
800000	10670	6435	.6031
850000	11282	6789	.6018
900000	11890	7157	.6019
950000	12487	7523	.6025
1000000	13094	7903	.6036

TABLE 3

x	$a = 2,\ b = 9$			$a = 5,\ b = 9$		
	$\pi(a, b; x)$	$F(a, b; x)$	$R(a, b; x)$	$\pi(a, b; x)$	$F(a, b; x)$	$R(a, b; x)$
200000	2994	1200	.4008	2988	1293	.4327
250000	3679	1440	.3914	3677	1559	.4240
300000	4328	1699	.3926	4341	1845	.4250
350000	5007	1965	.3925	4999	2096	.4193
400000	5651	2224	.3936	5647	2366	.4190
450000	6281	2471	.3934	6296	2628	.4174
500000	6916	2755	.3984	6945	2873	.4137
550000	7541	2987	.3961	7577	3124	.4123
600000	8176	3223	.3942	8204	3366	.4103
650000	8829	3472	.3932	8806	3607	.4096

Schmid [12]. Since precision problems in our implementation of these algorithms occur later for Dedekind type 2 fields than for type 1 fields, we were able to compute Table 2 out somewhat further than Table 3.

In Table 2 we notice that the surprisingly flat behavior of $R(-1, 9; x)$ for $1.5 \times 10^5 < x < 2 \times 10^5$, which was pointed out in [13], persists (although it does tend to decrease slightly) up to 10^6. These results, then, are still consistent with the conjectures made in [13]. In Table 3 we see that the values of $R(2, 9; x)$ and $R(5, 9; x)$ are coming closer together. This indicates that the peculiar behavior of these ratios for $x < 2 \times 10^5$ noted in Williams [9] was simply a result of this range of x values being too small.

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