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Supplement to Boundary Integral Solutions of the Heat Equations

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Appendix: Examples

Below, we present numerical results and, where appropriate, error estimates for five examples run in single precision on the Cray-1. In all cases except the first, we have used a known solution u to prescribe the boundary and initial data, and then measured the error in terms of u and its computed approximation U. More precisely, if $\{s_i\}$ is the set of B-spline knots associated (implicitly) with equation (3.2), and

$$u_{\max}(\tau) = \max_i |u(s_i,\tau)| ,$$

we define, for any value of τ, the maximum absolute and maximum percent error as

$$E_{\max}(\tau) = \max_i |U(s_i,\tau)-u(s_i,\tau)| ,$$

and

$$P_{\max}(\tau) = 100. * \frac{E_{\max}(\tau)}{U_{\max}(\tau)} ,$$

respectively. Our results may then be somewhat optimistic, since there remains the possibility of superconvergence at the s_i, in addition to which it may happen that large values of $u_{\max}$ generate cosmetically small $P_{\max}$.

In general, each computation, or *run*, consists of two calculations, or *phases*: the first, to compute the density function ζ, and the second to "recover" the desired approximation U and estimate the error.

For each run, we present the number, order, and underlying mesh of the B-splines used to approximate ζ in equation (3.2), and the number of points used in the quadratures along the boundary. As output for phase 1, we give the condition number of the matrix B in equation (3.3) and, for each time step, the maximum and minimum B-spline coefficients computed for the approximation to ζ. We note in passing that B never changes, so that we might possibly have increased our time savings by inverting it once instead of solving a linear system at each step as we have done here.

In the recovery calculation, phase 2, we present the values of E_{max} and P_{max} at each time step.

Example 1: $\zeta(s,\tau) \equiv \tau$ on the unit circle, where s is arclength

(no s dependence)

Of course here, and throughout, we are solving the homogeneous heat equation (2.1)

domain Ω	the unit disk
initial data:	$u(\mathbf{x},0) = 0 \ \forall \mathbf{x} \in \Omega$
boundary data:	$\frac{\partial u}{\partial \nu} = \frac{1}{2}[e^{-\frac{1}{2t}} I_0(\frac{1}{2t}) - 1]$,
discretization	8 B-splines of order 2 on a regular uniform mesh of 10 points.
integration rule·	7 point Gaussian quadrature
condition number of B:	2.816

In this instance, it can be shown that $\zeta(s,t) \equiv \tau$.[1] We compute ζ but do not bother to recover U or estimate the error as defined above. On the other hand, since we are using splines of order 2, we expect the maximum and minimum coefficients to be identical and equal to τ, which is essentially the case. After 50 time steps, the error in ζ is approximately:

$$\frac{.013}{2.5} = .52\% .$$

1 J McKenna private communication

TABLE 1

STEP	τ	MAXIMUM COEFFICIENT	MINIMUM COEFFICIENT
1	0.5000E-01	0.5005E-01	0.4963E-01
2	0.1000	0.1001	0.9987E-01
3	0.1500	0.1500	0.1500
4	0.2000	0.2002	0.2002
5	0.2500	0.2501	0.2501
6	0.3000	0.3005	0.3005
7	0.3500	0.3501	0.3501
8	0.4000	0.4008	0.4008
9	0.4500	0.4502	0.4502
10	0.5000	0.5011	0.5011
11	0.5500	0.5503	0.5503
12	0.6000	0.6016	0.6016
13	0.6500	0.6505	0.6505
14	0.7000	0.7021	0.7021
15	0.7500	0.7508	0.7508
16	0.8000	0.8028	0.8028
17	0.8500	0.8511	0.8511
18	0.9000	0.9035	0.9035
19	0.9500	0.9516	0.9516
20	1.000	1.004	1.004
21	1.050	1.052	1.052
22	1.100	1.105	1.105
23	1.150	1.153	1.153

STEP	τ	MAXIMUM COEFFICIENT	MINIMUM COEFFICIENT
24	1.200	1.206	1.206
25	1.250	1.254	1.254
26	1.300	1.306	1.306
27	1.350	1.354	1.354
28	1.400	1.407	1.407
29	1.450	1.455	1.455
30	1.500	1.507	1.507
31	1.550	1.556	1.556
32	1.600	1.608	1.608
33	1.650	1.656	1.656
34	1.700	1.708	1.708
35	1.750	1.757	1.757
36	1.800	1.808	1.808
37	1.850	1.857	1.857
38	1.900	1.909	1.909
39	1.950	1.958	1.958
40	2.000	2.009	2.009
41	2.050	2.058	2.058
42	2.100	2.109	2.109
43	2.150	2.158	2.158
44	2.200	2.210	2.210
45	2.250	2.258	2.258
46	2.300	2.311	2.311
47	2.350	2.359	2.359
48	2.400	2.412	2.412

STEP	τ	MAXIMUM COEFFICIENT	MINIMUM COEFFICIENT
49	2.450	2.459	2.459
50	2.500	2.513	2.513

Example 2: $\zeta(s,\tau) \equiv \tau \cos(s)$ on the unit circle

domain Ω — the unit disk

initial data: $u(\mathbf{x},0) = 0 \;\forall \mathbf{x} \in \Omega$

boundary data. $\frac{\partial u}{\partial \nu} = \frac{\cos(s)}{2}[e^{-\frac{1}{2t}} I_1(\frac{1}{2t}) - 1]$

where s is arc length.

discretization — 15 B-splines of order 4 on a regular uniform mesh of 19 points

integration rule: 7 point Gaussian quadrature

condition number of B. 41.35

The above problem results in the density function

$$\zeta(s,\tau) = \tau \cos(s)$$

and a solution whose values on the unit circle are given by

$$u(1,s,t) = \frac{e^{-\frac{1}{2t}}\cos(s)}{2}[I_0(\frac{1}{2t}) + I_1(\frac{1}{2t})],$$

where I_0 and I_1 are modified Bessel functions and $t = \tau^2$. In recovering the approximation U, we see errors on the order of 3 percent after 50 time steps

TABLE 2.1

Example 2, Phase 1

STEP	τ	MAXIMUM COEFFICIENT	MINIMUM COEFFICIENT
1	0 1000	-.1047	-.2111E-05
2	0.2000	-.2097	-.4205E-05
3	0.3000	-.3142	-.6257E-05
4	0.4000	-.4201	-.8360E-05
5	0.5000	-.5238	-.1042E-04
6	0.6000	-.6312	-.1255E-04
7	0.7000	-.7335	-.1455E-04
8	0.8000	-.8436	-.1671E-04
9	0.9000	-.9436	-.1863E-04
10	1.000	-1.057	-.2082E-04
11	1.100	-1.154	-.2263E-04
12	1.200	-1.271	-.2484E-04
13	1.300	-1.365	-.2652E-04
14	1.400	-1.484	-.2871E-04
15	1.500	-1.574	-.3028E-04
16	1.600	-1.694	-.3242E-04
17	1.700	-1.781	-.3389E-04
18	1.800	-1.901	-.3594E-04
19	1.900	-1.987	-.3734E-04
20	2.000	-2 105	-.3927E-04
21	2.100	-2.190	-.4062E-04

STEP	τ	MAXIMUM COEFFICIENT	MINIMUM COEFFICIENT
22	2.200	-2.306	-.4242E-04
23	2.300	-2.392	-.4373E-04
24	2.400	-2.504	-.4537E-04
25	2.500	-2.592	-.4668E-04
26	2.600	-2.701	-.4814E-04
27	2.700	-2.791	-.4946E-04
28	2.800	-2.896	-.5072E-04
29	2.900	-2.989	-.5208E-04
30	3.000	-3.091	-.5312E-04
31	3.100	-3.187	-.5452E-04
32	3.200	-3.285	-.5534E-04
33	3.300	-3.386	-.5680E-04
34	3.400	-3.480	-.5739E-04
35	3.500	-3.585	-.5891E-04
36	3.600	-3.676	-.5926E-04
37	3.700	-3.784	-.6086E-04
38	3.800	-3.872	-.6097E-04
39	3.900	-3.985	-.6265E-04
40	4.000	-4.070	-.6252E-04
41	4.100	-4.187	-.6430E-04
42	4.200	-4.270	-.6392E-04
43	4.300	-4.390	-.6582E-04
44	4.400	-4.471	-.6519E-04
45	4.500	-4.595	-.6722E-04
46	4.600	-4.674	-.6635E-04
47	4.700	-4.802	-.6852E-04
48	4.800	-4.880	-.6742E-04
49	4.900	-5.010	-.6976E-04
50	5.000	-5.087	-.6843E-04

TABLE 2.2

Example 2, Phase 2

STEP	ABSOLUTE ERROR	PERCENT ERROR
1	0.1487E-03	0.2642E+00
2	0.1418E-03	0.1270E+00
3	0.8026E-03	0.4855E+00
4	0.4471E-03	0.2071E+00
5	0.1224E-02	0.4672E+00
6	0.8539E-03	0.2831E+00
7	0.1487E-02	0.4442E+00
8	0.1419E-02	0.3925E+00
9	0.1704E-02	0.4447E+00
10	0.1963E-02	0.4900E+00
11	0.1929E-02	0.4649E+00
12	0.2180E-02	0.5113E+00
13	0.2028E-02	0.4654E+00
14	0.1795E-02	0.4047E+00
15	0.1846E-02	0.4101E+00
16	0.6964E-03	0.1529E+00
17	0.1334E-02	0.2899E+00
18	0.1069E-02	0.2303E+00
19	0.5656E-03	0.1209E+00
20	0.3290E-02	0.6991E+00
21	0.3484E-03	0.7363E-01

STEP	ABSOLUTE ERROR	PERCENT ERROR
22	0.5774E-02	0.1214E+01
23	0.1217E-02	0.2548E+00
24	0.8323E-02	0.1737E+01
25	0.1935E-02	0.4025E+00
26	0.1078E-01	0.2235E+01
27	0.2405E-02	0.4975E+00
28	0.1301E-01	0.2686E+01
29	0.2559E-02	0.5271E+00
30	0.1495E-01	0.3072E+01
31	0.2355E-02	0.4832E+00
32	0.1652E-01	0.3385E+01
33	0.1770E-02	0.3621E+00
34	0.1771E-01	0.3618E+01
35	0.7964E-03	0.1625E+00
36	0.1849E-01	0.3770E+01
37	0.5845E-03	0.1190E+00
38	0.1887E-01	0.3839E+01
39	0.2317E-02	0.4710E+00
40	0.1885E-01	0.3829E+01
41	0.4409E-02	0.8949E+00
42	0.1845E-01	0.3741E+01
43	0.6843E-02	0.1387E+01
44	0.1767E-01	0.3580E+01
45	0.9602E-02	0.1944E+01
46	0.1655E-01	0.3350E+01

STEP	ABSOLUTE ERROR	PERCENT ERROR
47	0.1266E-01	0.2562E+01
48	0.1510E-01	0.3053E+01
49	0.1601E-01	0.3236E+01
50	0.1334E-01	0.2695E+01

Example 3: Mixed Boundary Data on an Ellipse

domain Ω: the interior of the ellipse,

$$\xi = \xi_0 + (a \cos\theta, b \sin\theta) \ , \ 0 \leqslant \theta \leqslant 2\pi \ ,$$

where $a = 1.$, $b = .5$, and $\xi_0 = (2,0)$.

boundary data:

$$\frac{\partial u}{\partial \nu} + \beta u = g(\theta,t) \ ,$$

where

$$\beta(\theta,t) = \frac{\cos\theta}{1+t} \ ,$$

and we determine g by means of the solution

$$u = \frac{e^{-\|x\|^2/4t}}{t} \ .$$

discretization: 17 B-splines of order 4 on a regular uniform mesh of 21 points

integration rule: 7 point Gaussian quadrature

condition number of B: 64.99

After 50 time steps, the maximum percent error is on the order of 1 percent.

TABLE 3 1

Example 3, Phase 1

STEP	τ	MAXIMUM COEFFICIENT	MINIMUM COEFFICIENT
1	0.2500E-01	0.4454E-169	- 4226E-693
2	0.5000E-01	0.2129E-39	-.2357E-76
3	0.7500E-01	0.1196E-15	-.1976E-27
4	0.1000	0.1735E-07	-.3144E-14
5	0.1250	0.8231E-04	-.5002E-09
6	0.1500	0.6892E-02	-.1271E-06
7	0.1750	0.8731E-01	-.1624E-05
8	0.2000	0.4091	-.1254E-05
9	0.2250	1.087	0.7515E-05
10	0.2500	2.045	0.1706E-04
11	0.2750	3.091	-.1251E-04
12	0.3000	4.053	-.6392E-04
13	0.3250	4.820	-.8778E-04
14	0.3500	5.375	-.8661E-04
15	0.3750	5.711	-.1982E-04
16	0.4000	5.887	0.1304E-04
17	0.4250	5.911	0.7715E-04
18	0.4500	5.852	0.7497E-04
19	0.4750	5.705	0.1081E-03
20	0.5000	5.528	0.9011E-04
21	0.5250	5.305	0.1203E-03
22	0.5500	5.082	0 1222E-03
23	0.5750	4.837	0.1748E-03
24	0.6000	4.606	0.3689E-04
25	0.6250	4.365	0.3001E-03
26	0.6500	4.145	0.3173E-03
27	0.6750	3.921	0.5430E-03
28	0.7000	3.721	0.5477E-03
29	0.7250	3.519	0.9885E-03
30	0.7500	3 340	0.1142E-02
31	0.7750	3.160	0.1800E-02
32	0.8000	3.002	0.2161E-02
33	0.8250	2.843	0.3030E-02
34	0.8500	2.705	0.3634E-02
35	0.8750	2.564	0.4694E-02
36	0 9000	2.444	0.5552E-02
37	0.9250	2.320	0.6771E-02
38	0.9500	2.215	0.7872E-02
39	0.9750	2.106	0.9208E-02
40	1.000	2.014	0.1053E-01
41	1.025	1.918	0.1194E-01
42	1.050	1.838	0.1344E-01
43	1.075	1.752	0.1488E-01
44	1.100	1.682	0.1654E-01
45	1.125	1.606	0.1797E-01
46	1.150	1.544	0.1974E-01

STEP	τ	MAXIMUM COEFFICIENT	MINIMUM COEFFICIENT
47	1.175	1.477	0.2112E-01
48	1.200	1.422	0.2297E-01
49	1.225	1.362	0.2429E-01
50	1.250	1.314	0.2618E-01

TABLE 3 2

Example 3, Phase 2

STEP	ABSOLUTE ERROR	PERCENT ERROR
1	0.1060E-169	0.3458E+03
2	0.2272E-40	0.1527E+03
3	0.1193E-16	0.1345E+03
4	0.8419E-09	0.6062E+02
5	0.4258E-05	0.5912E+02
6	0.1016E-03	0.1530E+02
7	0.1936E-02	0.2080E+02
8	0.9054E-03	0 1876E+01
9	0.1119E-01	0.7906E+01
10	0.4030E-02	0.1375E+01
11	0.2099E-01	0.4328E+01
12	0.9479E-02	0.1372E+01
13	0.2653E-01	0.2988E+01
14	0.1262E-01	0.1190E+01
15	0.2766E-01	0.2301E+01
16	0.1319E-01	0.1007E+01
17	0.2623E-01	0.1891E+01
18	0.1248E-01	0.8683E+00
19	0.2381E-01	0.1627E+01
20	0.1136E-01	0.7723E+00
21	0.2124E-01	0.1450E+01

STEP	ABSOLUTE ERROR	PERCENT ERROR
22	0.1024E-01	0.7179E+00
23	0.1886E-01	0.1328E+01
24	0.9229E-02	0.6653E+00
25	0.1676E-01	0.1242E+01
26	0.8351E-02	0.6376E+00
27	0.1495E-01	0.1179E+01
28	0.7597E-02	0.6200E+00
29	0.1340E-01	0.1134E+01
30	0.6951E-02	0.6098E+00
31	0.1208E-01	0.1100E+01
32	0.6396E-02	0.6050E+00
33	0.1095E-01	0.1076E+01
34	0.5918E-02	0.6043E+00
35	0.9977E-02	0.1059E+01
36	0.5504E-02	0.6070E+00
37	0.9142E-02	0.1048E+01
38	0.5144E-02	0.6125E+00
39	0.8420E-02	0.1041E+01
40	0.4831E-02	0.6204E+00
41	0.7794E-02	0.1039E+01
42	0.4558E-02	0.6305E+00
43	0.7249E-02	0.1040E+01
44	0.4319E-02	0.6426E+00
45	0.6773E-02	0.1044E+01
46	0.4109E-02	0.6565E+00

STEP	ABSOLUTE ERROR	PERCENT ERROR
47	0.6356E-02	0.1052E+01
48	0.3925E-02	0.6723E+00
49	0.5989E-02	0.1062E+01
50	0.3762E-02	0.6898E+00

Example 4: Inhomogeneous Initial Data on the Unit Circle

domain Ω: the unit circle

boundary data:

$$\frac{\partial u}{\partial \nu} = g(s,t) ,$$

where we use the solution

$$u(x,t) = x_1^2 + x_2^2 + 4t .$$

discretization: 11 B-splines of order 2 on a regular uniform mesh of 13 points

integration rule: 5 point Gausian quadrature.

singularity isolation interval: .1

condition number of B: 4.403

The singularity isolation interval referred to above has to do with the size of the neighborhoods chosen about singularities in the integrands of v and $\frac{\partial v}{\partial \nu}$ when we evaluate the right hand side of (2.7). Specifying such neighborhoods makes it easier for *BQUAD* (see [7]) to do those integrations.

After 10 time steps, our error is about 1.4 percent.

TABLE 4.1

Example 4, Phase 1

STEP	τ	MAXIMUM COEFFICIENT	MINIMUM COEFFICIENT
1	0.1000	0.8600	0.8597
2	0.2000	1.258	1.258
3	0.3000	1.763	1.762
4	0.4000	2.397	2.397
5	0.5000	3.152	3.152
6	0.6000	4.066	4.066
7	0.7000	5.110	5.110
8	0.8000	6.339	6.339
9	0.9000	7.709	7.709
10	1.000	9.290	9.290

TABLE 4.2

Example 4, Phase 2

STEP	ABSOLUTE ERROR	PERCENT ERROR
1	0.3727E+00	0.3584E+02
2	0.2898E+00	0.2498E+02
3	0.2255E+00	0.1658E+02
4	0 1910E+00	0.1165E+02
5	0.1600E+00	0.8000E+01
6	0.1423E+00	0.5831E+01
7	0.1172E+00	0.3960E+01
8	0.1045E+00	0.2935E+01
9	0.8088E-01	0.1908E+01
10	0.7217E-01	0.1443E+01

Example 5: Nonstandard Domain

domain Ω· the interior of the curve (see diagram 5.1),

$$\xi = x_0 + r(\theta)(\cos\theta, \sin\theta) \quad , \quad 0 \leqslant \theta \leqslant 2\pi ,$$

where

$$r(\theta) = a + b \sin\theta + c \sin 2\theta ,$$

$$a = .5 , b = 1 , c = .2 , \text{ and } \xi_0 = (2,0)$$

discretization: 41 B-splines of order 2 on a regular uniform mesh of 43 points.

integration rule. 5 point Gaussian quadrature

condition number of B. 39.90

The percent error is .5317 percent after 29 time steps.

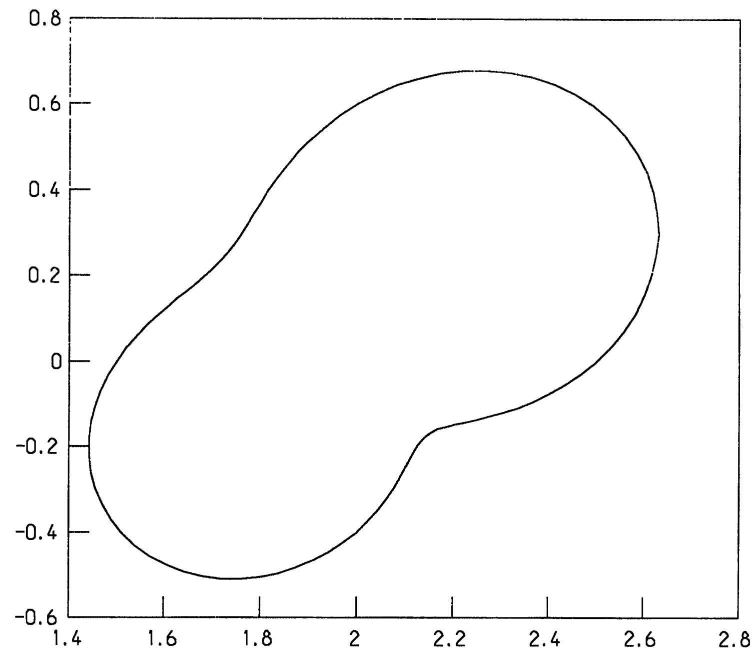

Diagram 5.1

TABLE 5 1

Example 5, Phase 1

STEP	τ	MAXIMUM COEFFICIENT	MINIMUM COEFFICIENT
1	0.2500E-01	0.3065E-33	-.6070E-159
2	0.5000E-01	0.1294E-05	-.4606E-25
3	0.7500E-01	0.7406E-01	-.3774E-15
4	0.1000	2.187	0.2762E-14
5	0 1250	8.049	-.3197E-13
6	0.1500	13.76	0.1273E-10
7	0.1750	16.77	-.2405E-09
8	0.2000	17.46	0 1405E-07
9	0.2250	16.67	0.7276E-06
10	0.2500	15.31	0.5118E-05
11	0.2750	13.69	0.3273E-04
12	0.3000	12.16	0.1775E-03
13	0.3250	10.69	0.2298E-03
14	0.3500	9.446	-.2721E-03
15	0.3750	8.306	0.7350E-03
16	0.4000	7.385	0.1148E-03
17	0.4250	6.552	0.1564E-02
18	0.4500	5.877	0.6180E-03
19	0.4750	5.248	0.2722E-02
20	0.5000	4.742	-.2404E-03
21	0.5250	4.264	0.4429E-02
22	0.5500	3.882	0.4048E-02
23	0.5750	3.513	0.6868E-02
24	0.6000	3.220	0.7268E-02
25	0.6250	2.933	0.1010E-01
26	0.6500	2 705	0.1039E-01
27	0.6750	2.478	0.1403E-01
28	0.7000	2.299	0.1429E-01
29	0.7250	2 116	0.1852E-01

TABLE 5.2

Example 5, Phase 2

STEP	ABSOLUTE ERROR	PERCENT ERROR
1	0.1607E-33	0.1315E+04
2	0.2986E-06	0.2525E+03
3	0.1280E-01	0.1238E+03
4	0.9493E-01	0.2289E+02
5	0.2807E+00	0.1468E+02
6	0.1190E+00	0.3065E+01
7	0.1604E+00	0.2945E+01
8	0.9433E-01	0.1487E+01
9	0.1042E+00	0.1558E+01
10	0.6389E-01	0.9604E+00
11	0.7230E-01	0.1129E+01
12	0.4216E-01	0.6979E+00
13	0.5021E-01	0.8914E+00
14	0.2934E-01	0.5624E+00
15	0.3609E-01	0.7496E+00
16	0.2169E-01	0.4891E+00
17	0.2705E-01	0.6620E+00
18	0.1686E-01	0.4475E+00
19	0.2105E-01	0.6057E+00
20	0.1358E-01	0.4229E+00
21	0.1690E-01	0.5683E+00

STEP	ABSOLUTE ERROR	PERCENT ERROR
22	0.1126E-01	0.4084E+00
23	0.1393E-01	0.5438E+00
24	0.9596E-02	0.4023E+00
25	0 1179E-01	0.5298E+00
26	0.8463E-02	0.4071E+00
27	0.1023E-01	0.5258E+00
28	0.7627E-02	0.4180E+00
29	0.9113E-02	0 5317E+00