

## Rate of Convergence of Shepard's Global Interpolation Formula

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**Abstract.** Given any data points  $x_1, \dots, x_n$  in  $\mathbf{R}^s$  and values  $f(x_1), \dots, f(x_n)$  of a function  $f$ , Shepard's global interpolation formula reads as follows:

$$S_p^0 f(x) = \sum_i f(x_i) w_i(x), \quad w_i(x) = |x - x_i|^{-p} / \sum_j |x - x_j|^{-p},$$

where  $|\cdot|$  denotes the Euclidean norm in  $\mathbf{R}^s$ . This interpolation scheme is stable, but if  $p > 1$ , the gradient of the interpolating function vanishes in all data points. The interpolation operator  $S_p^q$  is defined by replacing the values  $f(x_i)$  in  $S_p^0 f$  by Taylor polynomials of  $f$  of degree  $q \in \mathbf{N}$ . In this paper, we investigate the approximating power of  $S_p^q$  for all values of  $p$ ,  $q$  and  $s$ .

**1. Introduction.** In [12] D. Shepard introduced an interpolation scheme which is easily programmable and whose interpolating function can be written down explicitly. Given any arbitrarily spaced points  $x_1, \dots, x_n \in \mathbf{R}^s$  and values  $f(x_1), \dots, f(x_n)$  of a function  $f$ , the first version of Shepard's interpolation formula is given by

$$S_p^0 f(x) = \sum_{i=1}^n f(x_i) w_i(x)$$

with basis functions

$$w_i(x) = \frac{|x - x_i|^{-p}}{\sum_j |x - x_j|^{-p}}.$$

Here  $p > 0$ , and  $|\cdot|$  may denote any norm in  $\mathbf{R}^s$ , but for reasons of differentiability it is natural to use the Euclidean norm. The basis functions are not differentiable at the data points if  $p \leq 1$ ; otherwise, at least the first derivatives vanish. Generally, the parameter  $p$  is chosen to equal 2 so that, in addition, the basis functions are rational and infinitely differentiable. For a qualitative discussion of the parameter  $p$ , see Barnhill, Dube and Little [3], Gordon and Wixom [5], Lancaster and Salkauskas [7] and Poepfelmeier [10]. Since

$$w_i(x_j) = \delta_{ij}, \quad w_i(x) \geq 0 \quad \text{and} \quad \sum_i w_i(x) = 1,$$

the linear interpolation operator  $S_p^0$  is stable in the sense that

$$\min_i f(x_i) \leq S_p^0 f(x) \leq \max_i f(x_i).$$

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D. J. Newman and T. J. Rivlin [9] show that, if  $p = 2$ , then  $S_p^0$  is optimal in a certain sense among all ‘universally’ stable rational interpolation schemes. But the approximating power of a stable interpolation operator cannot exceed  $n^{-1/s}$ , as is shown in Section 3.

If  $p > 1$ , the interpolating function  $S_p^0 f$  has flat spots in the neighborhood of all data points. This drawback can be avoided by using Taylor polynomials of  $f$  of degree  $q$  instead of the values  $f(x_1), \dots, f(x_n)$ , if all partial derivatives  $D^\nu f(x_i)$  of  $f$  up to degree  $q$ , or approximations to them, are known:

$$S_p^q f(x) = \sum_{i=1}^n \sum_{|\nu| \leq q} \frac{1}{\nu!} D^\nu f(x_i) (x - x_i)^\nu w_i(x),$$

where  $\nu = (\nu_1, \dots, \nu_s)$  denotes a multi-index and  $|\nu| = \nu_1 + \dots + \nu_s$ . If  $q \geq 1$ ,  $S_p^q$  is no longer stable, but  $S_p^q f$  will approximate  $f$  (and its derivatives) much better than  $S_p^0 f$  does. Generalizations of Shepard’s global interpolation formula are described by Barnhill [2], McLain [6], Little [8], Schumaker [11] and in [3], [7] and [10].

In applications, the global character of Shepard’s interpolation formula is often undesirable. Further, the evaluation of  $S_p^q f(x)$  requires a considerable amount of work. These disadvantages are avoided by using local versions of Shepard’s formula. There the basic functions have ‘small’ compact support which may even depend on the local distribution of data points (see [2], [11] and [12]). Bash-Ayan [1] and Franke [4] tested Shepard’s global formula and several local versions and compared them to other multivariate interpolation schemes.

If the  $n$  data points are distributed in a homogeneous way throughout a given domain, and if  $n \rightarrow \infty$ , then  $S_p^q f$  will converge to the given function  $f$ . In [9], the rate of convergence of  $S_2^0$  is given in the univariate case. In this paper, we investigate the approximating power of  $S_p^q$  for all  $p$  and  $q$  and in every dimension  $s$ . Section 2 shows that  $p - s = q + 1$  or  $p - s > q + 1$  is a good choice of the parameter  $p$  if  $q$  is given. In Section 3, we prove (cf. [9], if  $s = 1$ ,  $q = 0$ ,  $p = 2$ ) that the results of our main theorem, Theorem 2.3, cannot be improved for any set of data points.

**2. Approximating Power of Shepard’s Global Interpolation Formula.** In the  $s$ -dimensional space  $\mathbf{R}^s$  let  $|\cdot|$  denote the Euclidean norm,  $\|\cdot\|$  the maximum norm, and let  $B_r(y)$  denote the closed cube  $\{x \in \mathbf{R}^s; \|x - y\| \leq r\}$  with center  $y$  and radius  $r$ . We use standard multi-index notation. In particular, given any multi-index  $\nu = (\nu_1, \dots, \nu_s) \in \mathbf{N}^s$ ,  $|\nu|$  denotes the sum  $\nu_1 + \dots + \nu_s$ , and not the Euclidean norm of the vector  $\nu$ .

Let  $f$  be a function sufficiently smooth in a domain  $A$  of  $\mathbf{R}^s$  such that the function values  $f(x_i)$  and all partial derivatives  $D^\nu f(x_i)$  up to some order  $q \in \mathbf{N}$  ( $|\nu| \leq q$ ) are known in  $n$  pairwise distinct data points  $x_1, \dots, x_n$ . Then, if  $p > 0$ , Shepard’s interpolation operator  $S_p^q$  is defined as

$$(2.1) \quad S_p^q f(x) = \sum_{i=1}^n \sum_{|\nu| \leq q} \frac{1}{\nu!} D^\nu f(x_i) (x - x_i)^\nu w_i(x)$$

with the  $i$ th basis function

$$(2.2) \quad w_i(x) = \frac{|x - x_i|^{-p}}{\sum_{j=1}^n |x - x_j|^{-p}} = \frac{\prod_{k \neq i} |x - x_k|^p}{\sum_{j=1}^n \prod_{k \neq j} |x - x_k|^p}.$$

Note that

$$w_i(x_j) = \delta_{ij}, \quad w_i(x) \geq 0, \quad \sum_{i=1}^n w_i(x) = 1,$$

and that  $w_i(x)$  is infinitely differentiable in  $\mathbf{R}^s$  except in the data points where all partial derivatives up to order  $p-1$  ( $p \in \mathbf{N}$ ) or  $[p]$  ( $p \notin \mathbf{N}$ ) exist and vanish. Thus, if  $q \leq p-1$  or  $q \leq [p]$ , the interpolating function  $S_p^q f$  is at least  $q$ -times continuously differentiable and

$$D^\nu S_p^q f(x_i) = D^\nu f(x_i) \quad (|\nu| \leq q, i = 1, \dots, n).$$

Assume that the domain  $A$  is compact and satisfies two conditions of regularity:

(A1) There is a number  $\gamma \geq 1$  such that any two points  $x$  and  $y$  in  $A$  are joined by a rectifiable curve  $\Gamma$  in  $A$  with length  $|\Gamma| \leq \gamma|x-y|$ .

(A2) There is a compact cone  $K$  such that for every  $x \in A$  there is a cone  $K(x) \subset A$  with vertex  $x$  and congruent with  $K$ .

Note that the cone  $K$  may be defined as  $K = \{\lambda y; y \in B_{r_1}(z), \lambda \geq 0\} \cap B_{r_2}(0)$  ( $r_1 > 0, r_2 > 0, 0 \notin B_{r_1}(z)$ ) and that the number  $\gamma$  in (A1) may be assumed to be an integer. The condition (A1) implies connectedness and excludes domains with too thin cusps directed into the interior  $\overset{\circ}{A}$  of  $A$ , while the cone property (A2) does not admit domains with too thin cusps directed outwards and guarantees that the closure of  $\overset{\circ}{A}$  equals  $A$ .

Next, we discuss some consequences of the condition (A1). According to H. Whitney [13], a function  $f: A \rightarrow \mathbf{R}$  is said to be of class  $C^q$  in  $A$  if and only if functions  $D^\nu f(x)$  and  $R_\nu(x; y)$  ( $|\nu| \leq q$ ) exist in  $A$  such that Taylor's formula holds in the following sense ( $x, y \in A, |\nu| \leq q$ )

$$(2.3) \quad D^\nu f(x) = \sum_{|\mu| \leq q-|\nu|} \frac{1}{\mu!} D^{\nu+\mu} f(y) (x-y)^\mu + R_\nu(x; y).$$

The remainder terms  $R_\nu(x; y)$  shall have the following property: Given any point  $z \in A$  and any  $\varepsilon > 0$  there is a  $\delta > 0$  such that

$$(2.4) \quad |R_\nu(x; y)| \leq \varepsilon |x-y|^{q-|\nu|} \quad \text{for every } x, y \in A, |x-z| < \delta, |y-z| < \delta.$$

(2.3) and (2.4) imply that  $f$  is continuous in  $A$  and, if  $q \geq 1$ , that  $f$  has continuous partial derivatives up to order  $q$  in the interior of  $A$  satisfying

$$D^\nu f(x) = \frac{\partial^{|\nu|} f(x)}{\partial x_1^{\nu_1} \dots \partial x_s^{\nu_s}}, \quad |\nu| \leq q, x \in A.$$

A function  $f$  is said to be of class  $C^{q,1}$  in  $A$  if and only if  $f$  is of class  $C^q$  in  $A$  and, additionally, the partial derivatives  $D^\nu f$  of  $f$  of order  $q$  ( $|\nu| = q$ ) are Lipschitz-continuous in  $A$ . In this case, the seminorm  $|\cdot|_{q,1}$  is defined as

$$|f|_{q,1} = \sup \left\{ \frac{|D^\nu f(x_1) - D^\nu f(x_2)|}{|x_1 - x_2|}; x_1, x_2 \in A, x_1 \neq x_2, |\nu| = q \right\}.$$

The results of H. Whitney [13] yield important estimates of the remainder terms  $R_\nu(x; y)$ .

LEMMA 2.1. *Let  $A$  satisfy (A1) and let  $f$  be of class  $C^{q,1}$  in  $A$ . Then, for every  $x, y \in A$ ,*

$$|R_\nu(x; y)| \leq c_{q-|\nu|} \gamma^{q-|\nu|} |x - y|^{q-|\nu|+1} |f|_{q,1},$$

where  $c_p = s^p / (p - 1)!$  ( $p > 0$ ) or  $c_p = 1$  ( $p = 0$ ).

If  $f$  is assumed to be only continuous in  $A$ , we will use its modulus of continuity  $\omega_f$ :

$$\omega_f(\delta) = \sup\{|f(x) - f(y)|; x, y \in A, |x - y| \leq \delta\}.$$

LEMMA 2.2. *Let  $A$  satisfy (A1), and let  $f \in C^0(A)$ . Then, for arbitrary positive numbers  $0 < \varepsilon < \delta$ ,*

$$\omega_f(\delta) \leq 2\gamma \frac{\delta}{\varepsilon} \omega_f(\varepsilon).$$

*Proof.* Let  $m$  be the integer defined by  $m\varepsilon < \delta \leq (m + 1)\varepsilon$ . Then, it is sufficient to prove that  $\omega_f((m + 1)\varepsilon) \leq \gamma(m + 1)\omega_f(\varepsilon)$  because  $m + 1 \leq 2\delta/\varepsilon$ . Let  $x, y \in A$  with  $|x - y| \leq (m + 1)\varepsilon$  and let  $\Gamma$  be a rectifiable curve in  $A$  joining  $x$  and  $y$  with length  $|\Gamma| \leq \gamma|x - y|$ . Parameterizing  $\Gamma$  by  $x(s) \in C^0[0, |\Gamma|]$ , where  $s$  denotes the length of the part of  $\Gamma$  between  $x$  and  $x(s)$ , we partition the interval  $[0, |\Gamma|]$  into  $\gamma(m + 1)$  consecutive parts  $[s_{i-1}, s_i]$  of length  $|s_i - s_{i-1}| \leq \delta$ . Noting that  $x = x(0)$  and  $y = x(|\Gamma|)$ , we get

$$|f(x) - f(y)| \leq \sum_{i=1}^{\gamma(m+1)} |f(x(s_i)) - f(x(s_{i-1}))| \leq \gamma(m + 1)\omega_f(\delta). \quad \square$$

Let  $X$  be a set of  $n$  pairwise distinct data points  $x_1, \dots, x_n$  in  $A$ . We set

$$r = \inf\{\rho > 0; \text{for every } x \in A, B_\rho(x) \text{ contains at least one element of } X\}$$

and

$$M = \sup_y \text{card}(B_r(y) \cap X).$$

Analogously to the half-open unit interval  $(0, 1]$ , the half-open unit cube in  $\mathbf{R}^s$  is defined as  $(0, 1]^s$ . Multiplying by  $2r$  and shifting, we get the half-open cube  $Q_r(x)$  with center  $x$  and radius  $r$  (with respect to the maximum norm). The finiteness of  $X$  implies for every  $x \in A$  that  $B_r(x)$  and  $Q_r(x)$ , if additionally  $Q_r(x) \subset \overset{\circ}{A}$ , contain at least one element of  $X$ . Our main theorem, Theorem 2.3, yields an estimate of  $\|S_p^q f - f\|_A$ , the supremum of  $S_p^q f - f$  in  $A$ .

THEOREM 2.3. *Let the compact domain  $A$  satisfy condition (A1) and, if  $p = s$ , also (A2) and let  $X = \{x_1, \dots, x_n\}$  be a set of  $n$  data points. If  $f$  is of class  $C^{q,1}$  in  $A$ , then*

$$(2.5) \quad \|S_p^q f - f\|_A \leq C\gamma^q M |f|_{q,1} \varepsilon_p^q(r),$$

where

$$(2.6) \quad \varepsilon_p^q(r) = \begin{cases} |\log r|^{-1}, & p = s, \\ r^{p-s}, & p - s < q + 1, p > s, \\ r^{p-s} |\log r|, & p - s = q + 1, \\ r^{q+1}, & p - s > q + 1, \end{cases}$$

and  $C$  is a positive constant independent of  $f$ ,  $\gamma$  and  $X$ . If  $f \in C^0(A)$ , then

$$(2.7) \quad \|S_p^0 f - f\|_A \leq C\gamma M\omega_f(\varepsilon_p^0(r)).$$

*Proof.* Define

$$(2.8) \quad s_p^q(x) = \frac{\sum_{i=1}^n \|x - x_i\|^{q+1-p}}{\sum_{i=1}^n \|x - x_i\|^{-p}}.$$

If  $f \in C^0(A)$ , then by Lemma 2.2 ( $\varepsilon(r) = \varepsilon_p^0(r)$ )

$$\begin{aligned} |(S_p^q f - f)(x)| &\leq \sum_{|x-x_i| \leq \varepsilon(r)} \omega_f(\varepsilon(r)) w_i(x) + \sum_{|x-x_i| > \varepsilon(r)} |f(x_i) - f(x)| w_i(x) \\ &\leq \omega_f(\varepsilon(r)) \left( 1 + \frac{2\gamma}{\varepsilon(r)} \sum_i |x - x_i| w_i(x) \right) \\ &\leq \omega_f(\varepsilon(r)) \gamma \left( 1 + \frac{C}{\varepsilon(r)} s_p^0(x) \right). \end{aligned}$$

If  $f \in C^{q,1}(A)$ , then by (2.3) and Lemma 2.1,

$$|(S_p^q f - f)(x)| \leq \sum_i |R_0(x; x_i)| w_i(x) \leq c_q \gamma^q |f|_{q,1} s_p^q(x).$$

Thus, it remains to be shown that for fixed  $x \in A$ ,  $x \neq x_i$  ( $1 \leq i \leq n$ ),

$$(2.9) \quad s_p^q(x) \leq MC\varepsilon_p^q(r)$$

with a positive constant  $C$  independent of  $x$  and  $X$ .

In order to prove (2.9) we need a disjoint covering of  $A$  by half-open cubes. Let  $T_j = T_{j,r}(x)$  be the half-open annulus with center  $x$  and radius  $2jr$  defined by

$$T_j = \bigcup_{\substack{k \in \mathbf{Z}^s \\ \|k\|=j}} Q_r(x + 2rk).$$

Note that  $T_0 = Q_r(x)$  and that

$$A \subset \bigcup_{j=0}^N T_j \quad \text{with} \quad N = \left\lceil \frac{d(A)}{2r} \right\rceil + 1,$$

where  $d(A) > 0$  denotes the diameter  $\sup\{\|y - z\|; y, z \in A\}$  of  $A$ . Evidently, the number of half-open cubes  $Q_r(\cdot)$  contained in  $T_j$  lies between  $2s(2j-1)^{s-1}$  and  $2s(2j+1)^{s-1}$ , thus

$$\text{card}(X \cap T_{j,r}(x)) \leq 2sM(2j+1)^{s-1} \quad (1 \leq j \leq N).$$

Further, note that

$$(2j-1)r \leq \|x - x_i\| \leq (2j+1)r \quad \text{for every } x_i \in X \cap T_{j,r}(x).$$

If (A2) is satisfied, we get a lower bound on  $\text{card}(X \cap T_j)$ . For there are a positive number  $r_0$  and positive integers  $j_0$  and  $j_r \geq \max(r_0/r, j_0)$  (each of them independent of  $x$ ) such that the number of half-open cubes  $Q_r(x + 2rk)$  with  $\|k\|=j$  contained in  $\dot{A} \cap K(x)$  is bounded below by  $[j/j_0]^{s-1}$  for every  $j_0 \leq j \leq j_r$ ,  $r \leq r_0$ . Thus,

$$\begin{aligned} \text{card}(X \cap T_{j,r}(x)) &\geq \left[ \frac{j}{j_0} \right]^{s-1} \quad \text{for every } j_0 \leq j \leq j_r, r \leq r_0 \\ &\quad \text{where } j_r \geq \max\left(\frac{r_0}{r}, j_0\right). \end{aligned}$$

Note that, if  $A$  is not empty and if  $M$  is bounded as  $n \rightarrow \infty$ ,  $r \rightarrow 0$ , then  $n = \text{card } X$  is of exact order  $r^{-s}$ . Defining the point  $x_{i_0} \in X$  by

$$\|x - x_{i_0}\| = \min_i \|x - x_i\|,$$

we get

$$(2.10) \quad \begin{aligned} s_p^q(x) &\leq \|x - x_{i_0}\|^p \left( \sum_{x_i \in T_0} \|x - x_i\|^{q+1-p} + \sum_{j=1}^N \sum_{x_i \in T_j} \|x - x_i\|^{q+1-p} \right) \\ &\leq Mr^{q+1} \left( 1 + C \sum_{j=1}^N j^{s-p+q} \right). \end{aligned}$$

*Case 1* ( $p > s$ ). If either  $p - s > q + 1$  or  $p - s = q + 1$  or  $p - s < q + 1$ , then  $\sum j^{s-p+q}$  is either bounded or diverges with  $O(\log N) = O(|\log r|)$  or diverges with  $O(N^{s-p+q+1}) = O(r^{p-s-q-1})$ . Thus (2.10) yields (2.9).

*Case 2* ( $p = s$ ). In this case we must examine the denominator of  $s_p^q(x)$  more carefully. Since  $\sum_{j_0}^j 1/j = O(\log j_r) = O(|\log r|)$ ,

$$\begin{aligned} \sum_{j=0}^N \sum_{x_i \in T_0} \|x - x_i\|^{-p} &\geq \sum_{x_i \in T_0} \|x - x_i\|^{-p} + \sum_{j=j_0}^{j_r} \left[ \frac{j}{j_0} \right]^{s-1} ((2j+1)r)^{-p} \\ &\geq \sum_{x_i \in T_0} \|x - x_i\|^{-p} + \frac{C}{r^p} |\log r|. \end{aligned}$$

Using the inequality

$$\frac{\sum a_i}{\sum b_i} \leq \sum \frac{a_i}{b_i} \quad (a_i \geq 0, b_i > 0),$$

we get

$$\begin{aligned} s_p^q(x) &\leq \sum_{x_i \in T_0} \|x - x_i\|^{q+1} + C_1 \frac{r^p}{|\log r|} \sum_{j=1}^N \sum_{x_i \in T_j} \|x - x_i\|^{q+1-p} \\ &\leq Mr^{q+1} \left( 1 + C_2 \frac{1}{|\log r|} \sum_{j=1}^N j^q \right). \end{aligned}$$

Finally, since

$$\sum_{j=1}^N j^q = O(N^{q+1}) = O(r^{-q-1}),$$

this last estimate yields (2.9).  $\square$

**THEOREM 2.4.** *Assume (A1) and let  $f$  be of class  $C^{q,1}$  in  $A$ ,  $q \geq 1$  and  $p > s + 1$ . Then, the first partial derivatives of  $S_p^q f$  converge to  $f$  uniformly in  $A$  as  $r \rightarrow 0$ . More precisely, if  $\mu$  denotes a multi-index with  $|\mu| = 1$ , then*

$$\|D^\mu(S_p^q f - f)\|_A \leq C\gamma^q M^2 |f|_{q,1} \frac{\varepsilon_p^q(r)}{r}.$$

*Proof.* Given any  $x \in A$ ,

$$\begin{aligned} D^\mu S_p^q f(x) - D^\mu f(x) &= \sum_i \left( \sum_{|\nu| \leq q-1} \frac{1}{\nu!} D^{\nu+\mu} f(x_i) (x-x_i)^\nu - D^\mu f(x) \right) w_i(x) \\ &\quad - \sum_i R_0(x; x_i) D^\mu w_i(x) \\ &= - \sum_i R_\mu(x; x_i) w_i(x) - \sum_i R_0(x; x_i) D^\mu w_i(x). \end{aligned}$$

Since

$$D^\mu w_i(x) = -p \frac{(x-x_i)^\mu |x-x_i|^{-p-2}}{\sum_j |x-x_j|^{-p}} + p \frac{1}{|x-x_i|^p} \frac{\sum_j (x-x_j)^\mu |x-x_j|^{-p-2}}{\left(\sum_j |x-x_j|^{-p}\right)^2},$$

Lemma 2.1 yields the estimate

$$\left| D^\mu (S_p^q f - f)(x) \right| \leq C \|f\|_{q,1} \gamma^q (s_p^{q-1}(x) + s_p^q(x) s_p^{-2}(x)).$$

Here  $s_p^{-2}(x)$  is defined by (2.8) with  $q = -2$ , and  $x \neq x_i$  ( $1 \leq i \leq n$ ). We also have

$$\begin{aligned} s_p^{-2}(x) &= \frac{\sum_i \|x-x_i\|^{-p-1}}{\sum_i \|x-x_i\|^{-p}} \\ &\leq \frac{1}{\|x-x_{i_0}\|} M \left( 1 + C_1 \sum_{j=1}^N j^{s-p-2} \right) \leq \frac{MC_2}{\|x-x_{i_0}\|}. \end{aligned}$$

Looking at the proof of the estimate (2.9) (see (2.10)), we have

$$s_p^q(x) s_p^{-2}(x) \leq CM^2 \frac{\varepsilon_p^q(r)}{r}.$$

Since for arbitrary values of  $p$ ,  $s$  and  $q$

$$s_p^{q-1}(x) \leq C_1 M \varepsilon_p^{q-1}(r) \leq C_2 M \frac{\varepsilon_p^q(r)}{r},$$

the theorem is proved.  $\square$

*Remark 2.5.* A similar result holds for higher derivatives: Given any multi-index  $\mu$  with  $|\mu| \leq q$  and  $p > s + |\mu|$ ,

$$\left\| D^\mu (S_p^q f - f) \right\|_A \leq C \gamma^q M^{|\mu|} \|f\|_{q,1} \frac{\varepsilon_p^q(r)}{r^{|\mu|}}.$$

Generally, the derivatives of  $f$  in the data points are not known. The next theorem guarantees the same rate of convergence if all derivatives needed are approximated to the right order.

**COROLLARY 2.6.** Assume (A1), let  $f$  be of class  $C^{q,1}$  in  $A$  and let  $d^\nu f(x_i)$  be approximations to  $D^\nu f(x_i)$ ,  $|\nu| \leq q$ ,  $x_i \in X$ , such that

$$d^\nu f(x_i) = D^\nu f(x_i) + O(r^{q+1-|\nu|}), \quad |\nu| \leq q, \quad r \rightarrow 0.$$

If  $\tilde{S}_p^q$  denotes the interpolation operator

$$\tilde{S}_p^q f(x) = \sum_i \sum_{|\nu| \leq q} \frac{1}{\nu!} d^\nu f(x_i) (x-x_i)^\nu w_i(x),$$

then

$$\|\tilde{S}_p^q f - f\|_A = M \cdot O(\epsilon_p^q(r)), \quad r \rightarrow 0.$$

*Proof.* It suffices to consider  $S_p^q f - \tilde{S}_p^q f$ . Now

$$\begin{aligned} |(S_p^q f - \tilde{S}_p^q f)(x)| &\leq C_0 \sum_{|\nu| \leq q} r^{q+1-|\nu|} \sum_i \|x - x_i\|^{|\nu|} w_i(x) \\ &\leq C_1 \sum_{|\nu| \leq q} r^{q+1-|\nu|} s_p^{|\nu|-1}(x). \end{aligned}$$

Since by the definition (2.8)  $s_p^{-1}(x) \equiv 1$ , let  $\epsilon_p^{-1}(r) \equiv 1$ , apply (2.9) and note that for all  $|\nu| \leq q$ ,  $r^{q+1-|\nu|} \epsilon_p^{|\nu|-1}(r) = O(\epsilon_p^q(r))$ .  $\square$

*Remark 2.7.* Analogously to Remark 2.5,

$$\|D^\mu(\tilde{S}_p^q f - f)\|_A = M^{|\mu|} \cdot O(\epsilon_p^q(r)/r^{|\mu|}), \quad r \rightarrow 0.$$

**3. Negative Results.** Theorem 2.3 gives no information if  $p < s$ . In this case Shepard’s formula fails to converge in the sense that there is no function  $\epsilon_p^q(r)$  such that  $\epsilon_p^q(r) \rightarrow 0$  as  $r \rightarrow 0$  and such that (2.5) holds. Theorem 3.2 deals with this case and, further, shows that Theorem 2.3 cannot be improved. In the following we will use the test functions

$$g_y^q(x) = |x - y|^{q+1} \quad (g_y = g_y^0),$$

some properties of which are listed in the next lemma.

**LEMMA 3.1.** *The function  $g_y^q$  is of class  $C^{q,1}$  in  $\mathbf{R}^s$ . If  $q = 0$ , then  $\omega_{g_y}(\delta) = \delta$  ( $\delta > 0$ ). If  $q \geq 1$ , then all partial derivatives of  $g_y^q$  exist at every point  $x \neq y$ , and up to order  $q + 1$  they are continuous and bounded functions in  $\mathbf{R}^s \setminus \{y\}$ . Further,*

$$\sum_{|\nu| \leq q} \frac{1}{\nu!} D^\nu g_y^q(x) (y - x)^\nu = (-1)^q |x - y|^{q+1}.$$

*Proof.* Let us only prove the last identity for the case  $y = 0$ . Using polar coordinates with  $r = r(x) = |x|$ , it is easily seen that for every  $j = 1, \dots, q$ ,

$$\begin{aligned} \frac{1}{j!} \frac{\partial^j}{\partial r^j} g_0^q(x) &= \frac{1}{j!} \sum_{i_1, \dots, i_j=1}^s \frac{\partial^j g_0^q(x)}{\partial x_{i_1} \cdots \partial x_{i_j}} \frac{\partial x_{i_1}}{\partial r} \cdots \frac{\partial x_{i_j}}{\partial r} \\ &= \frac{1}{r^j} \sum_{|\nu|=j} \frac{1}{\nu!} D^\nu g_0^q(x) x^\nu. \end{aligned}$$

Now insert this identity into Taylor’s formula for  $p(r) = r^{q+1} = g_0^q(x)$ :

$$0 = p(0) = \sum_{j=0}^q \frac{1}{j!} p^{(j)}(r) (-r)^j + (-r)^{q+1}. \quad \square$$

**THEOREM 3.2.** *Let the compact domain  $A \subset [0, 1]^s$  satisfy the cone property (A2), let  $X(n)$  denote an arbitrary set of  $n$  pairwise distinct data points  $\{x_1, \dots, x_n\}$  in  $A$  and let  $r = n^{-1/s}$ . Then, there is a positive constant  $c$  dependent only on  $p, s, q$  and  $A$  with the following property: given any set  $X(n)$  ( $n$  sufficiently large) there is a function  $g$  of class  $C^{q,1}$  in  $A$  such that*

$$(3.1) \quad \|S_p^q g - g\|_A \geq c |g|_{q,1} \epsilon_p^q(r).$$



Here  $\epsilon_p^q(r)$  is defined by (2.6) if  $p \geq s$  and  $\epsilon_p^q(r) = 1$  if  $p < s$ . Analogously, there is a function  $g \in C^0(A)$  such that

$$(3.2) \quad \|S_p^0 g - g\|_A \geq c \omega_g(\epsilon_p^0(r)).$$

*Proof.* The function  $g$  will be the test function  $g_y^q(x) = |x - y|^{q+1}$ , where the choice of the point  $y \in A$  depends on the set  $X(n)$ . Since  $|g_y^q|_{q,1}$  is bounded independently of  $y \in A$ , Lemma 3.1 implies that

$$\begin{aligned} \|S_p^q g_y^q - g_y^q\|_A &\geq |S_p^q g_y^q(y)| = \sum_i |y - x_i|^{q+1} w_i(y) \\ &\geq c_1 s_p^q(y) \geq c_0 |g_y^q|_{q,1} s_p^q(y) \end{aligned}$$

with positive constants  $c_0$  and  $c_1$ . Analogously, for  $g_y \in C^0(A)$ ,

$$\|S_p^0 g_y - g_y\|_A \geq c s_p^0(y) = c \omega_{g_y}(s_p^0(y)).$$

Thus, it suffices to show the existence of a positive constant  $c$  such that for every set  $X(n)$  ( $n$  sufficiently large) there is a point  $y \in A \setminus X(n)$  satisfying the inequality

$$s_p^q(y) \geq c \epsilon_p^q(r).$$

Let  $N(n)$  be an increasing function

$$N: \mathbf{N} \rightarrow \mathbf{N} \quad \text{with } N(n) \rightarrow \infty \text{ as } n \rightarrow \infty,$$

which will be defined later. By virtue of the transformation  $x \mapsto N(n)x$  it remains to show that

$$(3.3) \quad s_p^q(y) \geq c N^{q+1} \epsilon_p^q(r),$$

where now  $y$  and the data points  $x_1, \dots, x_n$  range through the set  $A_N = N(n)A \subset [0, N(n)]^s$ . If  $k \in \mathbf{N}^s$ , let  $Q(k)$  denote the half-open cube of radius  $\frac{1}{2}$  and center  $k$  and note that

$$A_N \subset \bigcup_{\substack{k \in \mathbf{N}^s \\ 0 \leq \|k\| \leq N}} Q(k).$$

Following the ideas of D. J. Newman and T. J. Rivlin [9] for the univariate case, we construct a point  $y \in A_N \setminus X(n)$  suitable for our purpose. Let

$$m = \text{card}(Q(l) \cap X(n)) = \min_{Q(k) \subset A_N} \text{card}(Q(k) \cap X(n)).$$

Since the number of half-open cubes  $Q(k) \subset A_N$  increases asymptotically like  $\mu(A_N) = \mu(A)N^s$ , as  $N \rightarrow \infty$  ( $\mu(A)$  denotes the Lebesgue measure of  $A$  and is positive because of (A2)),

$$(3.4) \quad m \leq \frac{n}{\frac{1}{2}\mu(A)N^s} \quad (N \text{ sufficiently large}).$$

Assume  $m \geq 1$ , the trivial case  $m = 0$  being considered later. Let

$$T = \left\{ x \in Q(l); \|x - l\| \leq \frac{1}{4}, \|x - x_i\| \geq \frac{1}{4}(2m)^{-1/s} \text{ for every } x_i \in X(n) \right\}$$

and note that

$$(3.5) \quad \mu(T) \geq \left(\frac{1}{2}\right)^s - m \left(\frac{1}{4}(2m)^{-1/s}\right)^s = 1/2^{s+1}.$$

In order to simplify the notation, define

$$\sigma_1(y) = \sum_{x_i \in Q(l)} \|y - x_i\|^{-p} \quad \text{and} \quad \sigma_2(y) = \sum_{x_i \notin Q(l)} \|y - x_i\|^{q+1-p}.$$

Case 1 ( $p > s$ ). Since

$$\begin{aligned} \int_T d\mu(x) \sum_{x_i \in Q(l)} |x - x_i|^{-p} &\leq m \int_{|x| \geq (2m)^{-1/s}/4} d\mu(x) |x|^{-p} \\ &= cm \int_{(2m)^{-1/s}/4}^{\infty} dr r^{s-p-1} = \frac{cm}{p-s} (4(2m)^{1/s})^{p-s}, \end{aligned}$$

$c$  being a positive constant arising from the introduction of polar coordinates, (3.5) guarantees the existence of a point  $y \in T$  such that

$$(3.6) \quad \sigma_1(y) \leq cm^{p/s} \quad (p > s).$$

Case 2 ( $p = s$ ). Since  $|x - x_i| \leq \sqrt{s}$  for every  $x \in T$  and  $x_i \in Q(l)$ ,

$$\begin{aligned} \int_T d\mu(x) \sum_{x_i \in Q(l)} |x - x_i|^{-p} &\leq m \int_{(2m)^{-1/s}/4 \leq |x| \leq \sqrt{s}} d\mu(x) |x|^{-p} \\ &= c_1 m \int_{(2m)^{-1/s}/4}^{\sqrt{s}} \frac{dr}{r} \leq c_2 m (1 + \log m). \end{aligned}$$

Thus, (3.5) yields the existence of a point  $y \in T$  such that

$$(3.7) \quad \sigma_1(y) \leq cm(1 + \log m) \quad (p = s).$$

Case 3 ( $p < s$ ).

$$\int_T d\mu(x) \sum_{x_i \in Q(l)} |x - x_i|^{-p} \leq mc_1 \int_0^{\sqrt{s}} dr r^{s-p-1} = mc_2,$$

thus there is a point  $y \in T$  such that

$$(3.8) \quad \sigma_1(y) \leq cm \quad (p < s).$$

If  $m = 0$ , the inequalities (3.6), (3.7) and (3.8) obviously hold for every  $y \in T$ :  $\sigma_1(y) = 0$ .

Following the proof of Theorem 2.3, let  $T_j$  be the half-open annulus with center  $l$  and radius  $j$  defined by

$$T_j = \bigcup_{\substack{k \in \mathbb{Z}^s \\ \|k-l\|=j}} Q(k).$$

Note that  $\frac{1}{4}j \leq \|y - x_i\| \leq 2j$  for every  $x_i \in X(n) \cap T_j$ , while  $\|y - x_i\| \geq \frac{1}{4}$  for every  $x_i \in X(n)$ . By virtue of (A2), we get a lower bound on  $\text{card}(X(n) \cap T_j)$ . For there are positive integers  $N_0$ ,  $j_0$  and  $j(N) \geq \max(N/N_0, j_0)$  (each of them independent of  $y$ ) such that the number of half-open cubes  $Q(k)$  with  $\|k - l\| = j$  contained in  $N \cdot K(y) \subset A_N$  is bounded below by  $[j/j_0]^{s-1}$  for every  $j_0 \leq j \leq j(N)$ ,  $N \geq N_0$ . Thus, by the definition of  $m$ ,

$$\begin{aligned} \text{card}(X(n) \cap T_j) &\geq m [j/j_0]^{s-1} \quad \text{for } j_0 \leq j \leq j(N), N \geq N_0 \\ &\quad \text{where } j(N) \geq \max(N/N_0, j_0). \end{aligned}$$

Hence

$$(3.9) \quad \sigma_2(y) \geq cm \sum_{j=j_0}^{j(N)} j^{s-p+q},$$

and, obviously,

$$(3.10) \quad s_p^q(y) \geq \frac{\sigma_2(y)}{\sigma_1(y) + 4^{q+1}\sigma_2(y)}.$$

Note that  $t/(a + bt)$  is strictly increasing in  $t > 0$  ( $a, b > 0$ ).

Case 1 ( $p > s$ ). If  $m > 0$ , insert (3.6) and (3.9) in (3.10) and cancel  $m$ . Then (3.4) implies that

$$(3.11) \quad s_p^q(y) \geq c \frac{\sum_{j=j_0}^{j(N)} j^{s-p+q}}{(n^{1/s}/N)^{p-s} + \sum_{j=j_0}^{j(N)} j^{s-p+q}}.$$

Case 1.1 ( $p - s > q + 1$ ). Define  $N(n) = [n^{1/s}]$ . Since  $\sum j^{s-p+q}$  is bounded, by (3.11),

$$s_p^q(y) \geq c \geq c \frac{N^{q+1}}{(n^{1/s})^{q+1}} = cN^{q+1}\epsilon_p^q(r).$$

Hence, (3.3) is proved if  $m \neq 0$ . Otherwise, (3.10) implies that  $s_p^q(y) \geq 4^{-q-1} \geq cN^{q+1}\epsilon_p^q(r)$ .

Case 1.2 ( $p - s = q + 1$ ). Define  $N(n) = [n^{1/s}/\log n]$  and note that asymptotically

$$\log N \sim \frac{1}{s} \log n, \quad (\log n)^{-q} \sim N^{q+1} \frac{\log n}{(n^{1/s})^{q+1}}$$

and that

$$\sum_{j=j_0}^{j(N)} j^{s-p+q} \geq c_1 \log j(N) \geq c_2 \log N.$$

Thus (3.11) implies that

$$s_p^q(y) \geq c_1 \frac{1}{(\log n)^{p-s-1} + 1} \geq c_2 (\log n)^{-q} \geq c_3 N^{q+1} \epsilon_p^q(r)$$

if  $m \neq 0$ . Otherwise, by (3.10),

$$s_p^q(y) \geq 4^{-q-1} \geq c_2 (\log n)^{-q} \geq c_3 N^{q+1} \epsilon_p^q(r).$$

Case 1.3 ( $p - s < q + 1$ ). Note that

$$\sum_{j=j_0}^{j(N)} j^{s-p+q} \geq cN^{s-p+q+1},$$

hence, by (3.11),

$$s_p^q(y) \geq c \frac{N^{q+1}}{(n^{1/s})^{p-s} + N^{q+1}}.$$

Now define  $N(n) = [(n^{1/s})^{(p-s)/(q+1)}]$  in order to get

$$s_p^q(y) \geq cN^{q+1} \frac{1}{(n^{1/s})^{p-s}} = cN^{q+1} \epsilon_p^q(r)$$

if  $m \neq 0$ . Otherwise (3.10) yields the same result.

Case 2 ( $p = s$ ). This time,

$$\sum_{j=j_0}^{j(N)} j^{s-p+q} \geq cN^{q+1}.$$

If  $m > 0$  insert (3.7) and (3.9) in (3.10) and cancel  $m$ . Then (3.4) implies that

$$s_p^q(y) \geq c \frac{N^{q+1}}{1 + \log \frac{n^{1/s}}{N} + N^{q+1}}.$$

Define  $N(n) = [(\log n)^{1/(q+1)}]$  such that

$$\log \frac{n^{1/s}}{N} \sim \frac{1}{s} \log n.$$

Hence,

$$s_p^q(y) \geq c \geq c \frac{N^{q+1}}{\log n} = cN^{q+1} \epsilon_p^q(r),$$

the same result being true if  $m = 0$ .

Case 3 ( $p < s$ ). Insert (3.8) and (3.9) in (3.10) to get

$$s_p^q(y) \geq 2c \frac{\sum_{j=j_0}^{j(N)} j^{s-p+q}}{1 + \sum_{j=j_0}^{j(N)} j^{s-p+q}} \geq c$$

if  $m \neq 0$  and  $N$  is sufficiently large. If  $m = 0$ ,  $s_p^q(y) \geq c$  as well. Thus, in this case, the function  $N(n)$  is defined as a constant  $N_1$  sufficiently large, and we get (3.1) and (3.2) ( $c > 0$ ):

$$\|S_p^q g_y^q - g_y^q\|_A \geq c |g_y^q|_{q,1}, \quad \|S_p^q g_y - g_y\|_A \geq c \omega_{g_y}(1). \quad \square$$

*Remark 3.3.* The estimate (3.1) is trivial if  $p - s > q + 1$ . Since  $\mu(A) > 0$ , there is a positive constant  $c$  such that for all sets  $X(n)$  of  $n$  data points of  $A$  there is a point  $y \in A$  with

$$\|y - x_i\| \geq \frac{c}{n^{1/s}} \quad \text{for all } x_i \in X(n).$$

If  $v_i(x) \geq 0$ ,  $\sum v_i \equiv 1$  and  $v_i(x_j) = \delta_{ij}$ , then the approximating power of the interpolating operator

$$S^q f(x) = \sum_{i=1}^n \sum_{|\nu| \leq q} \frac{1}{\nu!} D^\nu f(x_i) (x - x_i)^\nu v_i(x)$$

cannot exceed  $r^{q+1}$ . For, if  $q \geq 1$ ,

$$\|S^q g_y^q - g_y^q\|_A \geq \sum_i |y - x_i|^{q+1} v_i(x) \geq cr^{q+1} \geq c_1 |g_y^q|_{q,1} r^{q+1}.$$

Analogously, the estimate  $\|S^0 g_y - g_y\|_A \geq c \omega_{g_y}(r)$  shows that there exists no stable interpolation operator  $S^0$  in  $C^0(A)$  whose approximating power is better than  $O(n^{-1/s})$ .

*Example 3.4.* Let  $A$  be the following two-dimensional compact domain which fails to satisfy the cone property (A2):

$$A = \{(x, y) \in [0, 1]^2; x \leq y^2\}.$$

For every  $N \in \mathbb{N}$ , consider the set  $X$  of data points

$$X = \left\{ (x_i, y_j) = \left( \frac{i}{N}, \frac{j}{N} \right) \in A; i, j = 0, 1, \dots, N \right\},$$

thus  $r = 1/2N$  and  $M = 4$ . We note that if  $j < \sqrt{N}$ , no data point except for  $(0, y_j)$  lies on the straight line  $y = y_j$ , and if  $j \geq \sqrt{N}$ , exactly  $[j^2/N] + 1$  data points will do. Thus  $\text{card } X$  is asymptotic to  $\frac{1}{3}N^2$ . Let  $f(x, y) = g_{(0,r)}^q(x, y)$  and apply Lemma (3.1) to get

$$\left| (S_p^q f - f)(0, r) \right| \geq c_1 r^{q+1} \frac{\sum_{1 \leq j < \sqrt{N}} j^{q+1-p} + r \sum_{\sqrt{N} \leq j \leq N} j^{q+3-p}}{\sum_{1 \leq j < \sqrt{N}} j^{-p} + r \sum_{\sqrt{N} \leq j \leq N} j^{2-p}}.$$

The preceding inequality holds with the direction reversed, if  $c_1$  is replaced by a positive constant  $c_2$ . If  $p = 2$  and  $r \rightarrow 0$ , the denominator is bounded, while the numerator behaves like  $r^{-q-1}$ . Thus Shepard's formula fails to converge like  $\varepsilon_2^q(r) = |\log r|^{-1}$ :

$$\|S_2^q f - f\|_A \geq c \quad (r \rightarrow 0). \quad \square$$

*Example 3.5.* Let  $A$  be the unit interval  $[0, 1] \subset \mathbb{R}^1$ , and let  $(j_n)$  be a sequence in  $\mathbb{N}$  converging to  $+\infty$  ( $n \rightarrow \infty$ ) such that  $1 < j_n < n$  and  $M_n = [j_n^p] = O(n)$ . The set  $X$  will be defined as follows:  $x_i = i/n$  for  $1 \leq i \leq n$ , while all the data points  $x_i$  for  $n < i \leq n + M_n$  are arbitrarily spaced in the open interval  $(x_{j_n}, x_{j_n+1})$ , thus  $r = 1/2n$  and  $M = M_n + 2$ . Then Lemma 3.1 implies that  $(f = g^q)$

$$\begin{aligned} \left| (S_p^q f - f)(y) \right| &\geq \frac{\sum_{i=3}^{n+M_n} |x - x_i|^{q+1-p}}{2(1/r)^p + \sum_{i=3}^{n+M_n} |x - x_i|^{-p}} \\ &\geq cr^{q+1} \frac{\sum_{i=3}^n i^{q+1-p} + M_n j_n^{q+1-p}}{\sum_{i=3}^n i^{-p} + M_n j_n^{-p}}. \end{aligned}$$

*Case 1.1* ( $p - 1 > q + 1$ ). Since  $\sum i^{-p}$  and  $\sum i^{q+1-p}$  are bounded, we get

$$\|S_p^q f - f\|_A \geq c |f|_{q,1} \varepsilon_p^q(r) M_n^{(q+1)/p} \quad (r \rightarrow 0, n \rightarrow \infty).$$

*Case 1.2* ( $p - 1 = q + 1$ ).  $\sum i^{-p}$  is bounded, while  $\sum i^{q+1-p}$  diverges like  $\log n$ . Choosing  $(j_n)$  such that

$$\frac{M_n^{(q+1)/p}}{\log n} \rightarrow +\infty \quad \text{if } n \rightarrow \infty,$$

we get

$$\|S_p^q f - f\|_A \geq c |f|_{q,1} \varepsilon_p^q(r) \frac{M_n^{(q+1)/p}}{\log n} \quad (r \rightarrow 0, n \rightarrow \infty).$$

In both cases Shepard's formula fails to converge like  $\varepsilon_p^q(r)$ , thus showing that the factor  $M$  in (2.5) and (2.7) is inevitable.  $\square$

*Remark 3.6.* If  $f(x, y) = y^{q+1}$  in Example 3.3 and  $f(x) = x^{q+1}$  in Example 3.4, we get the same estimates as before. Essentially, this works because each term except the first in the numerator of  $S_p^q f - f$ , evaluated in  $(0, r)$  (Example 3.3) or in  $r$  (Example 3.4), has the same sign. This argument shows that the results of Theorem 2.3 cannot be improved for functions of higher differentiability, not even for polynomials.

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