MATHEMATICS OF COMPUTATION
VOLUME 46, NUMBER 174
APRIL 1986, PAGES 609-611

A Conjecture of Frobenius and the Sporadic Simple Groups, II*

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Abstract. A conjecture of Frobenius which has been reduced to the classification of finite simple groups is verified for the sporadic simple groups.

Let G be a finite group and n be a positive integer dividing $|G|$. Let $L_n(G) = \{x \in G \mid x^n = 1\}$. Then by a theorem of Frobenius [6] one knows that $|L_n(G)| = c_n n$ for some integer c_n. Frobenius conjectured that $L_n(G)$ forms a subgroup of G provided $|L_n(G)| = n$ (see [2]). Zemlin [25] has reduced the conjecture to the classification of finite simple groups which is now complete (see [8]). The author has verified the conjecture for the Fischer Griess monster F_1 and the Fischer baby monster F_2 in [24].

The purpose of this note is to prove the following

THEOREM. *The conjecture of Frobenius is true for all the sporadic simple groups.*

The proof of our theorem has been carried out in the following way with the use of a computer. Let G be one of the sporadic simple groups. By [24] we may assume that $G \neq F_1$ and $G \neq F_2$. Let $f(G,t)$ be the number of elements of order t in G and $\mathrm{Ord}(G) = \{\text{order of } x \mid x \in G\}$. Tables of $f(G,t)$ are given in the Appendix; see the supplements section at the end of this issue. For $f(G,t)$ the reader is referred to the following papers:

M_{11}, M_{22}, M_{23}	Burgoyne and Fong [1]
M_{12}, M_{24}	Frobenius [5]
J_1	Janko [14]
$HJ = J_2$	Hall and Wales [8]
$HJM = J_3$	Janko [15]
J_4	Janko [16]
HiS	Frame [4]
Suz	Wright [23]
McL, .3	Finkelstein [3]
Rud	Rudvalis [19]
HHM	Held [10]
LyS	Lyons [17]

Received June 21, 1982; revised March 19, 1984 and July 11, 1984.
1980 *Mathematics Subject Classification*. Primary 20D05.
Key words and phrases. Simple groups, a conjecture of Frobenius.
*Dedicated to Professor Nagayoshi Iwahori on his 60th birthday.

0025-5718/86 $1.00 + $.25 per page

ON	O'Nan [18]
.1	Wilson [21]
.2	Wilson [22]
$M(22)$	Hunt [11]
$M(23)$	Hunt [12]
$M(24)'$	Hunt [13]
F_3	Smith [20]
F_5	Harada [9]

We look for all the subsets Ω of $\mathrm{Ord}(G)$ satisfying the following two conditions:

(a) If t is a member of Ω, then Ω contains all divisors of t. In particular *1* is always a member of Ω.

(b) For the subset Ω of $\mathrm{Ord}(G)$ in (a), $\sum_{t\in\Omega} f(G,t)$ divides $|G|$.

A PASCAL program effectively generates such a subset Ω with the use of a recursive concept. Special add and divide routines are used for explicit calculation with very large digit numbers. Then we have only six possibilities for Ω:

(i) $\Omega = \{1\}$,

(ii) $\Omega = \mathrm{Ord}(G)$,

(iii) $\Omega = \{1,2,3,4,5,10,11\}$ and G is the Mathieu group M_{12},

(iv) $\Omega = \{1,2,3,5,7,11,19\}$ and G is the Janko group J_1,

(v) $\Omega = \{1,2,3,4,5,6,7,8,12,13,14,26,29\}$ and G is the Rudvalis group Rud,

(vi) $\Omega = \{1,2,3,4,5,6,7,8,13,14,16,26,29\}$ and G is the Rudvalis group Rud.

It follows from the Table of $f(G,t)$ in the Appendix that

$$\sum_{t\in\Omega} f(G,t) = \begin{cases} 1, & \text{if } \Omega = \{1\}, \\ |G|, & \text{if } \Omega = \mathrm{Ord}(G), \\ |G|/2, & \text{otherwise.} \end{cases}$$

Since $n = |L_n(G)| = \sum_{t\in\Omega} f(G,t)$ for some Ω satisfying the two conditions (a) and (b), we have $n = 1$ or $n = |G|$. This verifies the conjecture for G. The proof of our theorem is complete.

Acknowledgment. The author thanks Sachiko Yamaki for her help with the computer calculations.

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