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A Lower Bound for the Class Number of Certain Cubic Number Fields

By Günter Lettl

Abstract. Let K be a cyclic number field with generating polynomial

$$X^3 - \frac{a-3}{2}X^2 - \frac{a+3}{2}X - 1$$

and conductor m. We will derive a lower bound for the class number of these fields and list all such fields with prime conductor $m = (a^2+27)/4$ or $m = (1+27b^2)/4$ and small class number.

1. Introduction. Let h_m denote the class number of the cyclotomic field $\mathbf{Q}(\zeta_m)$, and h_m^+, the class number of its maximal real subfield $\mathbf{Q}(\cos(2\pi/m))$. It is a well-known conjecture of Vandiver that $p \nmid h_p^+$ holds for all primes $p \in \mathbf{P}$. This is a customary assumption for proving the second case of Fermat's Last Theorem (for more details see Washington [16]). Since h_p^+ grows slowly ($h_p^+ = 1$ for $p < 163$ with the use of the Generalized Riemann Hypothesis (GRH), van der Linden [10]), for no p with $h_p^+ > 1$ the exact value of h_p^+ is known without using GRH. Masley suggested that perhaps $h_p^+ < p$ always holds, but a counterexample was found in [3], [12]. The class number of each real subfield of $\mathbf{Q}(\zeta_p)$ divides h_p^+, and in this way one can find primes with $h_p^+ > 1$. Using the quadratic subfield, Ankeny, Chowla and Hasse [1] showed that $h_p^+ > 1$ if p belongs to certain quadratic sequences in $\mathbf{N}$, and S.-D. Lang [9] and Takeuchi [15] found more such sequences. Similar results were obtained for h_{4p}^+ by Yokoi [17]. Using the cubic subfield of $\mathbf{Q}(\zeta_p)$, which has been thoroughly investigated (e.g., [2], [5], [8]), the theorem of the present paper yields the following results:

If a is an odd integer, $a > 23$, and $p = (a^2+27)/4$, a prime, then $h_p^+ > 1$.

If b is an odd integer, $b > 7$, and $p = (1+27b^2)/4$, a prime, then $h_p^+ > 1$.

A conjecture about primes in quadratic sequences (Hardy and Wright [7, I.2.8]) implies that there exist infinitely many primes p of each of these two forms, because one can write

$$\frac{a^2+27}{4} = \left(\frac{a-3}{2}\right)^2 + 3\left(\frac{a-3}{2}\right) + 9$$

and

$$\frac{1+27b^2}{4} = 3\left(\frac{3b-1}{2}\right)^2 + 3\left(\frac{3b-1}{2}\right) + 1.$$

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2. Class Number Bounds and Main Results. Let K be a cyclic cubic number field with conductor m and class number h. It is well known that m is the product of distinct primes, which are congruent to 1 mod (6), and of 9, if 3 ramifies in K. The class number h is congruent to 1 mod (3), if m is a prime or $m = 9$, and h is divisible by 3 otherwise. Set $f(s) = L(s, \chi) \cdot L(s, \bar{\chi})$ for $s \in \mathbf{C}$, where χ and $\bar{\chi}$ are the nontrivial cubic Dirichlet characters modulo (m) belonging to K. Since the discriminant of K equals m^2, the analytic class number formula yields

$$h = \frac{m \cdot f(1)}{4 \cdot R}, \tag{1}$$

where R is the regulator of K. Moser [11] showed that for prime conductors, $h < m/3$ holds, so cubic fields will never lead to a contradiction to Vandiver's conjecture. Our aim is to establish a lower bound for the class number of a special family of cubic fields and to list all fields of some special types with prime conductor and small class number. From a result of Stark [14] one can deduce $f(1) > c/\log m$, where c is effectively computable, but this bound is not suited for our purposes. From the results of the next section we will obtain:

(2) *If K is a cyclic cubic number field with conductor $m > 10^5$, then $f(1) > 0.023 \cdot m^{-0.054}$.*

The harder problem is to find an upper bound for the regulator, which is only achieved for the following family of cyclic cubic fields. The polynomial

$$f_a(X) = X^3 - \frac{a-3}{2}X^2 - \frac{a+3}{2}X - 1, \qquad a \in \mathbf{N} \text{ odd}, \tag{3}$$

is irreducible over $\mathbf{Q}$, has discriminant $D(f_a) = ((a^2 + 27)/4)^2$, and if ε is a zero of f_a, the other zeros are $\varepsilon' = -1/(\varepsilon + 1)$ and $\varepsilon'' = -(\varepsilon + 1)/\varepsilon$. Therefore, f_a is a generating polynomial of a cyclic cubic field K with conductor m, and we define $k \in \mathbf{N}$ by $\sqrt{D(f_a)} = (a^2 + 27)/4 = km$.

We call the field K *of type* A, if $k = 1$, and *of type* B, if $k = 27$ and $a = 27b$ with $b \in \mathbf{N}$ odd, $b \neq 1$ (in this case we have $m = (1 + 27b^2)/4$). It is well known that fields of type A or B have relatively large class numbers (see, for example, the tables of Gras [5]). Shanks [13] states that for cubic fields of type A with prime conductor "a rough mean value for h is given by $h \approx 12m/35(\log m)^2$".

LEMMA 1. *Let K be a cyclic cubic field with generating polynomial f_a, conductor m and regulator R. Then,*

$$4R < \left(\tfrac{1}{2}\log D(f_a)\right)^2 = (\log(km))^2. \tag{4}$$

Proof of Lemma 1. Since the zeros $\varepsilon, \varepsilon', \varepsilon''$ of f_a are units of K, we can estimate the regulator of K by $R \leqslant \operatorname{Reg}(\{\varepsilon, \varepsilon'\}) =: R'$, if $R' \neq 0$ (see Lemma 4.15 in [16]). Choosing

$$\varepsilon = \frac{a - 3 + 4\sqrt{km} \cdot \cos\left(1/3 \cdot \arctan\left(\sqrt{27}/a\right)\right)}{6} \sim \sqrt{km}$$

with the principal value of arctan, we obtain

$$R' = \left| \det \begin{pmatrix} \log|\varepsilon| & \log|\varepsilon'| \\ \log|\varepsilon'| & \log|\varepsilon''| \end{pmatrix} \right|$$

$$= (\log|\varepsilon + 1|)^2 - \log|\varepsilon + 1| \log|\varepsilon| + (\log|\varepsilon|)^2.$$

Series expansions yield

$$R' = \frac{1}{4}(\log km)^2 - \frac{3\log(km)}{2km} + \frac{3}{4km} + O\left(\frac{\log km}{(km)^2}\right)$$

and elementary calculus explicitly gives (4).

With (2) and Lemma 1 we immediately obtain from (1):

Let K be a cyclic cubic field with conductor $m > 10^5$ and with generating polynomial f_a. Then,

$$h > 0.023 \frac{m^{0.946}}{(\log km)^2}. \tag{5}$$

THEOREM. (a) *Let K be a cyclic cubic field of type* A *with prime conductor m. Then $h < 16$ holds only for the following values of m:*

h	m
1	7, 13, 19, 37, 79, 97, 139
4	163, 349, 607, 709, 937
7	313, 877, 1129, 1567, 1987, 2557
13	1063

(b) *Let K be a cyclic cubic field of type* B *with prime conductor m. Then $h < 43$ holds only for the following values of m:*

h	m
1	61, 331
4	547, 1951
7	2437, 3571
13	9241
28	4219, 25117
31	23497
37	8269

Proof of the Theorem. From (5) we obtain $h > 14$ for fields of type A with $m \geqslant 169339$, and $h > 37.2$ for fields of type B with $m > 10^6$. It is well known (see, e.g., Gras [4]) that primes $q \equiv -1 \bmod (3)$ divide the class number of a cyclic cubic field only with an even exponent. The table of class numbers of Shanks [13], and Table 1 below, complete the proof of the theorem.

TABLE 1
Class numbers of cyclic cubic fields of type **B** *with prime conductor* $m < 10^6$

b	$m = \frac{1+27b^2}{4}$	h	b	$m = \frac{1+27b^2}{4}$	h
			173	202021	$316 = 2^2 \cdot 79$
3	61	1	185	231019	$343 = 7^3$
7	331	1	189	241117	$1216 = 2^6 \cdot 19$
9	547	$4 = 2^2$	191	246247	$175 = 5^2 \cdot 7$
17	1951	$4 = 2^2$	193	251431	$247 = 13 \cdot 19$
19	2437	7	199	267307	$196 = 2^2 \cdot 7^2$
23	3571	7	205	283669	541
25	4219	$28 = 2^2 \cdot 7$	221	329677	$316 = 2^2 \cdot 79$
33	7351	$49 = 7^2$	227	347821	331
35	8269	37	231	360187	$1732 = 2^2 \cdot 433$
37	9241	13	235	372769	$553 = 7 \cdot 79$
39	10267	$49 = 7^2$	243	398581	$1075 = 5^2 \cdot 43$
45	13669	109	259	452797	769
59	23497	31	261	459817	$2257 = 37 \cdot 61$
61	25117	$28 = 2^2 \cdot 7$	297	595411	$2299 = 11^2 \cdot 19$
91	55897	$133 = 7 \cdot 19$	299	603457	739
95	60919	193	301	611557	$889 = 7 \cdot 127$
105	74419	$688 = 2^4 \cdot 43$	303	619711	$1156 = 2^2 \cdot 17^2$
115	89269	211	305	627919	$1552 = 2^4 \cdot 97$
117	92401	$532 \quad 2^2 \cdot 7 \cdot 19$	341	784897	$688 = 2^4 \cdot 43$
123	102121	307	347	812761	769
129	112327	$604 = 2^2 \cdot 151$	361	879667	$688 = 2^4 \cdot 43$
131	115837	$148 = 2^2 \cdot 37$	367	909151	787
137	126691	97	371	929077	$1588 = 2^2 \cdot 397$
147	145861	$652 = 2^2 \cdot 163$	373	939121	661
159	170647	$628 = 2^2 \cdot 157$	383	990151	$532 = 2^2 \cdot 7 \cdot 19$

The class numbers of Table 1 were calculated with a "Sirius 1 Personal Computer", using the analytic class number formula (1). We also used that for fields of type B the roots of f_a are already fundamental units, and therefore $R = R'$ can be calculated with the explicit formula for ε, given in the proof of Lemma 1. In the following way it can be proved that ε is a fundamental unit:

Let K be a field of type B with generating polynomial f_a, $a = 27b$ and $m = (1 + 27b^2)/4$. Hasse [8] investigated the arithmetic of cyclic cubic fields, using the Gauss sums of the corresponding Dirichlet characters. With Hasse's notation, every integer $\alpha \in K$ can be written as $\alpha = [x, y]$ with $x \in \mathbf{Z}$, $y \in \mathbf{Z}[\rho]$, where $\rho^2 + \rho + 1 = 0$, and $x \equiv y \bmod (1 - \rho)$. If α is a unit of K, $N(\alpha) = 1$ implies $x^3 \equiv 27 \bmod (m)$ and $|x| \leqslant 2\sqrt{my\bar{y}}$ (Satz 8, [8]). For the roots of f_{27b} we have $\varepsilon = [(27b - 3)/2, 3i\sqrt{3}\,]$ and its conjugates. Since Godwin's conjecture about fundamental units holds for cyclic cubic fields with $m > 9$ (see Gras [6]), we have to show:

> There exists no unit $\alpha = [x, y] \in K$, $\alpha \neq \pm 1$, with $my\bar{y} = \frac{1}{2}\operatorname{tr}(\alpha - \alpha')^2 < \frac{1}{2}\operatorname{tr}(\varepsilon - \varepsilon')^2 = 27m$, where tr denotes the trace from K to $\mathbf{Q}$.

Suppose the contrary. Then $x^3 \equiv 27 \bmod (m)$ and $|x| < 2\sqrt{27m}$ imply $x \in \{3, (27b - 3)/2, -(27b + 3)/2\}$ for $b \geqslant 7$. Considering $0 \equiv x \equiv y \bmod (1 - \rho)$ and $y\bar{y} < 27$ yields only a few possibilities for $y \in \mathbf{Z}[\rho]$, and one can check that for each of these y, $N(\alpha) = 1$ has no solution $\alpha \neq 1$. For small values of b, one can consult the table in [5].

In the same way, but with much less computation, one can prove that for $k = 1$ (type A) and $k = 3$ the roots of f_a are also fundamental units. In these cases one has $\varepsilon = [(a-3)/2, \pm 1]$ with $(a-3)/2 \equiv \pm 1 \bmod (3)$, and $\varepsilon = [(9b-3)/2, i\sqrt{3}]$, respectively.

3. A Lower Bound for $L(1,\chi) \cdot L(1,\bar{\chi})$. Let m be the conductor of a cyclic cubic field K, χ and $\bar{\chi}$ the nontrivial cubic Dirichlet characters modulo m associated with K, and $f(s) = L(s,\chi)L(s,\bar{\chi})$. To find a lower bound for $f(1)$, we first need an upper bound for $|f(s)|$ in a disk in $\mathbf{C}$ containing 1. Consider $C = C(\mu,\rho) = \{s \in \mathbf{C} \,|\, |s-\mu| < \rho\}$ with $1 < \mu$ and $\mu - 1 < \rho < \mu$, and set $\sigma_0 = \mu - \rho$. Let $s = \sigma + it \in \mathbf{C}$. For $\sigma > 0$ we have the representation

$$L(s,\chi) = \sum_{n=1}^{m-1} \frac{\chi(n)}{n^s} + s \cdot \int_m^\infty \frac{S(x,\chi)}{x^{s+1}}\,dx \quad \text{with } S(x,\chi) = \sum_{1 \leqslant n < x} \chi(n)$$

(see [16, p. 211]). The inequality of Pólya-Vinogradov [16, Lemma 11.8] states that $|S(x,\chi)| < \sqrt{m} \cdot \log m$. For $s \in C(\mu,\rho)$, the function $|s|/\sigma = 1/\cos(\arg s)$ attains its maximum $\mu/\sqrt{\mu^2-\rho^2}$ if s is the point of contact of a tangent of C through 0. Combining these results, we obtain for every $s \in C(\mu,\rho)$:

$$\begin{aligned}|L(s,\chi)| &< 1 + \int_1^m \frac{1}{x^{\sigma_0}}\,dx + |s|\sqrt{m} \cdot \log m \int_m^\infty \frac{1}{x^{\sigma+1}}\,dx \\ &< \frac{1}{1-\sigma_0} m^{1-\sigma_0} + \frac{\mu}{\sqrt{\mu^2-\rho^2}} \log m \cdot m^{0.5-\sigma_0}.\end{aligned}$$

Since $\log x/\sqrt{x}$ is monotone decreasing for $x \geqslant e^2$, we conclude that for $m \geqslant m_0 \geqslant e^2$,

$$(6) \qquad |f(s)| < c_1 \cdot m^{2-2\sigma_0}$$

holds for all $s \in C(\mu,\rho)$, with

$$c_1 = \left(\frac{1}{1-\sigma_0} + \frac{\mu}{\sqrt{\mu^2-\rho^2}} \cdot \frac{\log m_0}{\sqrt{m_0}}\right)^2.$$

LEMMA 2. *If K is a cyclic cubic number field with conductor m, then $f(1) > c_6 \cdot m^{-c_7}$, with c_6, $c_7 > 0$ as given in the course of the proof. Furthermore, c_7 can be made arbitrarily small.*

Proof of Lemma 2. The proof follows mainly Washington [16, pp. 212–214]. Let $\zeta(s)$ be the Riemann zeta function and $\zeta_K(s) = \zeta(s)f(s)$ the zeta function of the cyclic cubic field K with conductor m. If $s = \sigma + it \in \mathbf{C}$, we have

$$\zeta_K(s) = 1 + \sum_{n=2}^\infty \frac{a_n}{n^s} \quad \text{for } \sigma > 1,$$

with $a_n \geqslant 0$, and $a_n \geqslant 1$ if n is a cube. Developing ζ_K in a power series around $\mu > 1$ gives

$$\zeta_K(s) = \sum_{j=0}^\infty b_j(\mu - s)^j,$$

with

$$(7)\quad b_0 = \zeta_K(\mu) > \zeta(3\mu) > 1 \quad \text{and} \quad b_j = \frac{1}{j!}\sum_{n=2}^{\infty} (\log n)^j \cdot \frac{a_n}{n^\mu} > 0 \quad \text{for } j \geqslant 1.$$

The integral representation of $\zeta(s)$ for $\sigma > 0$ yields

$$|\zeta(s)| \leqslant \left|\frac{s}{s-1}\right| + |s|\int_1^\infty \frac{1}{u^{\sigma+1}}\,du = \left|\frac{s}{s-1}\right| + \frac{|s|}{\sigma}$$

and

$$|\zeta(s)| \leqslant \left|\frac{s}{s-1}\right| + |s| \cdot \sum_{n=1}^{\infty} \frac{1}{n^{\sigma+1}} \cdot \int_n^{n+1} (u - [u])\,du$$
$$< \left|\frac{s}{s-1}\right| + \frac{|s|}{2}\left(1 + \frac{1}{\sigma}\right).$$

Let $C = C(\mu, \rho)$, with $\mu - 1 < \rho < \mu$, be the disk with center μ and radius ρ, and denote its boundary by ∂C. Using (6), we get for all $s \in \partial C$:

$$(8)\quad \left|\zeta_K(s) - \frac{f(1)}{s-1}\right| \leqslant |\zeta(s)| \cdot |f(s)| + \frac{1}{|s-1|} \cdot |f(1)| < c_2 \cdot m^{2-2\sigma_0},$$

with

$$c_2 = c_1 \cdot \max_{s \in \partial C}\left(\frac{|s|+1}{|s-1|} + |s| \cdot \min\left\{\frac{1}{\sigma}, \frac{1}{2}\left(1 + \frac{1}{\sigma}\right)\right\}\right).$$

Since $\zeta_K(s) - f(1)/(s-1)$ is holomorphic in the whole complex plane, (8) holds for all $s \in C(\mu, \rho)$. Computing the coefficients of

$$\zeta_K(s) - \frac{f(1)}{s-1} = \sum_{j=0}^{\infty}\left(b_j - \frac{f(1)}{(\mu-1)^{j+1}}\right) \cdot (\mu - s)^j$$

with a Cauchy integral gives

$$\left|b_j - \frac{f(1)}{(\mu-1)^{j+1}}\right| = \left|\frac{1}{2\pi i}\int_{\partial C}\left(\zeta_K(s) - \frac{f(1)}{s-1}\right)\frac{ds}{(s-\mu)^{j+1}}\right| < \frac{c_2}{\rho^j} \cdot m^{2-2\sigma_0}.$$

For $0 < \sigma < 1$, the integral representation of $\zeta(s)$, and $f(\sigma) = |L(\sigma, \chi)|^2$, show that $\zeta_K(\sigma) \leqslant 0$. So for any α with $\sigma_0 < \alpha < 1$, and any $\nu \in \mathbf{R}^+$ with $1 < \nu$, we have

$$-\frac{f(1)}{\alpha - 1} \geqslant \zeta_K(\alpha) - \frac{f(1)}{\alpha-1} > \sum_{j=0}^{[\nu]-1}\left(b_j - \frac{f(1)}{(\mu-1)^{j+1}}\right) \cdot (\mu - \alpha)^j$$
$$- c_2 \cdot m^{2-2\sigma_0} \cdot \sum_{j=[\nu]}^{\infty}\left(\frac{\mu-\alpha}{\rho}\right)^j$$
$$\geqslant c_3 - \frac{f(1)}{\alpha-1} - \frac{f(1)}{1-\alpha}\left(\frac{\mu-\alpha}{\mu-1}\right)^\nu - c_2 \cdot m^{2-2\sigma_0}\left(\frac{\mu-\alpha}{\rho}\right)^{\nu-1} \cdot \frac{\rho}{\alpha - \sigma_0},$$

where $\sum_{j=0}^{[\nu]-1} b_j(\mu-\alpha)^j \geqslant c_3 \geqslant 1$.

From this inequality, we obtain

$$f(1) > c_3(1-\alpha)\left(\frac{\mu-1}{\mu-\alpha}\right)^\nu - c_2 \cdot m^{2-2\sigma_0} \cdot \frac{\rho^2(1-\alpha)}{(\mu-\alpha)(\alpha-\sigma_0)}\left(\frac{\mu-1}{\rho}\right)^\nu.$$

Choosing $\nu = c_4 \cdot \log m + c_5$, with

$$c_4 = \frac{2 - 2\sigma_0}{\log \frac{\rho}{\mu - \alpha}}$$

and

$$c_5 = \frac{\log \frac{c_2 \cdot \rho^2}{(\mu - \alpha)(\alpha - \sigma_0)} + \log\log \frac{\rho}{\mu - 1} - \log\log \frac{\mu - \alpha}{\mu - 1}}{\log \frac{\rho}{\mu - \alpha}},$$

gives $f(1) > c_6 \cdot m^{-c_7}$, with

$$c_6 = c_3(1 - \alpha)\left(\frac{\mu - 1}{\mu - \alpha}\right)^{c_5} - c_2 \cdot \frac{\rho^2(1 - \alpha)}{(\mu - \alpha)(\alpha - \sigma_0)} \cdot \left(\frac{\mu - 1}{\rho}\right)^{c_5}$$

and

$$c_7 = (2 - 2\sigma_0) \cdot \log \frac{\mu - \alpha}{\mu - 1} / \log \frac{\rho}{\mu - \alpha}.$$

Since $c_7 \to 0$ for $\alpha \to 1$, the proof of Lemma 2 is completed.

Numerical computations show that for $m_0 = 10^5$ good results can be obtained by choosing $\mu = 10$, $\rho = 9.9$ and $\alpha = 0.975$. With these values we obtain $c_2 = 10.8685$ and $\nu \approx 315$.

Using (7), and $a_n \geqslant 1$ for n a cube, we obtain the following estimations for c_3:

$$\sum_{j=0}^{[\nu]-1} b_j(\mu - \alpha)^j \geqslant \zeta_K(\mu) + \sum_{j=1}^{300} \frac{1}{j!} \sum_{n=2}^{\infty} \frac{a_n}{n^\mu}((\mu - \alpha)\log n)^j$$

$$> \zeta(3\mu) + \sum_{k=2}^{N_0} \frac{1}{k^{3\mu}} \sum_{j=1}^{300} \frac{1}{j!}((\mu - \alpha)3 \cdot \log k)^j$$

$$> 1 + \sum_{k=2}^{N_0} \frac{1}{k^{3\mu}} \left(k^{3(\mu - \alpha)} - \frac{((\mu - \alpha)3 \cdot \log k)^{301}}{301!} \cdot \frac{302}{302 - (\mu - \alpha)3 \cdot \log k} \right),$$

where $N_0 < e^{302/3(\mu - \alpha)}$. With the special values for μ, ρ and α, and $N_0 = 100$, we obtain

$$\sum_{j=0}^{[\nu]-1} b_j(\mu - \alpha)^j > \sum_{k=1}^{100} k^{-2.925} - 10^{-40} > 1.2175 = c_3.$$

These values yield $c_6 > 0.023$ and $c_7 < 0.054$, and thus (2) is proved.

Institut für Mathematik
Karl-Franzens-Universität
Halbärthgasse 1
A-8010 Graz
Österreich, Austria

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