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284457 Steiner Triple Systems of Order 19 Contain a Subsystem of Order 9

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Abstract. In this paper, we enumerate the (nonisomorphic) Steiner triple systems of order 19 which contain a subsystem of order 9. The number of these designs is precisely 284457. We also determine which of these designs also contain at least one subsystem of order 7, and how many. Exactly 13529 of them contain at least one subsystem of order 7.

1. Introduction. A *Steiner triple system* is a pair $(X, \mathbf{B})$, where X is a finite set of elements called *points*, and $\mathbf{B}$ is a set of three-subsets of X called *blocks*, such that every pair of points occurs in a unique block. We denote a Steiner triple system having v points by STS(v); v is called the *order* of the STS(v). It is well-known that an STS(v) exists if and only if $v \geqslant 1$, $v \equiv 1$ or $3 \pmod 6$.

Two STS(v), say $(X_1, \mathbf{B}_1)$ and $(X_2, \mathbf{B}_2)$, are said to be *isomorphic* if there exists a bijection π: $X_1 \to X_2$ such that $\{x, y, z\} \in \mathbf{B}$ implies $\{\pi(x), \pi(y), \pi(z)\} \in \mathbf{B}_2$. We denote by $N(v)$ the number of mutually nonisomorphic STS(v). $N(v)$ has been enumerated for $v \leqslant 15$; we have $N(1) = N(3) = N(7) = N(9) = 1$, $N(13) = 2$, and $N(15) = 80$. (See Mathon, Phelps, and Rosa [4] for a comprehensive investigation of these designs). At this point, an explosion occurs: it is known [5] that $N(19) > 2300000$. The number $N(19)$ is probably too large to ever be calculated exactly, so several researchers have investigated certain special classes of STS(19). Some of these are mentioned in [4].

We enumerate certain classes of STS(19) in this paper. First, we have to define some terminology. We say that $(Y, \mathbf{B})$ is a *subsystem* of an STS $(X, \mathbf{A})$ provided Y is contained in X and $\mathbf{B}$ is contained in $\mathbf{A}$. The subsystem will be an STS(w) for some w. We say that it is a sub-STS(w) to indicate that it is a subsystem of another STS.

The problem we investigate is the enumeration of (nonisomorphic) STS(19) which contain sub-STS(9). We denote the number of these designs by $N_9(19)$. The best previous bounds on $N_9(19)$ were due to Déherder [1]: he proved that $284399 \leqslant N_9(19) \leqslant 290000$. In this paper, we prove that $N_9(19) = 284457$. We also determine

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which of these designs also contain sub-STS(7), and how many. The results are summarized in Table 1.

TABLE 1
STS(19) which contain a sub-STS(9)

# sub-STS(7)	# nonisomorphic STS(19)
0	270928
1	12800
2	641
3	45
4	37
6	5
12	1
	284457

2. STS(19) and One-Factorizations. It is an easy exercise to show that if an STS(v) contains a sub-STS(w), then $v \geqslant 2w+1$. Also, if $v = 2w+1$, then there is at most one sub-STS(w) contained in an STS(v).

An STS($2w+1$) which contains a sub-STS(w) has a very special structure. First, some definitions are required. Let X be a set of n points (where n is even). A *one-factor* (of X) is a set of $n/2$ unordered pairs that partition X. A *one-factorization* (of X) is a pair $(X, \mathbf{P})$, where $\mathbf{P}$ is a set of $n-1$ one-factors of X, such that every pair of points is contained in one one-factor of $\mathbf{P}$. Also, two one-factorizations $(X_1, \mathbf{P}_1)$ and $(X_2, \mathbf{P}_2)$ are said to be *isomorphic* if there exists a bijection π: $X_1 \to X_2$ such that $\{\{\pi(x), \pi(y)\}: \{x, y\} \in P\} \in \mathbf{P}_2$ for all $P \in \mathbf{P}_1$.

Suppose $(X, \mathbf{A})$ is an STS($2w+1$) which contains a sub-STS(w), $(Y, \mathbf{B})$. For $y \in Y$, define $P_y = \{\{a, b\}: \{a, b, y\} \in \mathbf{A}\}$. Then, it is easy to see that $(X \setminus Y, \{P_y: y \in Y\})$ is a one-factorization of $X \setminus Y$. Conversely, if we are given a one-factorization $(X \setminus Y, \mathbf{P})$ and an STS $(Y, \mathbf{B})$ where $|X| = 2 \times |Y| + 1$, then we can construct an STS $(X, \mathbf{A})$ as follows: let π: $\mathbf{P} \to Y$ be any bijection, and define $\mathbf{A} = \mathbf{B} \cup \{\{a, b, y\}: \{a, b\} \in P \in \mathbf{P} \text{ and } y = \pi(P)\}$.

So, any STS(19) containing a sub-STS(9) can be constructed as above from a one-factorization on ten points and an STS(9). These ingredients have been enumerated. First, it is well-known that there is a unique STS(9) up to isomorphism, namely the affine plane of order 3. The nonisomorphic one-factorizations on 10 points were enumerated by Gelling [2]; there are 396 of them.

Suppose $\mathbf{F}$ is a one-factorization on points $0, 1, \ldots, 9$, and label the one-factors P_i, $1 \leqslant i \leqslant 9$, where $\{0, i\} \in P_i$, $1 \leqslant i \leqslant 9$. Let $\mathbf{S}$ be any STS(9) on point set $\{i'$: $1 \leqslant i \leqslant 9\}$. Since we are interested only in nonisomorphic STS(19), we can take the bijection π to be $\pi(P_i) = i'$, $1 \leqslant i \leqslant 9$. We denote the resulting STS(19) by $\mathbf{F} + \mathbf{S}$.

If we choose one $\mathbf{F}$ from each isomorphism class, and all possible distinct $\mathbf{S}$, we will construct all possible nonisomorphic STS(19) which contain a sub-STS(9). Hence, we can easily obtain an upper bound on $N_9(19)$. It is well-known that the unique STS(9) has an automorphism group of order 432; hence there are $9!/432 = 840$ distinct STS(9) on a specified point set. So, we have $N_9(19) \leqslant 396 \times 840 = 332640$.

Let us now consider the possibility that $\mathbf{F} + \mathbf{S}$ is isomorphic to $\mathbf{F}' + \mathbf{S}'$. Since an STS(19) contains at most one sub-STS(9), any isomorphism π induces an isomorphism of $\mathbf{F}$ to $\mathbf{F}'$, and of $\mathbf{S}$ to $\mathbf{S}'$. Since we chose one one-factorization from each isomorphism class, we must have $\mathbf{F} = \mathbf{F}'$, and π induces an automorphism of $\mathbf{F}$.

Now, suppose we are given $\mathbf{F}$ and an automorphism π of $\mathbf{F}$. For $1 \leqslant i \leqslant 9$, define $\pi'(i') = j'$, where $\pi(P_i) = P_j$. Then, $\mathbf{F} + \mathbf{S}$ is isomorphic to $\mathbf{F} + \mathbf{S}^{\pi'}$, for any STS(9), $\mathbf{S}$. Conversely, if $\mathbf{F} + \mathbf{S}$ is isomorphic to $\mathbf{F} + \mathbf{S}'$, then $\mathbf{F} = \mathbf{F}'$ and $\mathbf{S} = \mathbf{S}^{\pi'}$, where π is an automorphism of $\mathbf{F}$.

Suppose we fix a one-factorization $\mathbf{F}$. For two STS(9) $\mathbf{S}$ and $\mathbf{S}'$, we can define $\mathbf{S} \approx \mathbf{S}'$ if $\mathbf{S}' = \mathbf{S}^{\pi'}$ for some automorphism π of $\mathbf{F}$. Then $\approx$ is an equivalence relation. The number of nonisomorphic STS $\mathbf{F} + \mathbf{S}$ is precisely the number of equivalence classes of $\approx$. To facilitate counting equivalence classes, we use Burnside's lemma. For any permutation π' of $\{i'\colon 1 \leqslant i \leqslant 9\}$, define $\mathrm{fix}(\pi') = |\{\mathbf{S}\colon \mathbf{S}^{\pi'} = \mathbf{S}\}|$. Then, we have

LEMMA 2.1 (BURNSIDE). *For a one-factorization* $\mathbf{F}$, *the number of nonisomorphic designs* $\mathbf{F} + \mathbf{S}$ *is precisely* $\sum_{\pi \in G} \mathrm{fix}(\pi')/|G|$, *where* $G = \mathrm{Aut}(\mathbf{F})$ *is the automorphism group of* $\mathbf{F}$.

So, for a given one-factorization $\mathbf{F}$, we need to first determine the automorphism group of $\mathbf{F}$, and then for each $\pi \in \mathrm{Aut}(\mathbf{F})$, we must calculate $\mathrm{fix}(\pi')$. We now describe how to calculate the numbers $\mathrm{fix}(\pi')$.

We consider the action of the symmetric group S_9 on the symbols i', $1 \leqslant i \leqslant 9$, on the set of 840 distinct STS(9) on these points. First, the value $\mathrm{fix}(\pi')$ depends only on the cycle type (i.e., the conjugacy class in S_9) of π'. We can classify the 432 automorphisms of an STS(9) according to their cycle types. This is done in Table 2, where we use the notation $1^i 2^j 3^k \cdots$ to denote i cycles of length 1, j cycles of length 2, k cycles of length 3, etc. Any π' having a cycle type that does not occur in Aut(STS(9)) has $\mathrm{fix}(\pi') = 0$. For each cycle type that does occur, we can easily count the number c of conjugates in the group S_9. If there are b members of S_9 having a particular cycle type, then any π' having this cycle type has $\mathrm{fix}(\pi') = 840 \times b/c$. We record this information in Table 2.

TABLE 2
Calculation of $\mathrm{fix}(\pi')$

cycle type of π'	number in Aut(STS(9))	size of conjugacy class	$\mathrm{fix}(\pi')$
$1^1 8^1$	108	45360	2
$3^1 6^1$	72	20160	3
$1^1 2^1 6^1$	72	30240	2
$1^1 4^2$	54	11340	4
3^3	56	2240	21
$1^3 3^2$	24	3360	6
$1^1 2^4$	9	945	8
$1^3 2^3$	36	1260	24
1^9	1	1	840
	432		

It is now a simple matter to enumerate the STS(19) containing a sub-STS(9).

ALGORITHM.

For each nonisomorphic one-factorization **F** do:

1. Compute Aut(**F**);
2. Sum := 0;
3. for each π in Aut(**F**) do Sum := Sum + fix(π');
4. Sum := Sum/|Aut(**F**)|.

"Sum" is the number of nonisomorphic STS(19) obtained from the one-factorization **F**.

We present the results of our enumeration in Section 3. We note that the methods we have used are very similar to Déherder [1]; the main simplification is the use of Burnside's lemma.

3. Enumeration of STS(19) Containing a Sub-STS(9). As mentioned, the 396 one-factorizations of ten points were enumerated in [2]. We used a computer to find the automorphism group of each, and the cycle type of each automorphism. We record our results in Table 3, according to the size of the automorphism groups of the one-factorizations and the number of nonisomorphic designs that result. (More detailed information is presented in Table A in the Appendix). Hence, we have

THEOREM 3.1. *The number of nonisomorphic* $STS(19)$ *that contain a sub-*$STS(9)$ *is exactly* 284457.

TABLE 3

Enumeration of $STS(19)$ *containing sub-*$STS(9)$

Group order	# one-factorizations	# nonisomorphic designs		Total
1	298	298	840	250320
2	69	40	432	
		7	424	29488
		22	420	
3	5	4	294	1460
		1	284	
4	7	3	224	
		2	214	1524
		2	212	
6	6	1	154	
		4	152	906
		1	144	
8	3	3	108	324
9	1	1	98	98
12	2	2	84	168
16	1	1	61	61
18	1	1	56	56
40	1	1	24	24
54	1	1	19	19
432	1	1	9	9
	396			284457

We remark that Déherder enumerated all the designs resulting from the 389 one-factorizations with automorphism groups of order at most 9. Our results agree with his, except for one small error. From one of the one-factorizations (#36 on Gelling's list), 144 nonisomorphic STS(19) result, and not 152, as claimed in [1]. We have adjusted the bounds claimed in the introduction downward by 8, to correct this error.

4. Sub-STS(7) and Sub-One-Factorizations of Order 4. The second question we investigate is the enumeration of sub-STS(7) in the 284457 STS(19) which contain a sub-STS(9). First, it is not difficult to see that, if an STS(19) contains both a sub-STS(9) and a sub-STS(7), then these subdesigns intersect in a block of the design. The sub-STS(7) contains four points i, j, k, l not in the sub-STS(9), and three points m', n', p', which occur in a block of the sub-STS(9). In the one-factorization $\mathbf{F}$, $\{i, j\}$ and $\{k, l\}$ occur in a one-factor, as do $\{i, k\}$ and $\{j, l\}$, and $\{i, l\}$ and $\{j, k\}$ (these three one-factors are, in some order, P_m, P_n, and P_p). Such a configuration is called a sub-one-factorization of order 4, on the four points i, j, k, l. We determined, by computer, the occurrences of sub-one-factorizations of order 4 in the 396 one-factorizations of order 10. This information is presented in Table 4.

TABLE 4
Sub-one-factorizations of order 4

# sub-one-factorizations of order 4	# one-factorizations of order 10
0	278
1	81
2	24
3	5
4	5
6	2
12	1
	396

Let $\mathbf{G}$ be a sub-one-factorization of order 4 in $\mathbf{F}$ (a one-factorization of order 10). Define $B(\mathbf{G}) = \{i'$: $\mathbf{G}$ meets P_i in a one-factor$\}$. Then an STS(19), $\mathbf{F} + \mathbf{S}$, has a sub-STS(7) containing the points of $\mathbf{G}$ if and only if $B(\mathbf{G})$ is a block of $\mathbf{S}$.

For a one-factorization $\mathbf{F}$, we define $\mathrm{Conf}(\mathbf{F}) = \{B(\mathbf{G})$: $\mathbf{G}$ is a sub-one-factorization of order 4 in $\mathbf{F}\}$. We call $\mathrm{Conf}(\mathbf{F})$ the *configuration* induced by $\mathbf{F}$. $\mathrm{Conf}(\mathbf{F})$ is a set of three-subsets of $\{i'$: $1 \leqslant i \leqslant 9\}$. Also, it is not difficult to check that no pair of points can occur in more than one three-subset of a particular configuration. An STS(19), $\mathbf{F} + \mathbf{S}$, contains precisely $|\mathrm{Conf}(\mathbf{F}) \cap \mathbf{S}|$ sub-STS(7)s.

We want to count nonisomorphic STS(19). As before, we have the symmetric group S_9 acting on the 840 distinct STS(9) on points $\{i'$: $1 \leqslant i \leqslant 9\}$, and $\mathrm{Aut}(\mathbf{F})$ induces an equivalence relation $\approx$ on this set of STS(9). Since $|\mathrm{Conf}(\mathbf{F}) \cap \mathbf{S}|$ is constant within equivalence classes, we could choose one STS(9) from each equivalence class, and determine $|\mathrm{Conf}(\mathbf{F}) \cap \mathbf{S}|$ to count the sub-STS(7)s. But again, we can use Burnside's lemma to avoid counting the equivalence classes.

For a configuration $C = \mathrm{Conf}(\mathbf{F})$, an automorphism π in $G = \mathrm{Aut}(\mathbf{F})$, and an integer i such that $0 \leqslant i \leqslant |C|$, define $\mathrm{fix}(C, i, \pi)$ to be the number of STS(9) $\mathbf{S}$ such that:

(1) $|C \cap \mathbf{S}| = i$, and

(2) $\mathbf{S}$ is fixed by π.

Then, the number of nonisomorphic STS(19) $\mathbf{F} + \mathbf{S}$ which contain exactly i sub-STS(7) is precisely $\sum_{\pi \in G} \text{fix}(C, i, \pi)/|G|$.

So, we can do our enumeration, provided we can find the relevant numbers $\text{fix}(C, i, \pi)$. In general, these quantities depend on the cycle type of π, and the structure of C. Note that we must have $C^\pi = C$, since $\pi \in \text{Aut}(\mathbf{F})$. Hence, there are usually not too many possibilities to consider. We pursue this in the next section.

5. Enumeration of Sub-STS(7). First, we observe that the 278 one-factorizations with no sub-one-factorizations of order 4 cannot give rise to any STS(19) with a sub-STS(7). So, we begin with the 81 one-factorizations $\mathbf{F}$ that contain a unique sub-one-factorization of order 4. Each such $\mathbf{F}$ has Conf($\mathbf{F}$) consisting of one block of size three.

So, we first want to determine the numbers $\text{fix}(\{a', b', c'\}, i, \pi)$, where π has one of the 9 cycle types in Table 2 (i.e., π is an automorphism of some STS(9)), π fixes $\{a', b', c'\}$, and $i = 0,1$. These numbers are not too difficult to calculate. For example, consider the cycle type $3^1 6^1$, and suppose $\pi = (123)(456789)$. Then we must have $\{a, b, c\} = \{1, 2, 3\}$. From Table 2, we see that there are three STS(9)s which are fixed by π, and it is easy to check that all three contain the block $\{1', 2', 3'\}$. Hence, $\text{fix}(\{1', 2', 3'\}, 1, \pi) = 3$ and $\text{fix}(\{1', 2', 3'\}, 0, \pi) = 0$.

The remaining fix numbers can be calculated without difficulty. We present them in Table 5. For each cycle type, we give a particular π with that cycle type. We then list all possible blocks B that could be fixed by π, and the fix numbers for each. For convenience, we will henceforth omit the prime (′) markings when discussing configurations.

TABLE 5
Fix numbers for configurations having one block

Cycle type	π	B	$\text{fix}(B, 1, \pi)$	$\text{fix}(B, 0, \pi)$
$1^1 8^1$	(1)(23456789)	none		
$3^1 6^1$	(123)(456789)	$\{1,2,3\}$	3	0
$1^1 2^1 6^1$	(1)(23)(456789)	$\{1,2,3\}$	2	0
$1^1 4^2$	(1)(2345)(6789)	none		
3^3	(123)(456)(789)	$\{1,2,3\}$	3	18
		$\{4,5,6\}$	3	18
		$\{7,8,9\}$	3	18
$1^3 3^2$	(1)(2)(3)(456)(789)	$\{1,2,3\}$	6	0
		$\{4,5,6\}$	6	0
		$\{7,8,9\}$	6	0
$1^1 2^4$	(1)(23)(45)(67)(89)	$\{1,2,3\}$	8	0
		$\{1,4,5\}$	8	0
		$\{1,6,7\}$	8	0
		$\{1,8,9\}$	8	0
$1^3 2^3$	(1)(2)(3)(45)(67)(89)	$\{1,2,3\}$	24	0
		$\{1,4,5\}$, etc.	8	16
1^9	(1)(2)(3)(4)(5)(6)(7)(8)(9)	$\{1,2,3\}$, etc.	120	720

Of the 81 one-factorizations which contain a unique sub-one-factorization of order 4, 68 have trivial automorphism groups, and hence each of these gives rise to 120 STS(19) which contain a sub-STS(7), for a total of $68 \times 120 = 8160$. Ten of the remaining 13 one-factorizations have groups of order 2; 8 of these produce 72 designs (each) with subdesigns of order 7, and the remaining 2 each yield 64. The 3 remaining one-factorizations have groups of order 6, and each contribute 26 of these designs. So the 13 one-factorizations with nontrivial automorphism groups produce a total of $8 \times 72 + 2 \times 64 + 3 \times 26 = 782$ STS(19) containing a sub-STS(7). (For details, see Tables B and C in the Appendix). So, we have

THEOREM 5.1. *The* 81 *one-factorizations which contain precisely one sub-one-factorization of order* 4 *produce exactly* 8942 *STS*(19) *which contain a sub-STS*(9) *and a sub-STS*(7).

We now consider the 24 one-factorizations of order 10 that contain exactly two sub-one-factorizations of order 4. First, we remark that these all have configurations isomorphic to $\{\{a, b, c\}, \{a, d, e\}\}$. This is because any configuration which contains two nonintersecting blocks must in fact contain a third block disjoint from the first two.

Examining the automorphisms which occur, we find that there are only four nonisomorphic possibilities. These are summarized in Table 6.

TABLE 6
Fix numbers for configurations having two blocks

Cycle type	π	representative configuration	fix numbers
$1^3 2^3$	(1)(2)(3)(45)(67)(89)	$\{\{1,2,3\}, \{1,4,5\}\}$, etc.	$\mathrm{fix}(C,2,\pi) = 8$ $\mathrm{fix}(C,1,\pi) = 16$ $\mathrm{fix}(C,0,\pi) = 0$
		$\{\{1,4,6\}, \{1,5,7\}\}$, etc.	$\mathrm{fix}(C,2,\pi) = 4$ $\mathrm{fix}(C,1,\pi) = 0$ $\mathrm{fix}(C,0,\pi) = 20$
$1^1 2^4$	(1)(23)(45)(67)(89)	$\{\{1,2,3\}, \{1,4,5\}\}$, etc.	$\mathrm{fix}(C,2,\pi) = 8$ $\mathrm{fix}(C,1,\pi) = 0$ $\mathrm{fix}(C,0,\pi) = 0$
1^9	(1)(2)(3)(4)(5)(6)(7)(8)(9)	$\{\{1,2,3\}, \{1,4,5\}\}$, etc.	$\mathrm{fix}(C,2,\pi) = 24$ $\mathrm{fix}(C,1,\pi) = 192$ $\mathrm{fix}(C,0,\pi) = 624$

The nine automorphism-free one-factorizations which contain two sub-one-factorizations of order 4 are listed in Table B. Each of these gives rise to 24 STS(19) containing two sub-STS(7)s and 192 STS(19) containing one sub-STS(7). The remaining 15 one-factorizations have automorphism groups of order 2 or 6. The sub-STS(7)s arising from them are presented in Table C. Summarizing these, we have

THEOREM 5.2. *The* 24 *subfactorizations which contain exactly two sub-one-factorizations of order* 4 *give rise to* 444 *STS*(19) *which contain a sub-STS*(9) *and two sub-STS*(7)*s, and* 3176 *STS*(19) *which contain a sub-STS*(9) *and one sub-STS*(7).

There are five one-factorizations that contain exactly three sub-one-factorizations of order 4. One of these (#20) has a configuration consisting of three mutually intersecting three-subsets, and the other four have configurations consisting of three mutually disjoint three-subsets. The one-factorization #20 has a trivial automorphism group. The fix numbers are easily calculated, and we obtain the results shown in Table C.

The four one-factorizations that have three disjoint three-subsets for their configurations have automorphism groups of orders 3, 6, 18, and 54. There are several fix numbers which must be calculated. These are presented in Table 7. We note that all the relevant fix numbers $\mathrm{fix}(C, 2, \pi) = 0$, since any STS(9) containing two specified disjoint blocks must also contain a third block disjoint from the first two. Also, we should point out that for some cycle types in Table 7, there could (conceivably) be different (nonisomorphic) ways in which the specified π intersects the configuration C. For example, if $\pi = (123)(456)(789)$, then there are two nonisomorphic possibilities for C: $\{\{1,2,3\}, \{4,5,6\}, \{7,8,9\}\}$ and $\{\{1,4,7\}, \{2,5,8\}, \{3,6,9\}\}$. Only the first of these occurs, so we do not calculate fix numbers for the second possibility.

TABLE 7

Fix numbers for configurations consisting of three disjoint blocks

Cycle type	π	representative configuration	fix numbers
$3^1 6^1$	(123)(456789)	123,468,279, etc.	$\mathrm{fix}(C,3,\pi) = 3$ $\mathrm{fix}(C,1,\pi) = 0$ $\mathrm{fix}(C,0,\pi) = 0$
$1^1 2^1 6^1$	(1)(23)(456789)	123,468,579, etc.	$\mathrm{fix}(C,3,\pi) = 2$ $\mathrm{fix}(C,1,\pi) = 0$ $\mathrm{fix}(C,0,\pi) = 0$
3^3	(123)(456)(789)	123,456,789, etc.	$\mathrm{fix}(C,3,\pi) = 3$ $\mathrm{fix}(C,1,\pi) = 0$ $\mathrm{fix}(C,0,\pi) = 18$
$1^3 3^2$	(1)(2)(3)(456)(789)	123,456,789, etc.	$\mathrm{fix}(C,3,\pi) = 6$ $\mathrm{fix}(C,1,\pi) = 0$ $\mathrm{fix}(C,0,\pi) = 0$
$1^1 2^4$	(1)(23)(45)(67)(89)	123,468,579, etc.	$\mathrm{fix}(C,3,\pi) = 2$ $\mathrm{fix}(C,1,\pi) = 6$ $\mathrm{fix}(C,0,\pi) = 0$
$1^3 2^3$	(1)(2)(3)(45)(67)(89)	123,468,579 etc.	$\mathrm{fix}(C,3,\pi) = 6$ $\mathrm{fix}(C,1,\pi) = 18$ $\mathrm{fix}(C,0,\pi) = 0$
1^9	(1)(2)(3)(4)(5)(6)(7)(8)(9)	123,456,789, etc.	$\mathrm{fix}(C,3,\pi) = 12$ $\mathrm{fix}(C,1,\pi) = 324$ $\mathrm{fix}(C,0,\pi) = 504$

There remain eight one-factorizations to consider. Five of these contain 4 sub-one-factorizations of order 4, two contain 6, and one contains 12. The configuration of this last one-factorization is in fact an STS(9), and the automorphism group is Aut(STS(9)). So, the determination of $|\mathrm{Conf}(\mathbf{F}) \cap \mathbf{S}|$ for various STS(9), $\mathbf{S}$, is equivalent to counting STS(9) which intersect a fixed STS(9) (namely, Conf(F)) in a specified number of blocks, under the action of Aut(STS(9)). These numbers were calculated in [3]. We obtain the numbers in Table C.

The seven one-factorizations having configurations of size 4 or 6 were dealt with as follows. In order to minimize the possibility of error in calculating the fix numbers, we simply used the computer. For each automorphism π, we generated all 840 STS(9) and counted how many were fixed by π. For many of these, we also calculated the fix numbers by hand, and the results agreed in all cases. We obtained the results of Table C.

Summarizing the STS(19) obtained from the one-factorizations with configurations of size at least three, we have

THEOREM 5.3. *From the* 13 *one-factorizations with configurations of size at least* 3, *we obtain* N_i *STS*(19) *which contain a sub-STS*(9) *and i sub-STS*(7), *where* $N_1 = 682$, $N_2 = 197$, $N_3 = 45$, $N_4 = 37$, $N_6 = 5$, *and* $N_{12} = 1$.

The overall distribution of sub-STS(7) in the STS(19) was presented in Table 1 in the Introduction. We have

THEOREM 5.4. *Of the* 284457 *nonisomorphic STS*(19) *which contain a sub-STS*(9), *exactly* 13529 *contain at least one sub-STS*(7).

As a final remark, we note that it would be possible to determine the automorphism groups of the 284457 STS(19), using similar techniques.

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APPENDIX

TABLE A

One-factorizations with nontrivial automorphism groups

One-factorization number	group order	#STS(19)	group generators (action on one-factors)
1	432	9	(1)(28653974)
			(1)(6)(7)(285)(394)
			(132)(487)(569)
2	12	84	(1)(89)(274365)
			(1)(26)(37)(45)(89)
3	6	154	(1)(2)(3)(45)(68)(79)
			(1)(4)(5)(297)(386)
4	2	432	(1)(8)(9)(27)(36)(45)
5	16	61	(1)(29473856)
			(1)(6)(7)(24)(35)(89)
6	2	432	(1)(2)(3)(45)(68)(79)
7	6	152	(132)(487569)
8	4	224	(1)(2)(3)(49)(58)(67)
			(3)(4)(9)(12)(56)(78)
9	4	224	(1)(2)(3)(46)(57)(89)
			(2)(4)(6)(13)(59)(78)
10	2	432	(1)(2)(3)(45)(68)(79)
11	18	56	(123)(497685)
			(4)(7)(8)(123)(596)

(continues)

(continued)

One-factorization number	group order	# STS(19)	group generators (action on one-factors)
12	12	84	(154)(298367)
			(1)(23)(45)(67)(89)
13	6	152	(132)(478965)
17	2	432	(1)(4)(8)(27)(36)(59)
18	2	432	(1)(2)(3)(49)(58)(67)
19	2	432	(1)(2)(3)(45)(68)(79)
21	2	432	(1)(2)(3)(45)(68)(79)
22	8	108	(1)(29385746)
23	2	432	(1)(2)(3)(45)(68)(79)
24	2	432	(1)(2)(3)(45)(68)(79)
36	6	144	(1)(23)(497586)
38	3	284	(4)(7)(8)(132)(596)
44	2	432	(1)(8)(9)(23)(47)(56)
49	4	212	(1)(2)(5)(6)(8)(34)(79)
			(1)(3)(4)(7)(9)(25)(68)
51	2	432	(1)(2)(3)(45)(68)(79)
53	2	432	(1)(4)(9)(26)(35)(78)
58	2	432	(1)(2)(3)(45)(68)(79)
61	2	432	(1)(2)(3)(48)(59)(67)
66	2	420	(1)(2)(3)(6)(9)(45)(78)
92	2	432	(1)(4)(7)(29)(38)(56)
95	2	432	(1)(2)(3)(45)(68)(79)
98	2	432	(1)(2)(3)(45)(68)(79)
100	2	432	(1)(4)(6)(29)(38)(57)
101	2	432	(1)(2)(6)(39)(48)(57)
105	2	432	(2)(5)(7)(16)(38)(49)
114	2	432	(1)(3)(6)(24)(59)(78)
125	2	420	(2)(3)(4)(6)(8)(19)(57)
132	6	152	(132)(478596)
135	2	432	(1)(2)(3)(45)(68)(79)
136	2	432	(1)(2)(3)(49)(58)(67)
147	2	420	(1)(3)(5)(7)(9)(24)(68)
148	2	432	(1)(2)(3)(49)(58)(67)
150	4	214	(1)(2736)(4958)
153	2	420	(1)(2)(3)(6)(9)(45)(78)
165	2	420	(1)(2)(5)(6)(7)(34)(89)
182	2	420	(1)(2)(5)(6)(7)(34)(89)
192	2	432	(1)(2)(6)(39)(48)(57)
193	2	420	(1)(2)(3)(6)(8)(45)(79)
194	2	432	(1)(2)(8)(36)(47)(59)
195	2	432	(2)(3)(9)(16)(47)(58)
199	2	420	(1)(2)(3)(7)(9)(48)(56)
201	2	420	(1)(2)(3)(4)(6)(59)(78)
202	2	432	(1)(2)(7)(35)(46)(89)
203	2	432	(1)(2)(4)(35)(67)(89)
204	2	432	(1)(2)(4)(35)(67)(89)
214	2	424	(1)(23)(45)(67)(89)
234	4	224	(1)(2)(3)(45)(68)(79)
			(1)(4)(5)(23)(69)(78)
243	2	432	(2)(7)(9)(15)(34)(68)
254	2	424	(8)(16)(27)(35)(49)
259	2	420	(1)(3)(4)(8)(9)(25)(67)
269	2	420	(1)(3)(4)(8)(9)(25)(67)
285	2	420	(1)(2)(3)(6)(9)(45)(78)

One-factorization number	group order	# STS(19)	group generators (action on one-factors)
290	54	19	(186345279)
			(7)(48)(162539)
292	2	420	(1)(2)(5)(7)(8)(34)(69)
297	2	420	(2)(4)(7)(8)(9)(15)(36)
299	2	420	(1)(2)(3)(6)(7)(49)(58)
305	2	432	(1)(3)(8)(24)(56)(79)
315	2	432	(1)(2)(8)(36)(47)(59)
316	2	432	(1)(3)(6)(24)(58)(79)
324	2	424	(1)(23)(45)(67)(89)
328	8	108	(1)(27385946)
329	4	214	(1)(2938)(4756)
330	8	108	(1)(2938)(4756)
			(1)(4)(5)(6)(7)(23)(89)
331	2	432	(2)(7)(9)(15)(38)(46)
332	2	432	(3)(5)(9)(18)(24)(67)
347	2	424	(1)(23)(47)(59)(68)
355	2	420	(1)(2)(3)(6)(9)(45)(78)
360	2	432	(2)(3)(9)(16)(45)(78)
362	3	294	(132)(476)(589)
363	3	294	(184)(265)(397)
364	2	420	(1)(2)(3)(6)(8)(45)(79)
367	2	432	(2)(3)(9)(16)(47)(58)
374	2	432	(1)(2)(4)(36)(59)(78)
375	2	424	(5)(16)(28)(39)(47)
376	2	420	(1)(2)(3)(6)(9)(48)(57)
378	2	424	(1)(25)(34)(69)(78)
380	6	152	(132)(498576)
381	3	294	(172)(395)(468)
382	3	294	(182)(375)(496)
388	2	420	(2)(3)(5)(7)(9)(16)(48)
389	2	420	(1)(2)(3)(6)(8)(47)(59)
390	2	420	(4)(5)(6)(8)(9)(13)(27)
391	2	420	(1)(2)(3)(7)(9)(48)(56)
392	2	432	(2)(3)(9)(16)(45)(78)
394	2	424	(1)(23)(45)(67)(89)
395	4	212	(1)(2)(3)(6)(7)(45)(89)
			(1)(4)(5)(8)(9)(23)(67)
396	40	24	(3)(1276)(4859)
			(45)(89)(16327)

TABLE B
Automorphism-free one-factorizations containing one or two sub-one-factorizations of order 4

One sub-one-factorization
14,27,30,32,34,35,37,40,41,42,43,45,47,50,55,62,63,64,65,67,69,71,73, 77,81,84,87,88,94,96,99,102,103,104,106,109,113,115,117,120,122,123, 124,128,130,133,137,140,155,157,159,162,181,185,196,197,205,218,229, 247,252,256,260,276,283,301,308,343
Two sub-one-factorizations
15,16,25,26,29,48,52,164,255

TABLE C

Other one-factorizations which contain at least one sub-one-factorization of order 4

one-factorization number	Configuration	# STS(19) which contain i sub-STS(7)					
		$i = 1$	2	3	4	6	12
1	123,145,167,189 246,258,279,349 357,368,478,569	1	1	2	1	1	1
2	123,145,167,189 246,357	30	15	4	3	2	
3	123,145,167,189 279,368	54	28	6	4	2	
4	123,145,167,189	136	50	0	6		
5	123,145,167,189	20	7	0	16		
6	123,145	104	16				
7	123,145,279,368	56	12	2	2		
8	123,349	56	12				
9	123,189,246,357	85	21	3	3		
10	123,345	104	16				
11	123,569,478	21	0	5			
12	123,145,569,478	31	9	2	2		
13	123	26					
17	123,167	96	14				
18	123,349	104	16				
19	123,145	104	16				
20	123,189,368	78	54	6			
21	123,145	104	16				
23	123,145	104	16				
24	123	72					
36	123,478,569	55	0	5			
38	123,478,569	108	0	8			
44	123,189	104	16				
51	123	72					
58	123,145	104	16				
61	123	72					
95	123,145	104	16				
98	123,145	104	16				
100	123,189	96	14				
132	123	26					
135	123,145	104	16				
136	123	72					
148	123	72					
234	123,145	56	12				
243	279	72					
254	358	64					
290	123,569,478	7	0	2			
331	279	72					
332	359	72					
347	123	64					
380	123	26					

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