MATHEMATICS OF COMPUTATION
VOLUME 46, NUMBER 174
APRIL 1986, PAGES S15–S20

Supplement to A Convergent 3-D Vortex Method With Grid-Free Stretching

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In this supplement we discuss the more special consistency estimates (C4)-(C6) arising from the fact that the integral kernels have average value zero. This section is not necessary for the proof of convergence and may be omitted without loss of continuity.

Our strategy will be to rewrite the discretization error before applying the quadrature estimate in such a way that the new integrand has improved L^1-estimates. In (C4) we need to compare

$$v(x_i) = -\frac{1}{4\pi} \int f(r/\delta) \frac{x_i - x'}{r^3} \times \omega(x')dx'$$

with the discretization

$$v_i^h = -\frac{1}{4\pi} \sum_j f(r_{ij}/\delta) \frac{x_i - x_j}{r_{ij}^3} \times \omega(x_j)h^3;$$

here $r = |x_i - x'|$, $r_{ij} = |x_i - x_j|$, and both expressions can be interpreted in Lagrangian coordinates by setting $x' = \Phi^t(\alpha)$ and $x_j = \Phi^t(\alpha_j)$, $\alpha_j = jh$. It is understood that the time t is fixed. We will think of x_i as fixed also, and for convenience we assume α is translated so that $\alpha_i = 0$, i.e., $x_i = \Phi^t(0)$. It is also convenient to set $x' = x_i - y$, $x_j = x_i - y_j$. With $\sigma(y) = -\omega(x_i - y)/4\pi$, we now rewrite the two velocity expressions as

$$\text{(A.1)} \qquad v = \int f(r/\delta)\, \frac{y}{r^3}\, \sigma(y)dy, \qquad r = |y| ,$$

$$\text{(A.2)} \qquad v^h = \sum_j f(r_j/\delta)\, \frac{y_j}{r_j^3}\, \sigma(y_j)h^3, \qquad r_j = |y_j| .$$

(It will not be important in this argument to distinguish components of the vector quantities, and we ignore them for simplicity of notation.) The coordinate mapping corresponding to the y-variable is $\Psi(\alpha) = x_i - \Phi(\alpha)$, so that $\Psi(0) = 0$. The sum (A.2) is a trapezoidal rule approximation to the integral (A.1) with $y = \Psi(\alpha)$; that is, the difference has the form

$$E[F] \equiv \sum_j F(\alpha_j)h^3 - \int F(\alpha)d\alpha$$

where F is the integrand of (A.1) with $y = \Psi(\alpha)$.

We estimate $E[F]$ in two stages: first we derive the improved estimate supposing that y in the velocity kernel depends linearly on α near $\alpha = 0$ and then show that the error in neglecting the nonlinear part of Ψ is of the same order. Let $A = (\partial\Psi/\partial\alpha)(0)$ be the Jacobian matrix at the origin. Then $\Psi(\alpha) = A\alpha + \theta_0(\alpha)\alpha^2$ for some smooth function $\theta_0(\alpha)$ given by the integral form of the remainder in the Taylor expansion. Since $\det A = 1$, there exists ρ_1 so that

$$|\Psi(\alpha) - A\alpha| \le |\Psi(\alpha)|/2 \quad \text{for} \quad |\alpha| \le 2\rho_1 .$$

Now let $\mu(\rho)$ be a function so that $\mu = 0$ for $\rho \le \rho_1$, $0 \le \mu \le 1$, and $\mu = 1$ for $\rho \ge 2\rho_1$. We define a new coordinate mapping

$$\Psi_0(\alpha) = A\alpha + \mu(|\alpha|)\theta_0(\alpha)\alpha^2 ;$$

then $\Psi_0(\alpha)$ equals $A\alpha$ for $|\alpha| \le \rho_1$ and $\Psi(\alpha)$ for $|\alpha| \ge 2\rho_1$. By choosing ρ_1 small enough, we can assume that Ψ_0 is invertible. Later we will use the fact that

$$\text{(A.3)} \qquad |\Psi(\alpha) - \Psi_0(\alpha)| \le |\Psi(\alpha)|/2, \quad \text{all } \alpha .$$

We decompose $E[F]$ as

$$E[F] = E[F_0] + E[F_1]$$

where

$$F_0(\alpha) = f(r_0/\delta) y_0 \, r_0^{-3} \, \sigma(y)$$

with $y_0 = \Psi_0(\alpha)$, $r_0 = |y_0|$, $y = \Psi(\alpha)$, and $F_1 = F - F_0$. To take advantage of the oddness of the kernel, we write σ as a sum of two terms, one even in α and the second vanishing at 0. Let $\nu(\alpha)$ be a smooth radial function with support in $\{\alpha: |\alpha| < \rho_1\}$ such that $\nu(0) = 1$, and write $\sigma = \sigma_0 \circ \Psi^{-1} + \sigma_1$, with $\sigma_0(\alpha) = \nu(\alpha)\sigma(0)$. Let $F_0 = F_{00} + F_{01}$ in the corresponding way. Since Ψ_0 is linear on the support of ν, F_0 is an odd function of α. Consequently, both the sum and the integral in $E[F_{00}]$ vanish identically. (We assume here that the α-grid is extended if necessary to cover the support of ν; this does not affect the final result since the original grid covers the support of ω_0.)

The improved estimate for F_{01} is possible because $\sigma_1(0) = 0$ and the singularity of the integrand is therefore reduced. By the composition $y = \Psi(\Psi_0^{-1} y_0)$ we can regard σ_1 as a function of y_0 and write

$$\sigma_1(y) = y_0 \cdot \sigma^{\#}(y_0), \quad \sigma^{\#}(y_0) = \int_0^1 \nabla(\sigma_1 \circ \Psi \circ \Psi_0^{-1})(s y_0) ds .$$

Also, by assumption (F1), $f(\eta) = |\eta|^3 f^{\#}(\eta)$ for a smooth function $f^{\#}$, and after substituting we have

$$F_{01} = \delta^{-3} r_0^3 f^{\#}(y_0/\delta) y_0 \, r_0^{-3} \cdot y_0 \, \sigma^{\#}(y_0) = \delta^{-3} y_0^2 f^{\#}(y_0/\delta)\sigma^{\#}(y_0)$$

$$\equiv \delta^{-1} g(y_0/\delta)\sigma^{\#}(y_0), \quad g(\eta) = \eta^2 f^{\#}(\eta) .$$

In order to apply the quadrature lemma we need to estimate derivatives of this integrand in L^1, as a function of α. It is enough to estimate the derivatives with respect to y_0. We have

$$D^k\{\delta^{-1} g(y_0/\delta)\} = \delta^{-1-k}(D^k g)(y_0/\delta)$$

so that

$$\int_{|y_0|<\delta} |D^k\{\delta^{-1} g(y_0/\delta)\}| dy_0 \le C\delta^3 \delta^{-1-k} = C\delta^{2-k}.$$

On the other hand, for $|y_0| > \delta$ we write

$$F_{01} = f(r_0/\delta) y_0^2 \, r_0^{-3} \, \sigma^{\#}(y_0)$$

and use the rapid decrease of Df (assumption (F4)). We have

$$|D^k f(\rho)| \le C_k \rho^{-k}, \quad \rho \ge 1$$

so that

$$|D^k\{f(r_0/\delta)y_0^2\, r_0^{-3}\}| \le Cr_0^{-k-1}$$

and

$$|D^k\, F_{01}(y_0)| \le Cr_0^{-k-1}$$

for $r_0 \ge \delta$. Thus

$$\int_{|y_0|>\delta} |D^k F_{01}|dy_0 \le \begin{cases} C\delta^{2-k}, & k \ge 3 \\ C|\log \delta|, & k = 0,1,2 \end{cases},$$

and combining the two parts we have

$$\int |D^k F_{01}|dy_0 \le C\delta^{2-k}, \quad k \ge 3 .$$

It now follows from the quadrature lemma that

$$\text{(A.4)} \qquad |E[F_{01}]| \le C\delta^2(h/\delta)^{\ell}, \quad \ell \ge 4.$$

It remains to show that $E[F_1]$ has the same bound as above; that is, we have to estimate the quadrature error for the integrand

$$F_1(\alpha) = \{f(r/\delta)yr^{-3} - f(r_0/\delta)y_0r_0^{-3}\}\sigma(y)$$

where as before

$$y = \Psi(\alpha), \quad y_0 = \Psi_0(\alpha), \quad r = |y|, \quad r_0 = |y_0| .$$

Define

$$g(\eta) = f(|\eta|)\eta|\eta|^{-3};$$

then

$$F_1 = \delta^{-2}\{g(y/\delta) - g(y_0/\delta)\}\sigma(y) .$$

We can express the difference of g-values as

$$g(\eta_1) - g(\eta_0) = \tilde{g}(\eta_1,\eta_0)(\eta_1 - \eta_0),$$

$$\tilde{g} = \int_0^1 \nabla g(\eta_0 + s(\eta_1 - \eta_0))ds.$$

Now, from the rapid decrease property (F4) of f, we have

$$|D^k g(\eta)| \le C|\eta|^{-2-k},$$

provided η is bounded away from zero. A similar property holds for $\tilde{g}$: suppose that $|\eta_1 - \eta_0| \le |\eta_1|/2$, in analogy with (A.3), and also that η_1 is bounded away from zero. Then

$$|\eta_0 + s(\eta_1 - \eta_0)| \ge |\eta_1|/2, \quad 0 \le s \le 1$$

and therefore

$$|D^k \tilde{g}(\eta_1, \eta_0)| \le C\, |\eta_1|^{-3-k}, \tag{A.5}$$

where the derivatives can be with respect to η_0 or η_1.

Our integrand now has the form

$$F_1 = \delta^{-3}(y - y_0)\tilde{g}(y/\delta, y_0/\delta)\sigma(y) .$$

Recall that $y - y_0 = \alpha^2\theta(\alpha)$, where $\theta(\alpha) = (1 - \mu(|\alpha|))\theta_0(\alpha)$; we can therefore write

$$F_1 = \delta^{-3}\, \alpha^2\, \theta(\alpha)\tilde{g}(y/\delta, y_0/\delta)\sigma(y) .$$

Since y/δ, y_0/δ are bounded for $|\alpha| < \delta$, we have

$$\int_{|\alpha|<\delta} |D^k_\alpha F_1|d\alpha \le C\delta^{-3} \cdot \delta^2 \cdot \delta^{-k} \cdot \delta^3 = C\delta^{2-k} .$$

We need to estimate the same quantity for $|\alpha| \ge \delta$. Since $D\Psi^{-1}$ is uniformly bounded, there exists c_0 so that $|\alpha| \ge \delta$ implies $|y|, |y_0| \ge c_0|\alpha|$. Recalling (A.3), we deduce from (A.5) that

$$|D^k_\alpha \hat{g}(y/\delta, y_0/\delta)| \le C\delta^{-k}(\delta/|\alpha|)^{k+3} = C\delta^3/|\alpha|^{k+3} .$$

Thus

$$|D^k_\alpha F_1| \le C\delta^{-3} \cdot \delta^3\, |\alpha|^2|\alpha|^{-(k+3)} = C|\alpha|^{-k-1},$$

and

$$\int_{|\alpha|\ge\delta} |D^k_\alpha F_1|d\alpha \le C\int_\delta^R \rho^{-k-1}\rho^2 d\rho \le C\delta^{2-k},$$

provided $k \ge 3$. The L^1-bounds we have now established for $D^k F_1$ imply that

$$|E[F_1]| \le C\delta^2(h/\delta)^\ell, \quad \ell \ge 4 , \tag{A.6}$$

again by the quadrature lemma, and (C4) follows by combining (A.4), (A.6) with the moment estimate, as in the verification of (C1).

It was noted in the Introduction that the smoothed version of the rate-of-strain tensor $\nabla^s u$ has the form (see (1.9))

$$\int \sum_{m,n} c_{k\ell mn} G_{\delta,\ell m}(x - x')\omega(x',t)_n \, dx'$$

for the (k,ℓ) component, where repeated second derivatives only occur in the combination $G_{\delta,\ell\ell} - G_{\delta,mm}$. If we again write $f(\rho) = \rho^3 f^{\#}(\rho)$, then, with subscripts denoting components,

$$G_{\delta,\ell m}(x) = -\frac{1}{4\pi} \frac{x_\ell x_m}{r\,\delta^4} (f^{\#})'(r/\delta), \qquad \ell \neq m$$

and

$$G_{\delta,\ell\ell}(x) - G_{\delta,mm}(x) = -\frac{1}{4\pi} \frac{x_\ell^2 - x_m^2}{r\,\delta^4} (f^{\#})'(r/\delta) .$$

Thus each kernel has average value zero. This property is retained on a rectangular grid, in the sense that $\nabla^s u^h(\alpha_i,0)$ would be zero if $\omega_0(\alpha_j)$ were replaced by a radial function of $\alpha_j - \alpha_i$: Under the linear transformation $x_\ell \to -x_\ell$ in the first case or $x_\ell \to x_m$, $x_m \to x_\ell$ in the second case, the grid is mapped to itself in such a way that lengths are preserved but the sign of the kernel is reversed. With this observation, the estimate (C5) for the discretization of $\nabla^s u$ at time zero can be derived by an argument like the estimation of $E[F_0]$ above. The same estimate seems unlikely to hold at later time; the average zero property on the grid is not preserved here under a linear coordinate transformation, as it was in the case of the velocity kernel, since a different symmetry is involved.

However, we may take advantage of the zero average of the kernel to reduce the order of singularity of the integrand before discretizing. This can be done in principle in the following way. Suppose that Ω has been chosen so that each point in the support of ω_0 has a neighborhood of radius R_{00} inside Ω, and the initial markers $\{\alpha_i\}$ cover Ω. Then by the smoothness of the flow there is some R_0, depending on R_{00} and T, so that each point in the support of $\omega(\cdot,t)$ has a neighborhood of radius R_0 in $\Phi^t(\Omega)$ for $0 \le t \le T$. Let $\varsigma(r)$ be a smooth function so that $\varsigma(0) = 1$ and $\varsigma(r) = 0$ for $r \ge R_0$. We need only evaluate $\nabla^s u$ at points $x_i(t)$ inside the support of $\omega(\cdot,t)$; for such points we can replace $\omega(x',t)$ in the integrand above by $\omega(x',t) - \varsigma(|x - x'|)\omega(x,t)$ without changing the value, thereby reducing the order of the singularity at $x' = x$. We now use the discretization

$$\nabla_0^s u^h(x_i,t) \doteq \sum_j \sum_{m,n} c_{k\ell mm} G_{\delta,\ell m}(x_i - x_j)(\omega_j - \varsigma_{ij}\omega_i)_n h^3$$

where $\varsigma_{ij} = \varsigma(|x_j - x_i|)$. The improved consistency estimate (C6) for this version can be verified as in the treatment of above (with y_0 replaced by y). We have not checked that the stability estimate analogous to (S2) holds in this case, although it appears likely that it does.