MATHEMATICS OF COMPUTATION
VOLUME 46, NUMBER 174
APRIL 1986, PAGES S21-S27

Supplement to On Weighted Chebyshev-Type Quadrature Formulas

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6. PROOFS OF THE THEOREMS

Fundamental for our considerations are Newton's well-known identities [7, p. 104] :

Let $t_1,\ldots,t_n$ be a solution of

$$(6.1)\qquad \sum_{i=1}^{n} t_i^{\,j} = u_j\ ,\quad j = 1,2,\ldots,n\ ,$$

for given values $u_j \in \mathbb{R}$. Let

$$(6.2)\qquad g(t) := \prod_{i=1}^{n} (t-t_i) := t^n + \sum_{i=1}^{n} c_i\, t^{n-i}\ .$$

Then

$$(6.3)\qquad \begin{aligned} c_1 &= -u_1 \\ c_1u_1 + 2c_2 &= -u_2 \\ c_1u_2 + c_2u_1 + 3c_3 &= -u_3 \\ &\ \vdots \\ c_1u_{n-1} + c_2u_{n-2} + \ldots + c_{n-1}u_1 + n\,c_n &= -u_n\ . \end{aligned}$$

If, conversely, the coefficients c_i are a solution of the system (6.3) then the roots t_i of the corresponding function $g(t)$ in (6.2) satisfy (6.1).

With respect to (1.3) the following holds for all $Q_n \in T(n,d)$ and $j = 1,\ldots,d$

$$(6.4)\qquad \sum_{i=1}^{n} x_i^{\,j} = \frac{n\,m_j}{m_o}\ .$$

Let $P_n(Q_n)$ be that polynomial of degree n whose roots are the nodes of Q_n

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$$(6.5)\qquad P_n(Q_n)(x) := \prod_{i=1}^{n} (x-x_i)$$

$$:= x^n + \sum_{i=1}^{d} a_i\, x^{n-i} + \sum_{i=d+1}^{n} b_i\, x^{n-i} .$$

It follows from (6.3) and (6.4) that the coefficients a_i in (6.5) of all $P_n(Q_n)$ with $Q_n \in T(n,d)$ are identical. Conversely, every P_n of type (6.5) with these coefficients a_i and only real roots represents a formula $Q_n \in T(n,d)$.

Let H_n be that polynomial of type (6.2) whose coefficients (6.3) are uniquely given by (6.4) with $j=1,\dots,n$:

$$(6.6)\qquad H_n(x) := x^n + \sum_{i=1}^{n} e_i\, x^{n-i} .$$

It follows from (6.3) for $Q_n \in T(n,d)$ and the corresponding $P_n(Q_n)$

$$R_n\,[p_{d+1}] = \int_a^b w(x)\, x^{d+1}\, dx - \frac{m_o}{n} \sum_{i=1}^{n} x_i^{d+1}$$

$$(6.7)\qquad = m_{d+1} + \frac{m_o}{n} \{(d+1)\, b_{d+1} + \sum_{i=1}^{d} a_i \frac{n}{m_o} m_{d+1-i}\}$$

$$= \frac{m_o}{n} (d+1)\, (b_{d+1} - e_{d+1}) .$$

$Q_n \in T(n,d)$ is therefore E-minimal (E-maximal) in $T(n,d)$ if and only if the coefficient b_{d+1} of $P_n(Q_n)$ is minimal (maximal) under the restriction of only real roots of $P_n(Q_n)$ [11].

Corresponding to the sequence $S(Q_n)$ (see (2.4)) we define a sequence $V(Q_n)$ as follows. Let k be the number of sign changes (2.5) of $S(Q_n)$. If there is a sign change between $s_i(Q_n)$ and $s_{i+\ell}(Q_n)$, $0<i<i+\ell<r$, the pair (u,v) is given by

$$(u,v) := (x_{n+1-i-\ell}, x_{n-i}) .$$

The sequence $V(Q_n) := \{(u_i,v_i)\}_{i=1}^{k}$ consists of all these pairs ordered by $u_i \geq u_{i+1}$. By this definition, it is possible that $u_i = v_i = u_{i+1} = v_{i+1}$ and for $k = 0$ that $V(Q_n)$ is empty. We define $K(Q_n) \in \{0,1\}$ by

$$(6.8)\qquad K(Q_n) := \begin{cases} 0, & \text{if all elements of } S(Q_n) \text{ are zero} \\ 1, & \text{if an element of } S(Q_n) \text{ is different from zero.} \end{cases}$$

<u>Lemma 1 :</u> Let $Q_n \in T(n,d)$. Let k be the number of sign changes of $S(Q_n)$. If

$$(6.9)\qquad k + K(Q_n) < n-d,$$

then Q_n is not E-extremal in $T(n,d)$.

Proof: We have to show the existence of $\bar{Q}_n$, $\tilde{Q}_n \in T(n,d)$ with

$$(6.10)\qquad \bar{R}_n[p_{d+1}] < R_n[p_{d+1}] < \tilde{R}_n[p_{d+1}] .$$

One has $k = 0$ for $d = n-1$ and $k \leq n-d-2$ for $d < n-1$. If $d = n-1$ we choose two real numbers $y_o > x_n$ and $y_1 < x_1$. If $d < n-1$ we choose $(n-d)$ real numbers $y_o \geq y_1 \geq \dots \geq y_{n-d-1}$ by means of the sequence $V(Q_n) = \{(u_i,v_i)\}_{i=1}^{k}$ as follows

$$y_i \in (u_i,v_i) \text{ and } y_i \neq x_j\ (j=1,\dots,n), \text{ for } u_i \neq v_i$$

$$y_i = u_i \qquad \text{for } u_i = v_i$$

$$y_o > x_n$$

$$y_{n-d-1} < \dots < y_{k+1} < x_1 .$$

Let α be the first nonzero element of $S(Q_n)$. If all elements of $S(Q_n)$ are zero we define $\alpha := -1$. We consider polynomials $\bar{h}$, $\tilde{h}$ of degree $n-d-1$ given by

$$\bar{h}(x) := -\alpha \sum_{i=0}^{n-d-2} (x-y_i),$$

(6.12)

$$\tilde{h}(x) := \alpha \sum_{i=1}^{n-d-1} (x-y_i).$$

For $d = n-1$ we obtain $\bar{h} \equiv -\alpha$ and $\tilde{h} \equiv \alpha$. First, we consider the case $\alpha = -1$. In every interval (y_{2i+1}, y_{2i}), $i = 0,1,\ldots$, by definition of $S(Q_n)$ and $V(Q_n)$ all relative minima of $P_n(Q_n)$ are negative. Let M_1 be the (negative) maximum of all these minima. A similar argument yields the positivity of all relative maxima in every interval (y_{2i}, y_{2i-1}). We denote by M_2 the (positive) minimum of all these maxima. Therefore the following numbers $\bar{m}$, $\tilde{m}$ are positive,

$$\bar{m} := \frac{\min\{|M_1|, M_2\}}{\max_{x\in[y_{n-d-1}, y_o]} |\bar{h}(x)|},$$

(6.13)

$$\tilde{m} := \frac{\min\{|M_1|, M_2\}}{\max_{x\in[y_{n-d-1}, y_o]} |\tilde{h}(x)|}.$$

The polynomials $\bar{P}_n := P_n(Q_n) - \bar{m}\,\bar{h}$ and $\tilde{P}_n := P_n(Q_n) - \tilde{m}\,\tilde{h}$ have only real roots. We proof this for $\bar{P}_n$; for $\tilde{P}_n$ the proof is similar. We distinguish two cases.

1) We consider an interval (y_{r+1}, y_r). First we assume that neither y_r nor y_{r+1} coincides with a root of $P_n(Q_n)$. Let ℓ be the number of roots of $P_n(Q_n)$ in this interval. It follows from (6.12) that $\bar{h}$ is of constant sign in (y_{r+1}, y_r). This sign is negative for r even or zero, otherwise it is positive. In (y_{r+1}, y_r), $P_n(Q_n)$ has only roots of maximal multiplicity two by definition of $S(Q_n)$. The existence of a root with higher multiplicity would imply that y_r or y_{r+1} coincides with this root, contrary to our assumption. In the case that two roots of P_n coincide in $y \in (y_{r+1}, y_r)$ then $P_n(Q_n)$ has in y a relative minimum for odd r and a relative maximum otherwise. By choice of $\bar{m}$ the polynomials $P_n(Q_n)$ and $\bar{m}\,\bar{h}$ have ℓ points of intersection.

2) We now consider the case, that y_r coincides with a root of $P_n(Q_n)$. By (6.11) this is possible only for $k > 0$ and $n-d > 2$. The interval (y_{r+i}, y_{r-j}) is defined as follows.
Each $y_p \in (y_{r+i}, y_{r-j})$ is a root of $P_n(Q_n)$, and y_{r+i} as well as y_{r-j} are not roots of $P_n(Q_n)$. Especially y_o and y_{n-d-1} are not roots of $P_n(Q_n)$ by (6.11).
Let ℓ be the number of distinct y_p in $[y_{r+i}, y_{r-j}]$, i.e., the number of elements of the set $\{y_{r+i}, y_{r+i-1}, \ldots, y_{r-j+1}, y_{r-j}\}$. We denote these elements by z_p, $z_1 > z_2 > \ldots > z_\ell$. Let Z_p be the multiplicity of the roots of $\bar{h}$ at z_p and let W_p be the number of roots of $P_n(Q_n)$ in the interval (z_{p+1}, z_p) for $p = 1,\ldots,\ell-1$. We remark that $P_n(Q_n)$ has for $1 < p < \ell$ in z_p a root of multiplicity Z_p+2. $P_n(Q_n) - \bar{m}\,\bar{h}$ has in (y_{r+i}, y_{r-j}) the same number of roots as $P_n(Q_n)$ if $P_n(Q_n) - \bar{m}\,\bar{h}$ has in (z_{p+1}, z_p) for $p \neq 1$, $p \neq \ell-1$ at least W_p+2 roots and for $p = 1$ and $p = \ell-1$ at least W_p+1 roots.
This can be shown by similar argumentation as in case 1), considering the fact, that on the one hand for $1 < p < \ell-1$ the multiplicity of the root z_p of $P_n(Q_n)$ is equal to the multiplicity of the root of $\bar{h}$ in z_p plus two and that on the other hand there exists a $\varepsilon > 0$ so that $\bar{h}$ and $P_n(Q_n)$ are of same constant sign in $z_p-\varepsilon$ as well as in $z_p+\varepsilon$.

By these two cases all roots of $P_n(Q_n)$ have been taken into account once. So $P_n(Q_n)$ and $P_n(Q_n) - \bar{m}\,\bar{h}$ have the same number of real roots. $\bar{h}$ is a polynomial of degree n-d-1. Therefore the formula $\bar{Q}_n$ resulting from $\bar{P}_n = P_n(Q_n) - \bar{m}\,\bar{h}$ is in T(n,d). The leading coefficient of $\bar{m}\,\bar{h}$ is positive by virtue of (6.12). Therefore (6.7) implies the left part of (6.10). For $\alpha = 1$ the argumentation is similar. $\tilde{h}$ and $\bar{h}$ have to be changed.

Remark 1: The proof remains valid if in (6.13) the value $\bar{m}$ ($\tilde{m}$) is replaced by $\bar{m} := c\,\bar{m}$ ($\tilde{m} := c\,\tilde{m}$) with $c \in (0,1)$. Therefore (6.10) implies the fact that under the assumptions of Lemma 1 there is a positive $s \in \mathbb{R}$ so that for any r with

$$r \in [R_n[p_{d+1}] - s\ ,\ R_n[p_{d+1}] + s]$$

a formula $\tilde{Q}_n$ exists in T(n,d) with $\tilde{R}_n[p_{d+1}] = r$.

Remark 2: If $P_n(Q_n)$ has roots of multiplicity higher than one, then the maximal multiplicity of the roots of $P_n(Q_n) - \bar{m}\,\bar{h}$ or $P_n(Q_n) - \tilde{m}\,\tilde{h}$ with any $\bar{m}$ or $\tilde{m}$ chosen as in Remark 1) is lower than the maximal multiplicity before. For these formulas $\tilde{Q}_n$ and $\bar{Q}_n$, Equ. (6.9) is also valid. So if a formula $Q_n \in T(n,d)$ has property (6.9), then there exists an infinite number of formulas in T(n,d) with pairwise distinct nodes.

Let $\bar{Q}_n$ be an E-extremal formula in T(n,d). Lemma 1 implies that $S(\bar{Q}_n)$ has for d = n-1 at least one nonzero element and for d < n-1 at least n-d-1 changes of sign. With definition (2.4), this yields the proof for the first part of Theorem 2.

The next lemma follows from Theorem 1 of [5] together with Remark 1, there.

Lemma 2: Let Q_n and $\tilde{Q}_n$ be in T(n,d). Let $L := Q_n - Q_n$ with

$$L[f] := \sum_{i=1}^{m} A_i\, f(B_i),\ i=1,\dots,m,\ B_1 < B_2 < \dots < B_m\ . \qquad (6.14)$$

Let q be defined by $B_q := \max\{B_i \mid A_i \neq 0,\ i=1,\dots,m\}$ and let A be the following sequence

$$A := \{\sum_{i=1}^{k} A_i\}_{k=1}^{m}\ .$$

Then

1) A has at least d changes of sign.
2) If A has at most d changes of sign, then for every

$$f \in C^{d+1}[B_1,B_m],\ f^{(d+1)} \geq 0\ ,$$

$$\mathrm{sign}\,(A_q)\ \tilde{R}_n[f] \ \geq\ \mathrm{sign}\,(A_1)\ R_n[f]\ .$$

In the following we first consider the case d < n-1. Let $\bar{x}_i$ be the nodes of an E-extremal formula $\bar{Q}_n$ in T(n,d). Lemma 1 and (2.4) imply that there exist n-d nodes $\bar{x}_{g(i)}$ with $g(i) < g(i+1)$ for i=1,...,n-d-1 so that

$$\bar{x}_{g(i)} = \bar{x}_{g(i)+1}\ ,\qquad i=1,\dots,n-d\ ,$$
(6.15)
$$[g(i) + g(i+1)]\ \text{ is odd for }\ i=1,\dots,n-d-1\ .$$

Let $\bar{x}_{g(i)+1} \neq \bar{x}_{g(i+1)}$. Then the number of nodes between $\bar{x}_{g(i)+1}$ and $\bar{x}_{g(i+1)}$ is equal to $g(i+1)-g(i)-2$. Eq. (6.15) shows that this number is odd. Let this number be $2\ell(i)-1$. In case of $\bar{x}_{g(i)+1} = \bar{x}_{g(i+1)}$ - for avoiding multiple counting of the same node - we define $2\ell(i)-1 := -1$. Let $\ell(0) := g(1)-1$ and $\ell(n-d) := n-g(n-d)-1$. There follows

$$n = \ell(0) + \ell(n-d) + 2(n-d) + \sum_{i=1}^{n-d-1} (2\ell(i)-1)\ ,$$
(6.16)
$$d = 1 + \ell(0) + \ell(n-d) + 2\sum_{i=1}^{n-d-1} \ell(i)\ .$$

In Lemma 2 let for $Q_n \in T(n,d)$

$$L := Q_n - \bar{Q}_n ,$$

$$L[f] := \sum_{i=1}^{m} A_i f(B_i) .$$

For the nodes B_i we require

$$(6.17) \qquad B_1 < B_2 < \dots < B_m , \qquad \{B_1, B_2, \dots, B_m\} = \{x_1, \dots, x_n, \bar{x}_1, \dots, \bar{x}_n\} .$$

We see that some coefficients A_i may be zero. Let $A_{r,s}$ be the sequence

$$(6.18) \qquad A_{r,s} := \{ \sum_{j=1}^{k} A_j \}_{k=r}^{s} \qquad 1 \le r \le s \le m$$

and by (6.17)

$$B_{\bar{g}(i)} = \bar{x}_{g(i)} .$$

$\bar{x}_{g(i)} = \bar{x}_{g(i+1)}$, implies $\bar{g}(i) = \bar{g}(i+1)$ and therefore implies that $A_{\bar{g}(i),\bar{g}(i+1)}$ has no sign change, i.e., by the above definition of $\ell(i)$ this sequence has $2\ell(i)$ sign changes. In the case $\bar{g}(i) \neq \bar{g}(i+1)$ the number of nodes in the interval $(B_{\bar{g}(i)}, B_{\bar{g}(i+1)})$ is at least $2\ell(i)-1$. For an estimation of the number of sign changes of $A_{\bar{g}(i),\bar{g}(i+1)}$ we remark, that every element of the sequence A is of the type $c_j c$ with $c_j \in \mathbb{Z}$ and c defined in (1.4). The case $A_{r,r}$ positive and $A_{r+s,r+s}$ negative implies therefore that in the interval $(B_r, B_{r+s}]$ there are at least two nodes of $\bar{Q}_n$. This means that the sequence $A_{\bar{g}(i),\bar{g}(i+1)}$ has at most $2\ell(i)$ changes of sign. This number is reduced to $2\ell(i)-1$ if $A_{\bar{g}(i),\bar{g}(i)}$ or $A_{\bar{g}(i+1),\bar{g}(i+1)}$ is not negative. In the following this last remark is of importance in the case $\ell(n-d) = 0$ and $\bar{x}_n \ge x_n$, because $A_{\bar{g}(n-d),\bar{g}(n-d)} = A_{m,m} = 0$. So we have shown that the number of sign changes of the sequence $A_{\bar{g}(1),\bar{g}(n-d)}$ is at most

$$(6.19) \qquad 2 \sum_{i=1}^{n-d-1} \ell(i)$$

or even

$$(6.20) \qquad \{ 2 \sum_{i=1}^{n-d-1} \ell(i) \} -1 \qquad \text{if } \ell(n-d) = 0 \text{ and } \bar{x}_n \ge x_n \text{ and } \bar{g}(1) \neq \bar{g}(n-d).$$

We now consider the number of sign changes of the sequences $A_{1,\bar{g}(1)}$ and $A_{\bar{g}(n-d),m}$. We distinguish between the following cases.

	I)	Ia) $\ell(0)$ even or $\ell(0)=0$	II)	IIa) $\ell(n-d)$ even or $\ell(n-d)=0$
(6.21)		Ib) $\ell(0)$ odd		IIb) $\ell(n-d)$ odd
	1)	1a) $x_1 < \bar{x}_1$	2)	2a) $x_n < \bar{x}_n$
(6.22)		1b) $x_1 > \bar{x}_1$		2b) $x_n > \bar{x}_n$
		1c) $x_1 = \bar{x}_1$		2c) $x_n = \bar{x}_n$

By the same argumentation as above we get the following results. The number of sign changes of $A_{1,\bar{g}(1)}$ is at most

$$(6.23) \qquad \ell(0)+1 \qquad \text{for Ia1a) or Ib1b) if } A_{\bar{g}(1),\bar{g}(1)} < 0 .$$

In case of every other combination of I) and 1) the number of sign changes of $A_{1,\bar{g}(1)}$ is at most $\ell(0)$.

Because of $A_{m,m} = 0$ the number of sign changes of $A_{\bar{g}(n-d),m}$ is at most

$$(6.24) \qquad \ell(n-d) \qquad \text{for i) IIa2b) or IIa2a) \quad ii) } \ell(n-d) = 0 .$$

For any other combination of II) and 2) there are at most $\ell(n-d)-1$ sign changes of this sequence.

By virtue of (6.16) and (6.19), (6.20), (6.23), (6.24) the sequence A has at most d changes of sign. This number can only be achieved in the following cases

$$(6.24)\quad \begin{array}{ll} \text{A)} & x_n < \bar{x}_n,\ x_1 > \bar{x}_1,\ \ell(n-d)\text{ odd},\ \ell(0)\text{ odd}, \\ \text{B)} & x_n < \bar{x}_n,\ x_1 < \bar{x}_1,\ \ell(n-d)\text{ odd},\ \ell(0)\text{ even or }\ell(0)=0, \\ \text{C)} & x_n > \bar{x}_n,\ x_1 > \bar{x}_1,\ \ell(n-d)\text{ even or }\ell(n-d)=0,\ \ell(0)\text{ odd}, \\ \text{D)} & x_n > \bar{x}_n,\ x_1 < \bar{x}_1,\ \ell(n-d)\text{ even or }\ell(n-d)=0,\ \ell(0)\text{ even or }\ell(0)=0. \end{array}$$

So we have shown for every Q_n in $T(n,d)$ and for every $f \in C^{d+1}$, $f^{(d+1)} \geq 0$, with the help of Lemma 2, the following result

$$(6.25)\quad \begin{array}{ll} R_n[f] \geq \bar{R}_n[f] & \text{for } \ell(n-d)\text{ odd}, \\ R_n[f] \leq \bar{R}_n[f] & \text{for } \ell(n-d)\text{ even or } \ell(n-d)=0. \end{array}$$

The number $\ell(n-d)$ resp. $\ell(0)$ is independent of the choice of Q_n in $T(n,d)$. Therefore, $\bar{Q}_n$ is E-minimal for odd $\ell(n-d)$, and E-maximal for even $\ell(n-d)$, or $\ell(n-d)=0$. This proves Theorem 5.

(6.24) resp. (6.25) follows only from the fact that $S(\bar{Q}_n)$ has at least n-d-1 changes of sign. So, by means of Lemma 1, we have proven the first part of Theorem 2i. The second part of Theorem 2i follows from the definition of $\ell(n-d)$ and (2.4).

In the cases A and D in (6.24), d is odd; in the other two cases d is even. This proves Theorem 4 and therefore also Theorem 1.

In the case d = n-1, these theorems can be proven in the same way or with the help of the methods given in[4]. We remark that by Lemma 1 an E-extremal formula has at least one multiple node $\bar{x}_{g(1)}$.

Theorem 3i follows immediately from Theorem 2 and Remarks 1 and 2. To prove Theorem 3ii we consider the corresponding formula Q_n in Theorem 3i and ask for "E-extremal" formulas under the additional assumption $R_n[p_{d+1}] = r$. By the methods given above, it follows that these "E-extremal" formulas have at most d+1 distinct nodes (see Theorem 2).

Let $n > 2$ and therefore $d_n > 1$. Let $Q^{min}_{n,d}$ and $Q^{max}_{n,d}$ be the E-minimal and the E-maximal formula in $T(n,d_n)$. It follows from the definition of d_n and from Theorem 3 that

$$(6.26)\qquad 0 \in [R^{min}_{n,d}[p_{d_n+1}]\ ,\ R^{max}_{n,d}[p_{d_n+1}]]\ .$$

By definition, the E-optimal formula Q^{opt}_n must be either E-minimal or E-maximal. This proves Theorem 6 for $n > 2$.

For n = 1, T(1,1) has only one element, so the conclusion is valid. For n = 2, there remains the case $d_2=1$. For every Q_2 in T(2,1) $P(Q_2)$ is a parabola with positive leading coefficient. So there exists a unique E-maximal formula in T(2.1). This is E-optimal by the same arguments as above. An E-minimal formula doesn't exist.

For the proof of Theorem 7, we first consider the case that Q^{opt}_n is the E-minimal formula in $T(n,d_n)$. It follows from (6.26) and Theorem 5 that

$$(6.27)\qquad R^{opt}_n\ [p_{d_n+1}] > 0\ ,$$

$$(6.28)\qquad R^{opt}_n\ [f] \leq R_n[f]$$

for every Q_n in T(n,d) and every $f \in C^{d_n+1}$, $f^{(d_n+1)} \geq 0$. (6.27) and (6.28) imply the assertion. For E-maximal Q^{opt}_n the inequality signs in (6.27) and (6.28) have to be reversed.

It remains to prove Corollary 3a for n = 12. One has $d_{12}=9$ and $R^{opt}_{12}[p_{10}] > 0$. By Theorem 5 and the symmetry of Q^{opt}_{12} we have to show

(6.29) $$R_{12}^{opt}\,[p_{2i+10}] \geq 0$$

for every $i \in \mathbb{N}$. With Peano's represention of the remainder term - cf. [3,p.39] - there follows

(6.30) $$\frac{1}{2}\,\frac{(2i)!}{(2i+10)!}\,R_{12}^{opt}\,[p_{2i+10}] = \int_a^b x^{2i}\,K_{10}^{opt}(x)\,dx\ .$$

K_{10}^{opt} denotes the Peano kernel of highest degree with respect to Q_{12}^{opt}. K_{10}^{opt} has in (0,b) only one change of sign and is negative at the orign - see [1,p.65]. The positivity of R_{12}^{opt} therefore implies

(6.31) $$\int_0^b K_{10}^{opt}(x)\,dx > 0\ .$$

From (6.30) and (6.31) the inequality (6.29) follows in view of the monotonicity of p_{2i} in (0,b).