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Logarithmic Convexity for Discrete Harmonic Functions and the Approximation of the Cauchy Problem for Poisson's Equation

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Abstract. Logarithmic convexity type continuous dependence results for discrete harmonic functions defined as solutions of the standard C^0 piecewise-linear approximation to Laplace's equation are proved. Using this result, error estimates for a regularization method for approximating the Cauchy problem for Poisson's equation on a rectangle are obtained. Numerical results are presented.

1. Introduction. This paper will examine numerical methods for approximating the following Cauchy problem for Poisson's equation. Let $\Omega = [0,1] \times [0,1]$, let Γ denote the boundary of Ω, let Σ denote the open segment of Γ lying on the x-axis, and let Σ' denote the open segment of Γ along $y = 1$. Then, given functions f, g_1, and g_2, and positive parameters M, ε_1, and ε_2, we seek a function u that satisfies:

$$(1) \qquad \begin{aligned} &-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{when } x = 0, \text{or } x = 1, \\ &|u - g_1|_{1,\Sigma} \leqslant \varepsilon_1, \quad |u_n - g_2|_{0,\Sigma} \leqslant \varepsilon_2, \quad |u|_{0,\Sigma'} \leqslant M. \end{aligned}$$

Here $|\cdot|_{m,*}$ denotes the mth Sobolev norm on the line segment $*$. The L_2 bound on Σ' stabilizes what would otherwise be an ill-posed problem. Such stability questions for elliptic Cauchy problems have been discussed extensively by Payne (cf. [8], [9], and [10]). However, these results have only been derived for continuous problems, such as Problem (1). If we wish to analyze discrete methods for approximating Problem (1), or more general elliptic Cauchy problems, these continuous stability results are not sufficient. This is because, although it is possible to derive error estimates using the continuous theory, such estimates tend to be pessimistic. Furthermore, constraints on the method to ensure stability will be excessive.

Previous work on the numerical approximation of the Cauchy problem for Poisson's equation includes the work of Douglas [4], Cannon [1], Cannon and Miller [2], and Cannon and Douglas [3]. In these papers, stability and error estimates are derived for special numerical schemes. Franzone and Magenes [5] have presented detailed numerical and experimental work on an application of the Cauchy problem

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in electrocardiology. In their paper, the Cauchy problem was approximated using finite elements and was solved by a least squares penalty technique similar to the one analyzed in the present paper. A different approach was considered by Houde Han [7], who posed the Cauchy problem as a variational inequality and proved convergence (but not order estimates) for a finite element discretization of the problem.

The first concern of this paper is to prove that particular discrete harmonic functions satisfy logarithmic convexity type results. As a consequence, we will be able to prove stability in a discrete version of Problem (1). Our proofs are guided by the ideas of Payne [10]. Having derived these stability results, we will then prove error estimates for a discrete least squares method for approximating Problem (1). We recognize that our least squares method is possibly not the ideal method for solving Problem (1). However, unlike more specialized methods, the numerical method proposed here generalizes to more complex elliptic operators, and more complex domains. By concentrating on this model problem, we are able to give an analysis which avoids many technical problems and highlights the type of results to be expected of the method.

The layout of this paper is as follows. In the remainder of the introduction we will define some notation to be used throughout the paper. In Section 2, we will define a particular finite element space on Ω, and then prove that certain discrete harmonic functions satisfy logarithmic convexity type results. Although we will only provide detailed proofs of our theorems for solutions of Laplace's equation, we will indicate how the proofs can be generalized to prove logarithmic convexity of solutions of the more general problem

$$-\nabla \cdot (a(x)\nabla u) = 0 \quad \text{in } \Omega.$$

This would allow us to analyze a more general problem than Problem (1) in which an x-dependent diffusivity is included. For the sake of clarity, we do not pursue that here.

In Section 3, we summarize the approximation properties of a standard finite element method for approximating Poisson's equation. Then, in Section 4, we show how the results of Section 2 may be used to obtain error estimates for a particular least squares penalty method for approximating Problem (1). Finally, in Section 5, we present some results of numerical experiments with the algorithm of Section 4.

Directions for further work include the analysis of the Cauchy problem with a diffusivity depending on both x and y, the use of higher-order elements, and the generalization to more complex geometries.

Throughout this paper, we shall use the following notation. For N a positive integer, let $h = 1/N$, and define

$$\Sigma_n = \{(x, nh)\,|\,0 < x < 1\}.$$

Clearly, $\Sigma_0 = \Sigma$ and $\Sigma_N = \Sigma'$. We denote by $\|\cdot\|_m$, $|\cdot|_m$, and $|\cdot|_{m,\Sigma_n}$ the norms on $H^m(\Omega)$, $H^m(\Gamma)$, and $H^m(\Sigma_n)$, respectively.

For $m = 1$ or 2 and $p \geqslant 2$, we shall denote the norms in the Sobolev spaces $W^{m,p}(\Omega)$ and $W^{m-1/p,p}(\Gamma)$ by $\|\cdot\|_{m,p}$ and $|\cdot|_{m-1/p,p}$, respectively. We shall also consider traces in the Sobolev spaces $W^{m-1/p,p}(\Sigma_n)$ (for $n = 0$ and N) and denote the norms in these spaces by $|\cdot|_{m-1/p,p,\Sigma_n}$. Definitions of these spaces and their relationships can be found in Grisvard [6].

We also denote by $H_0^1(\Omega)$, the set of functions in $H^1(\Omega)$ vanishing on Γ and by $H_0^1(\Sigma_n)$ the set of functions in $H^1(\Sigma_n)$ vanishing at $x = 0$ and $x = 1$.

Finally, we shall use the notation $(\cdot, \cdot)$ for the L^2-inner product on Ω and $\langle \cdot, \cdot \rangle_\Lambda$ for the L^2-inner product on Λ when Λ is either Γ, Σ, or Σ'.

2. Logarithmic Convexity and Stability. First, we shall define the finite element spaces to be used in this paper. Let τ_h be the uniform triangulation of Ω, consisting of right isosceles triangles with two sides of length $h = 1/N$ (where N is a positive integer), oriented as in Figure 1.

Let $S_h \subset H^1(\Omega)$ denote the finite element space of all continuous piecewise-linear functions on τ_h, and let $S_h^0 = \{v_h \in S_h | v_h = 0 \text{ on } \Gamma\}$. In this section, we are going to study the behavior of discrete harmonic functions. By discrete harmonic, we mean any function $w_h \in S_h$ which satisfies the following equations

$$(2) \qquad (\nabla w_h, \nabla \phi_h) = 0 \quad \text{for every } \phi_h \in S_h^0.$$

Notice that, because of the particular choice of τ_h and S_h, a discrete harmonic function w_h also satisfies the five-point difference operator at interior mesh points. Hence, if we let $W_i^j = w_h(x = ih, y = jh)$, and if $\mathbf{W}^j$ is the column vector with entries W_i^j, $1 \leqslant i \leqslant N - 1$, then

$$(3) \qquad \mathbf{W}^{n+1} - 2\mathbf{W}^n + \mathbf{W}^{n-1} = L\mathbf{W}^n.$$

Here L is the $(N - 1) \times (N - 1)$ symmetric, tridiagonal matrix with 2 down the main diagonal, and -1 down the off diagonals.

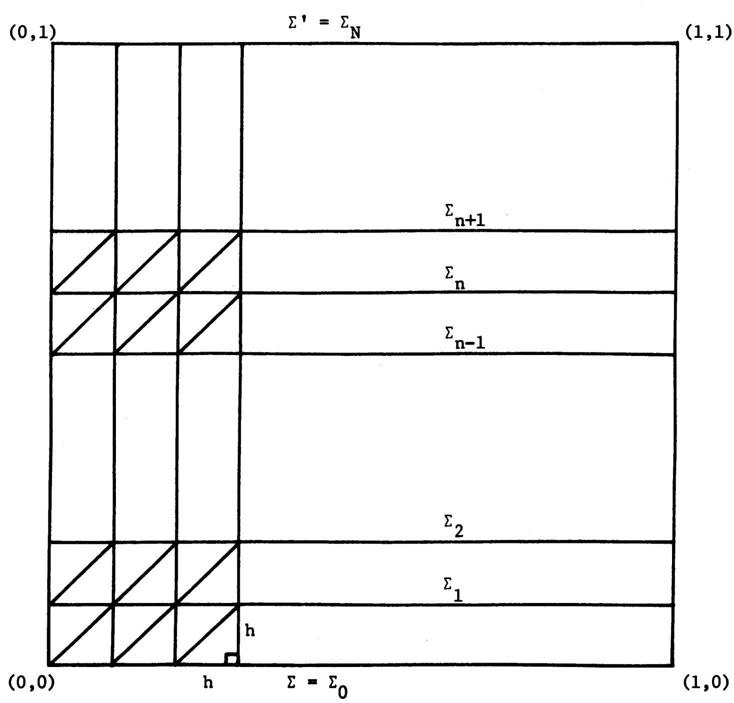


FIGURE 1

We shall be interested in estimating norms of w_h on strips of constant y. On these strips we will use both standard Sobolev norms and the following mesh-dependent norm. If $w_h \in S_h^0$, we define

$$|w_h|_{0,\Sigma_n,h} = \left(h(\mathbf{W}^n)^T\mathbf{W}^n\right)^{1/2}.$$

It is well known that for C^0 piecewise-linear finite element functions, the above mesh-dependent norm is equivalent to the standard Sobolev norm. Let

$$E(0) = \left[(\mathbf{W}^1)^T L\mathbf{W}^0 - (\mathbf{W}^1 - \mathbf{W}^0)^T(\mathbf{W}^1 - \mathbf{W}^0)\right]/h. \tag{4}$$

We can now state the main result of this paper.

THEOREM (1). *Let w_h satisfy Eq.* (2) *together with the boundary conditions that $w_h = 0$ at $x = 0$ and $x = 1$, and let $E(0)$ be given by* (4). *Then the following hold for $0 \leqslant n \leqslant N$.*

1. *If $E(0) \geqslant 0$,*

$$|w_h|_{0,\Sigma_n,h} \leqslant \left(|w_h|_{0,\Sigma,h}\right)^{(1-n/N)}\left(|w_h|_{0,\Sigma',h}\right)^{(n/N)}.$$

2. *If $E(0) < 0$,*

$$|w_h|^2_{0,\Sigma_n,h} \leqslant (1 - 2h^2)^{(n(n-N)/2)}\left(|w_h|^2_{0,\Sigma,h} + |E(0)|\right)^{(1-n/N)} \cdot\left(|w_h|^2_{0,\Sigma',h} + |E(0)|\right)^{(n/N)}.$$

Remark. Results similar to those in Theorem (1) hold in more general cases. For instance, let us consider an extension of Problem (2) involving a diffusion coefficient depending on x, so that the differential equation becomes

$$-\nabla\cdot(a(x)\nabla w) = 0 \quad \text{in } \Omega. \tag{5}$$

If the standard C^0 finite element discretization defined in this section is applied to this equation, the following equation analogous to Eq. (2) holds for generalized discrete harmonic functions $w_h \in S_h$,

$$(a\nabla w_h, \nabla\phi_h) = 0 \quad \text{for every } \phi_h \in S_h^0,$$

This equation implies that the nodal values for w_h satisfy the difference equation

$$A(\mathbf{W}^{n+1} - 2\mathbf{W}^n + \mathbf{W}^{n-1}) = L_a\mathbf{W}^n. \tag{6}$$

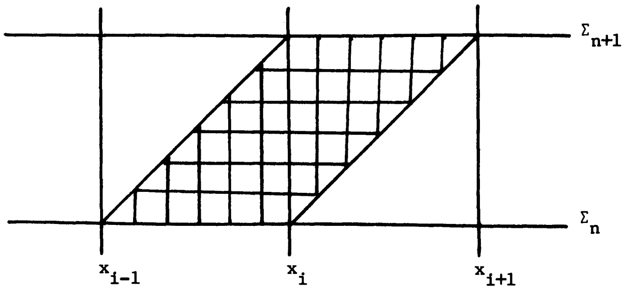
Here L_a is an $(N-1)\times(N-1)$ symmetric tridiagonal matrix depending on $a(x)$, and A is an $(N-1)\times(N-1)$ diagonal matrix with ith diagonal entry a_i defined as follows:

$$a_i = \int_{R_i} a(x)\,dA,$$

where R_i is the region shown in Figure 2.

For Eq. (5), the quantity corresponding to $E(0)$ is

$$E_a(0) = \left[(\mathbf{W}^1)^T L_a\mathbf{W}^0 - (\mathbf{W}^1 - \mathbf{W}^0)^T A(\mathbf{W}^1 - \mathbf{W}^0)\right]/h.$$



R_i ≡ Hatched Region

FIGURE 2

Using this definition, it is possible to prove an analogue of Theorem (1) for Eq. (5). In this case, $E(0)$ is replaced by $E_a(0)$, $|w_h|_{0,\Sigma_n,h}$ by $h^{1/2}((\mathbf{W}^n)^T A\mathbf{W}^n)^{1/2}$, $|w_h|_{0,\Sigma,h}$ by $h^{1/2}((\mathbf{W}^0)^T A\mathbf{W}^0)^{1/2}$, and $|w_h|_{0,\Sigma',h}$ by $h^{1/2}((\mathbf{W}^N)^T A\mathbf{W}^N)^{1/2}$. Obviously, such discrete L_2-norms are equivalent to the standard Sobolev norms provided the differential operator is uniformly elliptic.

It is also possible to prove theorems similar to Theorem (1) for different boundary conditions at $x = 0$ and $x = 1$. For instance, we can deal with homogeneous Neumann data on $x = 0$ and $x = 1$.

Before we prove Theorem (1), we shall define a useful discrete functional and derive some of its properties. Let

$$G_n = h(\mathbf{W}^n)^T\mathbf{W}^n + Q, \tag{7}$$

where Q is a nonnegative constant. G_n plays the same role in the discrete theory as the functional $G(t)$ of Payne [10, p. 20] plays in the continuous theory. Our first lemma offers an alternative characterization of $E(0)$.

LEMMA (1). *Let $E(0)$ be given by* (4). *Then*

$$(\mathbf{W}^{n+1})^T\mathbf{W}^{n-1} - (\mathbf{W}^n)^T\mathbf{W}^n = hE(0) \quad \textit{for } 1 \leqslant n \leqslant N-1.$$

Proof of Lemma (1). We use (3) to replace terms on the left-hand side of the above equality to prove the following:

$$\begin{aligned}
&(\mathbf{W}^{n+1})^T\mathbf{W}^{n-1} - (\mathbf{W}^n)^T\mathbf{W}^n \\
&\quad = [(L+2I)\mathbf{W}^n]^T\mathbf{W}^{n-1} - (\mathbf{W}^{n-1})^T\mathbf{W}^{n-1} - (\mathbf{W}^n)^T\mathbf{W}^n \\
&\quad = (\mathbf{W}^n)^T(\mathbf{W}^n - 2\mathbf{W}^{n-1} + \mathbf{W}^{n-2}) + 2(\mathbf{W}^n)^T\mathbf{W}^{n-1} \\
&\qquad - (\mathbf{W}^{n-1})^T\mathbf{W}^{n-1} - (\mathbf{W}^n)^T\mathbf{W}^n \\
&\quad = (\mathbf{W}^n)^T\mathbf{W}^{n-2} - (\mathbf{W}^{n-1})^T\mathbf{W}^{n-1}.
\end{aligned}$$

Iterating this equality proves that

$$(\mathbf{W}^{n+1})^T\mathbf{W}^{n-1} - (\mathbf{W}^n)^T\mathbf{W}^n = (\mathbf{W}^2)^T\mathbf{W}^0 - (\mathbf{W}^1)^T\mathbf{W}^1.$$

It remains to show that the expression on the right-hand side above is $hE(0)$. Again we use (3) to obtain

$$\begin{aligned}&(\mathbf{W}^2)^T\mathbf{W}^0-(\mathbf{W}^1)^T\mathbf{W}^1\\&\quad=\left[(L+2I)\mathbf{W}^1\right]^T\mathbf{W}^0-(\mathbf{W}^0)^T\mathbf{W}^0-(\mathbf{W}^1)^T\mathbf{W}^1=hE(0).\end{aligned}$$

This completes the proof of the lemma.

Our next lemma shows that G_n satisfies a discrete form of the standard second-order differential inequality of logarithmic convexity.

LEMMA (2). *Let G_n be defined by* (7). *Then*

$$G_{n+1}G_{n-1}-G_n^2\geqslant 2h^2E(0)G_n\quad \textit{for } 1\leqslant n\leqslant N-1.$$

Remark. If Eq. (1) is replaced by Eq. (5) (cf. the remark following Theorem (1)), we should redefine G_n as $G_n=h(\mathbf{W}^n)^TA\mathbf{W}^n+Q$. With this redefinition, Lemma (2) holds with $E(0)$ replaced by $E_a(0)$.

Proof of Lemma (2). We simply expand G_{n+1}, G_{n-1}, and G_n and use Lemma (1),

$$\begin{aligned}G_{n+1}G_{n-1}-G_n^2&=h^2\left((\mathbf{W}^{n+1})^T\mathbf{W}^{n+1}\right)\left((\mathbf{W}^{n-1})^T\mathbf{W}^{n-1}\right)-h^2\left((\mathbf{W}^n)^T\mathbf{W}^n\right)^2\\&\quad+hQ\left[(\mathbf{W}^{n+1})^T\mathbf{W}^{n+1}+(\mathbf{W}^{n-1})^T\mathbf{W}^{n-1}-2(\mathbf{W}^n)^T\mathbf{W}^n\right]\\&=h^2\left((\mathbf{W}^{n+1})^T\mathbf{W}^{n+1}\right)\left((\mathbf{W}^{n-1})^T\mathbf{W}^{n-1}\right)-h^2\left((\mathbf{W}^{n+1})^T\mathbf{W}^{n-1}\right)^2\\&\quad+h^4E(0)^2+2h^2E(0)\left[h(\mathbf{W}^n)^T\mathbf{W}^n+Q\right]\\&\quad+hQ\left[(\mathbf{W}^{n+1})^T\mathbf{W}^{n+1}+(\mathbf{W}^{n-1})^T\mathbf{W}^{n-1}-2(\mathbf{W}^{n+1})^T\mathbf{W}^{n-1}\right].\end{aligned}$$

The application of Schwarz's inequality and the arithmetic-geometric mean inequality finishes the proof.

Now we are ready to prove Theorem (1). Essentially, Lemma (2) proves that G_n is logarithmically convex. Then, we relate G_n to $|w_h|_{0,\Sigma_n,h}$.

Proof of Theorem (1). (1) Suppose $E(0)\geqslant 0$. Then Lemma (2) implies that $G_n^2\leqslant G_{n-1}G_{n+1}$. Using an induction argument on N, we now show that, for $0\leqslant n\leqslant N$,

$$G_n\leqslant G_0^{(1-n/N)}G_N^{(n/N)}. \tag{8}$$

The result is obvious for $N=2$. Now suppose that (8) holds for $N=m-1$. First we prove (8) for $N=m$, and $n=m-1$. By Lemma (2), $G_{m-1}\leqslant G_{m-2}^{(1/2)}G_m^{(1/2)}$, and so estimating G_{m-2} by estimate (8) with $N=m-1$ and $n=m-2$, we obtain

$$G_{m-1}\leqslant\left(G_0^{1/(m-1)}G_{m-1}^{((m-2)/(m-1))}\right)^{(1/2)}G_m^{(1/2)}.$$

Simplifying this expression proves the following:

$$G_{m-1}\leqslant G_0^{(1/m)}G_m^{(1-(1/m))}. \tag{9}$$

Now we can use this result to prove (8) for $N=m$, and $0\leqslant n\leqslant m-1$. Again by induction,

$$G_n\leqslant G_0^{(1-n/(m-1))}G_{m-1}^{(n/(m-1))}.$$

Finally, we use (9) to estimate G_{m-1} in the above expression. This completes the proof of (8) for $N=m$, and hence, by induction for all N.

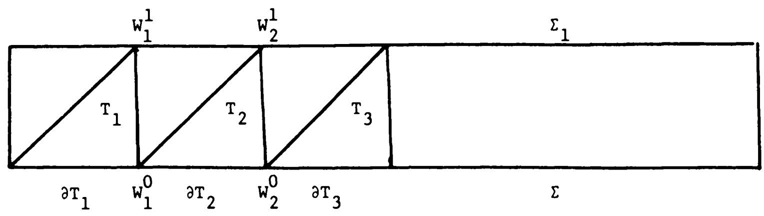


FIGURE 3

If we now take $Q = 0$, and note that $(\mathbf{W}^n)^T\mathbf{W}^n = |w_h|^2_{0,\Sigma_n,h}$, we have the first result of the theorem.

(2) Suppose $E(0) < 0$. Then, if we let $Q = |E(0)|$, Lemma (2) tells us that

$$G_{n+1}G_{n-1} - G_n^2 \geqslant \left[2h^2E(0)/Q\right]G_n^2 = -2h^2G_n^2 \quad (\text{since } G_n \geqslant Q).$$

Hence, if $F_n = (1 - 2h^2)^{-(n^2/2)}G_n$, it is easy to show that F_n satisfies $F_{n+1}F_{n-1} - F_n^2 \geqslant 0$, and so F_n satisfies an estimate like (8). Rewriting F_n in terms of w_h and $E(0)$ proves the second estimate of the theorem.

Our final lemma of this section estimates $E(0)$ in terms of norms of w_h. As one would expect from the continuous theory, this involves norms of w_h and $\partial w_h/\partial n$ on Σ.

LEMMA (3). *If $w_h \in S_h$, then*

$$|E(0)| \leqslant (3/2)|(w_h)_n|^2_{0,\Sigma} + 3|(w_h)_x|^2_{0,\Sigma}.$$

Proof of Lemma (3). We estimate the two terms of Eq. (4) directly. Let us label the triangles with edges on Σ by $T_1, T_2, \ldots, T_N$ (starting at $x = 0$), as shown in Figure 3. Then

$$(\mathbf{W}^1 - \mathbf{W}^0)^T(\mathbf{W}^1 - \mathbf{W}^0) \leqslant \sum_{i=1}^{N-1} (\mathbf{W}_i^1 - \mathbf{W}_i^0)^2 = h|(w_h)_n|^2_{0,\Sigma}. \tag{10}$$

To estimate the remaining term, we expand it and use the Schwarz inequality and the arithmetic-geometric mean inequality:

$$\begin{aligned}(\mathbf{W}^1)^TL\mathbf{W}^0 &= (\mathbf{W}^1 - \mathbf{W}^0)^TL\mathbf{W}^0 + (\mathbf{W}^0)^TL\mathbf{W}^0\\ &\leqslant (1/2)(\mathbf{W}^1 - \mathbf{W}^0)^T(\mathbf{W}^1 - \mathbf{W}^0) + (1/2)(\mathbf{W}^0)^TLL\mathbf{W}^0 + (\mathbf{W}^0)^TL\mathbf{W}^0\\ &\leqslant (1/2)h|(w_h)_n|^2_{0,\Sigma} + 3h|(w_h)_x|^2_{0,\Sigma}\end{aligned} \tag{11}$$

(since, if $\rho(\cdot)$ represents the spectral radius of a matrix, $\rho(L) \leqslant 4$). Combining (10) and (11) in (4) proves the lemma.

3. The Dirichlet Problem for Poisson's Equation. Suppose that for some $2 \leqslant p < \infty$, $f \in L^p(\Omega)$, $z_1 \in W_p^{2-1/p}(\Sigma) \cap H_0^1(\Sigma)$, and $z_2 \in W_p^{2-1/p}(\Sigma') \cap H_0^1(\Sigma')$. Let $z \in W_p^2(\Omega)$ be the solution of the Dirichlet Problem:

$$\begin{aligned} -\Delta z &= f \quad \text{in } \Omega,\\ z &= 0 \quad \text{if } x = 0 \text{ or } x = 1,\\ z &= z_1 \quad \text{on } \Sigma,\\ z &= z_2 \quad \text{on } \Sigma'.\end{aligned} \tag{12}$$

With S_h and S_h^0 defined as in the previous section, let $z_h \in S_h$ be defined as the solution of

$$(13)\qquad \begin{aligned} &(\nabla z_h, \nabla\phi_h) = (f, \phi_h) \quad \text{for every } \phi_h \in S_h^0,\\ &z_h = 0 \quad \text{if } x = 0 \text{ or } x = 1,\\ &z_h \text{ interpolates } z_1 \text{ on } \Sigma,\\ &z_h \text{ interpolates } z_2 \text{ on } \Sigma'. \end{aligned}$$

Some approximation properties of z_h are summarized in the following theorem.

THEOREM (2). *There exists a constant C independent of h, n, and z, such that*
(1) $\|z - z_h\|_1 \leqslant Ch\|z\|_2$.
(2) $\|z - z_h\|_0 \leqslant Ch^{2-1/p}\|z\|_{2,p}$.
(3) $|z - z_h|_{0,\Sigma_n} \leqslant Ch^{[3/2-1/(2p)]}\|z\|_{2,p}$.
(4) $|(z - z_h)_n|_{0,\Sigma} \leqslant Ch^{[1-1/p]}\|z\|_{2,p}$.

Proof of Theorem (2). Property (1) can be found in [12]. Property (2) is proved using a minor modification of the usual duality argument. Let $w \in H_0^1(\Omega)$ satisfy $-\Delta w = z - z_h$ in Ω, and let $w_I \in S_h^0$ interpolate w. Then

$$\begin{aligned} \|z - z_h\|_0^2 &= (\nabla(w - w_I), \nabla(z - z_h)) - \langle z - z_h, w_n\rangle_\Gamma \\ &\leqslant \|\nabla(w - w_I)\|_0\|\nabla(z - z_h)\|_0 + |z - z_h|_0|w_n|_0 \\ &\leqslant Ch^2\|w\|_2\|z\|_2 + Ch^{[2-1/p]}\left[|z_1|_{2-1/p,p,\Sigma} + |z_2|_{2-1/p,p,\Sigma'}\right]\|w\|_2 \\ &\leqslant Ch^{[2-1/p]}\|z\|_{2,p}\|z - z_h\|_0. \end{aligned}$$

To prove property (3), we first observe that for $w \in H^1(\Omega)$ with $w = 0$ for $x = 0$ and $x = 1$,

$$(14)\qquad |w|_{0,\Sigma_n}^2 \leqslant C\|w\|_0\|w\|_1,$$

where C is a constant independent of w and n. This is easily proved by considering

$$\int_0^1 \int_0^{nh} \frac{\partial}{\partial y}(w^2 y)\, dy\, dx \quad \text{for } nh \geqslant 1/2,$$

and a similar integral for $nh \leqslant 1/2$. Estimate (3) follows from inequality (14) and parts (1) and (2) of Theorem (2).

To prove property (4) we let $z_I \in S_h$ interpolate z. Then, using the inverse properties of S_h, we get

$$\begin{aligned} |(z - z_h)_n|_{0,\Sigma} &\leqslant |(z - z_I)_n|_{0,\Sigma} + |(z_I - z_h)_n|_{0,\Sigma} \\ &\leqslant |(z - z_I)_n|_{0,\Sigma} + Ch^{-1/p}\|z_I - z_h\|_{1,p}. \end{aligned}$$

Now define w by $w = z_I$ on Γ and $(\nabla w, \nabla\phi) = (f, \phi)$ for all $\phi \in H_0^1$. Then, $w - z_I \in H_0^1(\Omega) \cap W_p^1(\Omega)$, $z_h - z_I \in S_h^0$, and

$$(\nabla(z_h - z_I), \nabla\phi_h) = (\nabla(w - z_I), \nabla\phi_h) \quad \text{for all } \phi_h \in S_h^0.$$

Hence, by a result of Rannacher and Scott [11],

$$\begin{aligned} \|z_h - z_I\|_{1,p} &\leqslant C\|w - z_I\|_{1,p} \leqslant C\left[\|z - z_I\|_{1,p} + \|w - z\|_{1,p}\right] \\ &\leqslant C\left[\|z - z_I\|_{1,p} + |w - z|_{1-1/p,p,\Sigma} + |w - z|_{1-1/p,p,\Sigma'}\right] \end{aligned}$$

(since $w - z$ is harmonic). Thus,

$$\begin{aligned}|(z - z_h)_n|_{0,\Sigma} &\leqslant |(z - z_I)_n|_{0,\Sigma} + Ch^{-1/p}\Big[\|z - z_I\|_{1,p} + |z_1 - (z_1)_I|_{1-1/p,p,\Sigma} \\ &\qquad + |z_2 - (z_2)_I|_{1-1/p,p,\Sigma'}\Big] \\ &\leqslant Ch^{[1-1/p]}\|z\|_{2,p}.\end{aligned}$$

4. Error Estimates for a Numerical Method. Before we define the method to be analyzed, we will need to define one more finite element space. Let $M_h(\Sigma) \subset H_0^1(\Sigma)$ be the space of continuous piecewise-linear functions on the uniform mesh of length h on Σ, which vanish at the end points of Σ. In other words, let $M_h(\Sigma) = \{v_h|_\Sigma \mid v_h \in S_h \text{ and } v_h = 0 \text{ if } x = 0 \text{ or } 1\}$. The approximation and inverse properties of $M_h(\Sigma)$ are well known.

We can now define our approximate problem. Find $(\lambda_h, \mu_h) \in M_h(\Sigma) \times M_h(\Sigma')$ such that

$$J_\omega(\lambda_h, \mu_h) = \min_{(\sigma_h, \rho_h) \in M_h(\Sigma) \times M_h(\Sigma')} J_\omega(\sigma_h, \rho_h), \tag{15}$$

where $J_\omega(\sigma_h, \rho_h) = |g_1 - \sigma_h|_{1,\Sigma}^2 + |g_2 - (u_h(\sigma_h, \rho_h))_n|_{0,\Sigma}^2 + \omega^2|\rho_h|_{0,\Sigma'}^2$ and $u_h(\sigma_h, \rho_h)$ solves Problem (13) with $z_1 = \sigma_h$ and $z_2 = \rho_h$. The fixed positive parameter ω (the regularization or penalty parameter) will be discussed in more detail later.

To understand the relationship between Problem (15) above, and Problem (1), note that $u_h(\sigma_h, \rho_h)$ is an approximate solution of Poisson's equation with boundary data σ_h on Σ, and ρ_h on Σ'. We then seek the σ_h and ρ_h which give the best fit to the data while penalizing the growth of the approximate solution on Σ'.

Before proceeding to state and prove error estimates for this method, we will need to make some existence and regularity assumptions. We assume the following:

(R1) There exists a solution $u^* \in W_p^2(\Omega)$ to Problem (1) for some $p \geqslant 2$.

As a consequence of this, we get that if, $g_1^* = u^*|_\Sigma$ and $\mu^* = u^*|_{\Sigma'}$, then $g_1^* \in H_0^1(\Sigma) \cap W_p^{1-1/p}(\Sigma)$ and $\mu^* \in H_0^1(\Sigma') \cap W_p^{1-1/p}(\Sigma')$.

We shall let $g_2^* = (u^*)_n|_{\Sigma'}$.

THEOREM (3). *Let u^* satisfy hypothesis* (R1), *and let $u_h(\lambda_h, \mu_h)$ be constructed via the algorithm of this section* (*Problem* (15)). *Then there exist constants $C \geqslant 1$ and $C^* \geqslant 1$ such that for all $\omega < 1$,*

$$\begin{aligned}&|u^* - u_h(\lambda_h, \mu_h)|_{0,\Sigma_n} \\ &\quad\leqslant C^*h^{[3/2-1/(2p)]} \\ &\qquad + C(1 - 2h^2)^{(n(n-N)/4)}\big(\varepsilon_1 + \varepsilon_2 + C^*h^{[1-1/p]} + \omega\big(M + C^*h^{[2-1/p]}\big)\big)^{(1-n/N)} \\ &\qquad \cdot\big(M + \varepsilon_1 + \varepsilon_2 + C^*h^{[1-1/p]} + \omega^{-1}\big(\varepsilon_1 + \varepsilon_2 + C^*h^{[1-1/p]}\big) \\ &\qquad\qquad + \omega\big(M + C^*h^{[2-1/p]}\big)\big)^{(n/N)},\end{aligned}$$

where C is independent of h, ε_1, ε_2, M, and u^*, and C^* depends on $\|u^*\|_{2,p}$, but is also independent of h, ε_1, ε_2, and M.

Remark. Using the inequality $e^x \geqslant 1 + x$, first with $x = -2h^2$, and then with $x = 2h^2/(1 - 2h^2)$, we can show that

$$e^{(nh(1-nh)/2)} \leqslant (1 - 2h^2)^{(n(n-N)/4)} \leqslant e^{(nh(1-nh)/((1-2h^2)2))}.$$

COROLLARY. *Let* $\omega = (\varepsilon_1 + \varepsilon_2 + \bar{C}h^{[1-1/p]})/(M + \bar{C}h^{[2-1/p]})$, *where* $\bar{C}$ *is a constant, and let the hypotheses of Theorem* (3) *hold. Then, if* $y^n = nh$, *we have for* h *sufficiently small that*

$$\begin{aligned}|u^* - u_h(\lambda_h, \mu_h)|_{0,\Sigma_n} &\leqslant C^*h^{[3/2-1/(2p)]} + Ce^{(y^n(1-y^n)/2)}\left(\varepsilon_1 + \varepsilon_2 + C^*h^{[1-1/p]}\right)^{(1-y^n)} \\ &\quad \cdot\left(M + \varepsilon_1 + \varepsilon_2 + C^*h^{[1-1/p]}\right)^{y^n},\end{aligned}$$

where C *and* C^* *have the same dependence as in Theorem* (3).

Remarks. (1) This theorem, and its corollary, suggest that $h = O[(\varepsilon_1 + \varepsilon_2)^{p/(p-1)}]$ is the best balance between measurement and discretization error for this problem.

(2) In other instances of the Cauchy problem we may be able to do much better than this result suggests. For instance, suppose we know u_n exactly on Σ (i.e., $\varepsilon_2 = 0$). Then a better discrete method is to approximate a mixed boundary value problem, specifying the exact Neumann data on Σ, and adjusting a Dirichlet data function on Σ' to fit the Dirichlet data on Σ. A least squares penalty method like the one in the present section can be used to compute the solution. In this case, we find that we can use Theorem (1), part (1), in our estimates, which now look like the results of Theorem (3) and its corollary with all terms of the form $C^*h^{[1-1/p]}$ replaced by $C^*h^{[2-1/p]}$, and the term $(1 - 2h^2)^{(n(n-N)/4)}$ left out.

Before we prove Theorem (3), we must estimate $J_\omega(\lambda_h, \mu_h)$.

LEMMA (4). *Assuming hypothesis* (R1) *holds*,

$$J_\omega(\lambda_h, \mu_h) \leqslant \left(\varepsilon_1 + C^*h^{[1-1/p]}\right)^2 + \left(\varepsilon_2 + C^*h^{[1-1/p]}\right)^2 + \omega^2(M + C^*h^{[2-1/p]})^2,$$

where the constant C^* *has the same dependence as in Theorem* (3).

Proof of Lemma (4). Let $(g_1^*)_I \in M_h(\Sigma)$ be the interpolant of g_1^* and let $(\mu^*)_I \in M_h(\Sigma')$ be the interpolant of μ^*. Then,

$$\begin{aligned}J_\omega(\lambda_h, \mu_h) &\leqslant J_\omega\left((g_1^*)_I, (\mu^*)_I\right) \\ &= \left|g_1 - (g_1^*)_I\right|_{1,\Sigma}^2 + \left|g_2 - \left(u_h\left((g_1^*)_I, (\mu^*)_I\right)\right)_n\right|_{0,\Sigma}^2 + \omega^2\left|(\mu^*)_I\right|_{0,\Sigma'}^2.\end{aligned} \tag{16}$$

Notice that $u_h((g_1^*)_I, (\mu^*)_I) = u_h(g_1^*, \mu^*)$. Next, we estimate each term in (16). By adding and subtracting g_1^*, we find that

$$\begin{aligned}\left|g_1 - (g_1^*)_I\right|_{1,\Sigma} &\leqslant |g_1 - g_1^*|_{1,\Sigma} + \left|g_1^* - (g_1^*)_I\right|_{1,\Sigma} \\ &\leqslant \varepsilon_1 + Ch^{[1-1/p]}|g_1^*|_{2-1/p,p,\Sigma} \leqslant \varepsilon_1 + Ch^{[1-1/p]}\|u^*\|_{2,p}.\end{aligned}$$

In the same way,

$$\begin{aligned}\left|(\mu^*)_I\right|_{0,\Sigma'} &\leqslant |\mu^*|_{0,\Sigma'} + \left|\mu^* - (\mu^*)_I\right|_{0,\Sigma'} \\ &\leqslant M + Ch^{[2-1/p]}|\mu^*|_{2-1/p,p,\Sigma'} \leqslant M + Ch^{[2-1/p]}\|u^*\|_{2,p}.\end{aligned}$$

Finally, by Theorem (2), part (4),

$$\left|g_2 - \left(u_h\left(g_1^*, \mu^*\right)\right)_n\right|_{0,\Sigma} \leqslant \left|g_2 - u_n^*\right|_{0,\Sigma} + \left|\left(u^* - u_h\left(g_1^*, \mu^*\right)\right)_n\right|_{0,\Sigma}$$
$$\leqslant \varepsilon_2 + Ch^{[1-1/p]}\left\|u^*\right\|_{2,p}.$$

Putting the above estimates in (16) yields the desired estimate.

Now we can prove the error estimate of Theorem (3).

Proof of Theorem (3). By the triangle inequality,

$$\left|u^* - u_h(\lambda_h, \mu_h)\right|_{0,\Sigma_n} \leqslant \left|u^* - u_h\left(g_1^*, \mu^*\right)\right|_{0,\Sigma_n} + \left|u_h\left(g_1^*, \mu^*\right) - u_h(\lambda_h, \mu_h)\right|_{0,\Sigma_n}. \tag{17}$$

We can easily estimate the first term on the right-hand side of (17) by Theorem (2), part (3). To estimate the second term, we use Theorem (1). Since we cannot be sure of the sign of $E(0)$, we must use the more pessimistic estimate in Theorem (1), part (2). By Lemma (3), and the equivalence of norms, it suffices to perform the following estimates.

(a) $|u_h(g_1^*, \mu^*) - u_h(\lambda_h, \mu_h)|_{1,\Sigma}$. We use the fact that u_h interpolates its boundary data,

$$\left|u_h\left(g_1^*, \mu^*\right) - u_h(\lambda_h, \mu_h)\right|_{1,\Sigma} = \left|\left(g_1^*\right)_I - \lambda_h\right|_{1,\Sigma}$$
$$\leqslant \left|\left(g_1^*\right)_I - g_1^*\right|_{1,\Sigma} + \left|g_1^* - g_1\right|_{1,\Sigma} + \left|g_1 - \lambda_h\right|_{1,\Sigma}$$
$$\leqslant Ch^{[1-1/p]}\left\|u^*\right\|_{2,p} + \varepsilon_1 + \left(J_\omega(\lambda_h, \mu_h)\right)^{(1/2)}.$$

(b) $|u_h(g_1^*, \mu^*) - u_h(\lambda_h, \mu_h)|_{0,\Sigma'}$. In the same way as in part (a) of this proof, we estimate

$$\left|u_h\left(g_1^*, \mu^*\right) - u_h(\lambda_h, \mu_h)\right|_{0,\Sigma'}$$
$$= \left|\left(\mu^*\right)_I - \mu_h\right|_{0,\Sigma'} \leqslant \left|\left(\mu^*\right)_I - \mu^*\right|_{0,\Sigma'} + \left|\mu^*\right|_{0,\Sigma'} + \left|\mu_h\right|_{0,\Sigma'}$$
$$\leqslant Ch^{[2-1/p]}\left\|u^*\right\|_{2,p} + M + \omega^{-1}\left(J_\omega(\lambda_h, \mu_h)\right)^{(1/2)}.$$

(c) $|(u_h(g_1^*, \mu^*) - u_h(\lambda_h, \mu_h))_n|_{0,\Sigma}$. Here we must use Theorem (2), part (4),

$$\left|\left(u_h\left(g_1^*, \mu^*\right) - u_h(\lambda_h, \mu_h)\right)_n\right|_{0,\Sigma}$$
$$\leqslant \left|\left[u_h\left(g_1^*, \mu^*\right) - u\left(g_1^*, \mu^*\right)\right]_n\right|_{0,\Sigma} + \left|u_n^* - g_2\right|_{0,\Sigma} + \left|g_2 - \left[u_h(\lambda_h, \mu_h)\right]_n\right|_{0,\Sigma}$$
$$\leqslant Ch^{[1-1/p]}\left\|u^*\right\|_{2,p} + \varepsilon_2 + \left(J_\omega(\lambda_h, \mu_h)\right)^{1/2}.$$

Combining estimates (a), (b) and (c) with Theorem (1), part (2), using the equivalence of the discrete and standard Sobolev norms in (17), and then estimating $(J_\omega(\lambda_h, \mu_h))^{1/2}$ by Lemma (4) completes the proof.

5. Numerical Results. Once bases are chosen for $M_h(\Sigma)$ and $M_h(\Sigma')$, the discrete problem of Section 3 (Problem (15)) reduces to the minimization of a quadratic form. Taking first variations, we obtain a system of linear equations. More precisely, let G_h^Σ and G_h be the operators defined as follows: $G_h^\Sigma z_1 \in S_h$ solves Problem (13) with $z_2 = 0$ and $f = 0$, and $G_h z_2 \in S_h$ solves Problem (13) with $z_1 = 0$ and $f = 0$.

Then, by taking first variations, we see that Problem (15) is equivalent to finding $(\lambda_h, \mu_h) \in M_h(\Sigma) \times M_h(\Sigma')$ which satisfy the following problem:

$$
\begin{aligned}
&\langle g_1 - \lambda_h, \phi_h \rangle_\Sigma + \langle (g_1 - \lambda_h)_x, (\phi_h)_x \rangle_\Sigma \\
&\qquad + \langle g_2 - (u_h(\lambda_h, \mu_h))_n, (G_h^\Sigma \phi_h)_n \rangle_\Sigma = 0, \\
&\langle g_2 - (u_h(\lambda_h, \mu_h))_n, (G_h \psi_h)_n \rangle_\Sigma - \omega^2 \langle \mu_h, \psi_h \rangle_{\Sigma'} = 0 \\
&\qquad\qquad\qquad\qquad \text{for every } (\phi_h, \psi_h) \in M_h(\Sigma) \times M_h(\Sigma').
\end{aligned} \tag{18}
$$

Clearly, Problem (18) is just a linear system. The dimensions of $M_h(\Sigma)$ and $M_h(\Sigma')$ are both $N - 1$. Let us suppose that we number the nodes on Σ followed by the nodes on Σ'. Then Problem (18) induces the matrix problem

$$(D + \omega^2 I')\mathbf{x} = \mathbf{b}, \tag{19}$$

where

$$\mathbf{x} = \begin{bmatrix} [\lambda_h] \\ [\mu_h] \end{bmatrix},$$

with $[\lambda_h]$ and $[\mu_h]$ being vectors of the nodal values of λ_h and μ_h, respectively. Furthermore, D and I' have the structure

$$D = \left(\begin{array}{c|c} A & C \\ \hline C^T & B \end{array} \right), \qquad I' = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & I_{N-1} \end{array} \right),$$

where I_{N-1} is the $(N-1) \times (N-1)$ identity matrix and A and B are both symmetric $(N-1) \times (N-1)$ matrices. The ill-conditioning of this system comes from the near singularity of B.

The matrix D is dense and costly to compute, so one might wish to solve Problem (19) iteratively. To compute the action of $D + \omega^2 I'$ efficiently, or to compute D itself, we must be able to compute the adjoints of the operators $\partial G_h^\Sigma / \partial n$ and $\partial G_h / \partial n$. The following lemma allows us to do that.

LEMMA (5). *Let $P_h(\Sigma)$ consist of all piecewise-constant functions on the uniform mesh of size h on Σ. Let $v_h \in P_h(\Sigma)$ and let $z_h \in S_h$ be discrete harmonic and take on the boundary values $z_h = 0$ on $\Gamma \setminus \Sigma$ and $z_h(x = nh, y = 0) = \lim_{x \to nh} -v_h(x)$ for $n = 1, \ldots, N-1$. Then the following hold:*

(a) $\langle v_h, (G_h^\Sigma \phi_h)_n \rangle_\Sigma = \sum_{i=1}^{N-1} (Z_i^0 - Z_i^1) \phi_h(ih)$ *for every $\phi_h \in M_h(\Sigma)$, where $Z_i^j = z_h(x = ih, y = jh)$.*

(b) $\langle v_h, (G_h \phi_h)_n \rangle_\Sigma = \sum_{i=1}^{N-1} (Z_i^N - Z_i^{(N-1)}) \phi_h(ih)$ *for every $\phi_h \in M_h(\Sigma')$.*

Proof of Lemma (5). We will only prove the first equality. The second follows in the same way. Let $W_i^j = G_h^\Sigma \phi_h(x = ih, y = jh)$. Then

$$\langle v_h, (G_h^\Sigma \phi_h)_n \rangle_\Sigma = \sum_{i=1}^{N-1} Z_i^0 (W_i^0 - W_i^1).$$

Now by applying summation by parts to

$$\sum_{j=0}^{N-1} (Z_i^{j+1} - Z_i^j)(W_i^{j+1} - W_i^j) \quad \text{and} \quad \sum_{i=0}^{N-1} (Z_{i+1}^j - Z_i^j)(W_{i+1}^j - W_i^j)$$

and by using the known zero boundary data for z_h and $G_h^{\Sigma}\phi_h$, we may show that

$$
\begin{aligned}
&-\sum_{i=1}^{N-1}\left(W_i^1-W_i^0\right)Z_i^0\\
&\qquad-\sum\sum_{i,j=1}^{N-1}\left\{\left(W_i^{j+1}-2W_i^j+W_i^{j-1}\right)Z_i^j+\left(W_{i+1}^j-2W_i^j+W_{i-1}^j\right)Z_i^j\right\}\\
&\quad=-\sum_{i=1}^{N-1}\left(Z_i^1-Z_i^0\right)W_i^0\\
&\qquad-\sum\sum_{i,j=1}^{N-1}\left\{\left(Z_i^{j+1}-2Z_i^j+Z_i^{j-1}\right)W_i^j+\left(Z_{i+1}^j-2Z_i^j+Z_{i-1}^j\right)W_i^j\right\}.
\end{aligned}
$$

Using the fact that z_h and $G_h^{\Sigma}\phi_h$ are discrete harmonic completes the proof.

After applying Lemma (5), one step of the conjugate-gradient method can be computed by solving only two Dirichlet problems. Despite the possible advantages of using iterative methods, we have elected to solve Problem (19) by constructing D directly, for two reasons. First, we wished to solve Problem (19) for many ω and $\mathbf{b}$ on the same grid, and second, we did not want the stabilizing effects of iterative methods obscuring the effects of discretization error.

The continuous dependence result established in Theorem (3) is quite pessimistic, as are all logarithmic convexity type estimates. We view Theorem (3) as a theoretical justification of the fact that solving the underlying partial differential equation numerically does not significantly affect the continuous dependence of the solution on the Cauchy data. In numerical experiments we performed, the errors were much better than predicted by this theory. The results of some of these computations are presented below.

The true solution u^* of the Cauchy problem is taken to be the classical example of Hadamard,

$$u^*(x,y)=\operatorname{Sinh}(m\pi y)\operatorname{Sin}(m\pi x)/(m^2\pi^2),$$

for $m=1,2$, and 3. This solution becomes progressively more poorly behaved as m increases. The data g_1 and g_2 is obtained from u^* by adding a suitable random error to each discrete data value. This is arranged so that the functions g_1 and g_2 are in error by an amount at most ε (i.e., in Problem (1), $\varepsilon_1=\varepsilon_2=\varepsilon$). The constant M can be found analytically. Theorem (3) suggests that if we are to have any accuracy at points intermediate to Σ_0 and Σ_N, ε and h must be chosen small (for example, the error at $y=1/2$ is essentially governed by $(\varepsilon+h)^{1/2}$). In all the results reported, we have taken $\varepsilon=0.01$ and $h=0.02$. The corollary to Theorem (3) suggests that we take ω of the form

$$\omega=(2\varepsilon+\bar{C}h)/(M+\bar{C}h^2). \tag{20}$$

Since the choice of $\bar{C}$ is still open, we have attempted to test the sensitivity of the approximate solution to the choice of ω by performing computations for the values of $\bar{C}=.5$, 1. and 2. Tables 1–3 ($m=1,2,3$) give values of the relative error, defined by $|u^*-u_h|_{0,\Sigma_n}/|u^*|_{0,\Sigma_n}$, at $y=nh$, for $n=10,20,\ldots,50$, and each of the values of ω determined by the choice of $\bar{C}$ given above. As expected, the relative error in our approximations gets worse as m increases.

TABLE 1. ($m = 1$)

	Relative Error at y		
y	$w = 0.0363$ ($\bar{C} = .5$)	$w = 0.0483$ ($\bar{C} = 1.$)	$w = 0.0725$ ($\bar{C} = 2.0$)
0.2	0.0481	0.0782	0.171
0.4	0.0400	0.0623	0.128
0.6	0.0412	0.0597	0.121
0.8	0.0497	0.0606	0.120
1.0	0.0725	0.0672	0.120

TABLE 2. ($m = 2$)

	Relative Error at y		
y	$w = 0.00628$ ($\bar{C} = .5$)	$w = 0.00837$ ($\bar{C} = 1.$)	$w = 0.0125$ ($\bar{C} = 2.0$)
0.2	0.130	0.213	0.377
0.4	0.120	0.200	0.350
0.6	0.120	0.200	0.349
0.8	0.122	0.199	0.350
1.0	0.125	0.210	0.351

TABLE 3. ($m = 3$)

	Relative Error at y		
y	$w = 0.000608$ ($\bar{C} = .5$)	$w = 0.000811$ ($\bar{C} = 1.$)	$w = 00122$ ($\bar{C} = 2.0$)
0.2	0.237	0.342	0.543
0.4	0.234	0.337	0.533
0.6	0.238	0.340	0.535
0.8	0.242	0.344	0.537
1.0	0.247	0.347	0.540

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