MATHEMATICS OF COMPUTATION
VOLUME 47, NUMBER 175
JULY 1986, PAGES 225-252

Explicit Nearly Optimal Linear Rational Approximation with Preassigned Poles

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Abstract. This paper gives explicit rational functions for interpolating and approximating functions on the intervals $[-1,1]$, $[0,\infty]$, and $[-\infty,\infty]$. The rational functions are linear in the functions to be approximated, and they have preassigned poles. The error of approximation of these rationals is nearly as small as the error of best rational approximation with numerator and denominator polynomials of the same degrees. Regions of analyticity are described, which make it possible to tell *a priori* the accuracy which we can expect from this type of rational approximation.

1. Introduction and Summary. In this paper we attempt to give a constructive, affirmative answer to each of the following questions.

1. Given a function f and an interval I, is it possible to tell *a priori* whether or not one can accurately approximate f via a low-degree rational function?

2. Can such a rational function be easily constructed explicitly, so that one encounters no poles on the interval of approximation?

3. Can one use the Thiele algorithm to construct or evaluate this rational function?

4. Can one tell *a priori* when we can expect the Thiele algorithm, the ε-algorithm, or the Padé method to produce an accurate low-degree rational approximation?

5. Does the error of this rational function compare favorably with the error of the best possible rational approximation of the same degree?

Although we cannot give an affirmative answer to the above questions in all cases, we shall describe classes of analytic functions which house nearly all of the cases encountered by the author in applications, and for which the answer to each of the above questions is "Yes".

We shall develop a class of rational approximations for interpolation over $[-1,1]$, $[0,\infty]$, and $[-\infty,\infty]$. These rational approximations share many of the features of SINC methods summarized in [23]. The interpolation points of these rationals are the same as the SINC interpolation points, and the classes of functions which low-degree rationals approximate accurately are the same as the classes which the SINC functions approximate accurately.

Indeed, the error bounds for, e.g., approximation on $[-1,1]$ of functions analytic on the unit disc are the same as the SINC bounds, i.e., rationals have the same optimality properties as SINC methods. In using rationals instead of SINC functions, we lose many of the simple relations that SINC functions satisfy, such as

Received June 22, 1983; revised September 9, 1985.
1980 *Mathematics Subject Classification*. Primary 41A20, 41S25, 65B05, 65D15, 41A50.
* Supported by U.S. Army Research Contract DAAG-29-77-G-0319 and DAAG-29-83-K-0012.

orthogonality, and ease of getting other formulas, such as quadrature, approximations of transforms, approximations of derivatives, methods of solving differential equations, etc. However, the well-known rational function algorithms of Thiele [29] (the ρ-algorithm), Padé [17], Shanks [21] and Wynn [31] (the ε-algorithm) all share simple methods of prediction which are at this time not known for the SINC methods. This paper provides an understanding in that it enables us to tell *a priori* when we can expect these algorithms to work effectively.

The spaces of functions for which the rationals provide accurate approximations are described precisely in Section 2 of the paper. One such space consists, roughly, of functions analytic on an interval with possible singularities at end-points of the interval, such that the functions are of class Lip_α ($\alpha > 0$) on the closed interval. One encounters such functions in nearly all cases in applications.

The rational approximations of this paper have the following additional features.

(a) There are no poles on the interval of approximation.

(b) The rational functions are linear in f, the function that is being approximated.

(c) They are nearly optimal. More precisely, we prove the following result:

THEOREM 1.1. *Let* $1 < p < \infty$, *let* $p' = p/(p-1)$, *let* $\mathbf{U}$ *denote the unit disc in the complex plane, let* g *be in the Hardy space* $\mathbf{H}_p(\mathbf{U})$, *and let* $f(z) = (1 - z^2)g(z)$. *Let* $\mathbf{P}_n$ *denote the space of polynomials of degree n and set*

$$\delta_N = \sup_{g \in \mathbf{H}_p(\mathbf{U}), \|g\|_p = 1} \left\{ \inf_{\mu \in \mathbf{P}_{2N+2}, \sigma \in \mathbf{P}_{2N+1}} \left(\sup_{-1 \leqslant x \leqslant 1} \left| f(x) - \frac{\mu(x)}{\sigma(x)} \right| \right) \right\}, \tag{1.1}$$

where $\|g\|_p$ *denotes the usual norm in* $\mathbf{H}_p(\mathbf{U})$. *Then there exist positive constants* C_1, C_2, *and* N_0, *depending only on p such that for all* $N > N_0$,

$$C_1 N^{-1} \exp\left\{-\pi(2N/p')^{1/2}\right\} \leqslant \delta_N \leqslant C_2 N^{1/2} \exp\left\{-\pi\left(N/(2p')\right)^{1/2}\right\}. \tag{1.2}$$

In a recent very interesting paper, Burchard and Höllig [5] obtained essentially the same upper and lower bounds for the linear n-width of approximation in the same space. One may deduce from their results, moreover, that the constant in the exponent on the right-hand side of (1.2) is best possible for any rationals that are a linear combination of $2N + 1$ values of f. The rationals which we shall derive in this paper are of the form μ/σ in (1.1) and they approximate f on $[-1, 1]$ to within an error bounded by the right-hand side of (1.2).

A typical approximation result of the present paper is the following:

THEOREM 1.2. *Let f and g satisfy the conditions of Theorem* 1.1, *let N be a positive integer, and define* h, z_j *and* $B(z)$ *by*

$$h = \pi\left[p'/(2N)\right]^{1/2}, \tag{1.3}$$

$$z_j = \frac{e^{jh} - 1}{e^{jh} + 1}, \qquad B(z) = (1 - z^2) \prod_{j=-N}^{N} \frac{z - z_j}{1 - z_j z}. \tag{1.4}$$

Then

$$\max_{-1 \leqslant x \leqslant 1} \left| f(x) - \sum_{j=-N}^{N} \frac{f(z_j) B(x)}{(x - z_j) B'(z_j)} \right| \leqslant C_2 N^{1/(2p')} \exp\left\{-\pi\left[\frac{N}{2p'}\right]^{1/2}\right\}, \tag{1.5}$$

where C_2 *depends only on p.*

Due to their simplicity of construction and approximation properties, the rational function approximations of this paper play a similar role as the interpolation polynomials obtained by interpolation at the zeros of the Chebyshev polynomials play for polynomial approximation. In order to describe this role effectively, we return first to the case of Fourier series.

Let $R > 1$, and let $\mathbf{A}_R$ denote the annular region in the complex plane $\mathbf{C}$, $\mathbf{A}_R = \{w \in \mathbf{C}\colon R^{-1} < |w| < R\}$, let F be analytic in $\mathbf{A}_R$, and let c_j be determined from

$$c_j = \frac{1}{2N+1} \sum_{k=0}^{2N} F(\exp[i\theta_k]\exp[ij\theta_k]); \qquad \theta_k = \frac{2k\pi}{2N+1}. \tag{1.6}$$

Then

$$\max_{0 \leqslant \theta \leqslant 2\pi} \left| F(e^{i\theta}) - \sum_{j=-N}^{N} c_j e^{ij\theta} \right| = O(R^{-N}). \tag{1.7}$$

The bound on the right-hand side of (1.7) is best possible with regard to order, in that the number R cannot be replaced by a larger positive number, regardless of how the c_j are chosen.

In (1.7) we now consider only those functions F for which $F(w) = F(1/w)$ for all w in $\mathbf{A}_R$. We can then obtain a cosine polynomial approximation to F on the unit circle. The mapping

$$z = \tfrac{1}{2}(w + 1/w) \tag{1.8}$$

transforms the annulus $\mathbf{A}_R$ onto the ellipse $\mathbf{E}_R$ with foci at $z = \pm 1$ and sum of semiaxes equal to R. Conversely, if $f(z)$ is analytic and uniformly bounded in $\mathbf{E}_R$ then we can use (1.8) to get a new function $F(w)$ analytic in $\mathbf{A}_R$ with Fourier series expansion $F(e^{i\theta}) = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta}$, and where $c_{-k} = c_k$ for all integers k. If $T_N(x) = \cos(N\theta)$, where $x = \cos\theta$, and $x_k = \cos\{(2k-1)\pi/(2N)\}$, then

$$\max_{-1 \leqslant x \leqslant 1} \left| f(x) - \sum_{k=1}^{N} \frac{f(x_k)T_N(x)}{(x - x_k)T_N'(x_k)} \right| = O(R^{-N}), \tag{1.9}$$

where once again, the R in the $O(R^{-N})$ bound on the right-hand side cannot be replaced by a larger number, regardless of how a polynomial of degree $N - 1$ is chosen to approximate f on $[-1, 1]$. Indeed, Powell [19] has shown that the left-hand side of (1.9) is at most 4 times as large as the error of best "minimax" polynomial approximation for N at most 20, and at most 5 times as large for N up to 100.

Hence, instead of finding the polynomial which best approximates f on $[-1, 1]$, it is much easier to use the Chebyshev polynomial for which the interpolation points x_k are known explicitly to get an approximation which is nearly as good. The rational functions of this paper share this feature.

Notice that for the case of polynomial approximation above, we required a knowledge of a region of analyticity of f, a property which we can usually determine *a priori* in applications. Once we have identified such an ellipse $\mathbf{E}_R$ (resp. an annulus $\mathbf{A}_R$) we can be certain that polynomial (resp. Fourier polynomial) approximation will work very well on $[-1, 1]$ (resp. on $[0, 2\pi]$). From the point of view of approximation in applications, we can thus identify functions analytic in $\mathbf{E}_R$ (resp. $\mathbf{A}_R$) with polynomials (resp. Fourier polynomials), since they can be very accurately approximated with polynomials (resp. Fourier polynomials) of low degree.

Unfortunately, there is a drastic change in the rate of convergence of polynomial approximation in the case when the function to be approximated has a singularity on the interval of approximation, a situation often encountered in applications. For example, if $0 < \alpha < 1$, we have

$$\max_{-1 \leqslant x \leqslant 1} \left|(1 - x^2)^\alpha - p_N(x)\right| \geqslant \frac{C}{N^{2\alpha}}, \tag{1.10}$$

where $p_N(x)$ is any polynomial of degree N in x and C is a positive constant independent of N. If $\alpha = \frac{1}{4}$ we would have to take $N > 10^5$ to get three places of accuracy.

While for practical purposes functions with singularities on the interval of approximation cannot be identified with polynomials, there is, nevertheless, a class of functions with singularities on the interval of approximation, which we describe in this paper, and which lends itself to accurate rational approximation. Such a class includes the functions which we can accurately approximate with polynomials and for practical purposes, we can identify this class with rational functions. For example, by Theorem 1.2 above, given an integer $N > 0$, there exists a rational function $p_{2N+2}(x)/q_{2N+1}(x)$ with p_{2N+2} of degree $2N + 2$ in x and q_{2N+1} of degree $2N + 1$ in x, such that

$$\max_{-1 \leqslant x \leqslant 1} \left|(1 - x^2)^\alpha - \frac{p_{2N+2}(x)}{q_{2N+1}(x)}\right| < CN^{\alpha/2}\exp\left\{-\pi\left(\frac{\alpha N}{2}\right)^{1/2}\right\}. \tag{1.11}$$

We remark that by identifying classes of functions which can be approximated accurately by rational functions, we are identifying classes of functions for which we can expect (say) the Thiele algorithm to work well, provided that the interpolation points are suitably chosen. We shall later in this paper illustrate this also for the case of the ε-algorithm and the Padé method. Thus we are able to replace the intuitive feeling upon which scientists base their decision to use rational functions by a more deterministic approach. For example, we would be able to tell *a priori* that the Padé method used in [3] may be expected to be accurate.

Another practically important use of rational functions is in analytic continuation. For sake of illustration, let us momentarily return to the class of functions analytic and bounded in the ellipse $\mathbf{E}_R$ described above. Let us assume that f is known on $[-1, 1]$, and that we want to evaluate f at the point $\frac{1}{2} + \frac{1}{4}(R + 1/R)$ in the ellipse. This can be done by means of the polynomial in (1.9), the rate of convergence of the error to zero being $O(r^N)$, where $r = \{(a + 1)/2 + \sqrt{[a + (a^2 - 3)/4]}\}/R$, $a = (R + 1/R)/2$. On the other hand, if $p_N(x)$ is *any* polynomial approximation to $f(x) = c + (1 - x^2)^\alpha$ on $[-\frac{1}{2}, \frac{1}{2}]$ and we want to approximate $f(1) = c$ by evaluating $p_N(1)$, then we may expect $[f(1) - p_n(1)]$ to converge to zero very slowly, indeed, too slowly to be of any practical value. Since, however, we may identify $f(x) = c + (1 - x^2)^\alpha$ with a rational function for practical purposes, we can accurately evaluate $f(1)$ via a rational function by using values of x on $[-\frac{1}{2}, \frac{1}{2}]$ only.

As a more sophisticated example, let $u = u(x, y)$ be harmonic in the right halfplane, and assume that $u(0^+, y)$ is of class Lip_α ($\alpha > 0$) on a neighborhood of $y = 0$. It follows then that $u(x, 0)$ is analytic and bounded on a sector with vertex at the origin, and of class Lip_α on $[0, A]$, where $A > 1$ is arbitrary. That is, for practical purposes, we may identify $u(x, 0)$ with a rational function, and we can accurately approximate $u(0, 0)$ via a low-degree rational function, by using values of $u(x, 0)$ on, e.g., the interval $[1, A]$.

In the cases when the condition of accurate approximation are satisfied, it is thus possible to do accurate analytic continuation all the way to the boundary of analyticity, via a relatively low-degree rational function.

The Lip_α property of the function to be approximated is important from the point of view of applications; if f approaches zero too slowly in a neighborhood of a singularity, then it is necessary to choose the degree of the rational function to be very large, in order to achieve a desired accuracy. For example, for rational approximation on $[0, 1]$, if $f(x) - f(1) \approx c/[-\log(1 - x)]^\alpha$ as $x \to 1^-$, then it is just as difficult to approximate f on $[0, 1]$ by a rational function as it is to approximate $(1 - x^2)^\alpha$ on $[-1, 1]$ by a polynomial (see also Henrici [11, pp. 53–54]). We remark, however, that this difficulty can often be remedied by means of a transformation. For example, if we set $x = 1 - \exp[-1/z]$, we get $f(1 - \exp[-1/z]) - f(1) \approx cz^\alpha$, $z \to 0$, and we can now approximate the new function of z defined on the interval $[0, \infty]$ by a rational function.

We mention that a rational function of the same degree as that in (1.5) for approximating f on $[-1, 1]$ was previously constructed by the author [24], and moreover, the error bound in [24] is the same as that on the right-hand side of (1.5). However, whereas the interpolation points in [24] are the points (see also [6])

$$w_j = k^{1/2}\mathrm{sn}[(2j - 1)K/(2N); k], \tag{1.12}$$

the evaluation of the w_j is more difficult than the evaluation of the z_j in (1.4).

The same points z_j defined in (1.4) were also used by Peaceman and Rachford [18] to approximate the points w_j in (1.12) in their alternating direction method for obtaining approximate solutions to parabolic and elliptic partial differential equations.

For many problems of rational approximation one does not have analyticity in the unit disc $\mathbf{U}$, but rather in a smaller region $\mathbf{D}_d^2$ (see Figure 2.2 in Section 2), and we have therefore also considered this case. Although our error bounds for this case are not as small as the SINC bounds, we believe that the errors of the rationals of this paper do in fact have the same bounds as the corresponding SINC errors, and we therefore expect that the bounds of this paper may be improved for the case when $0 < d < \pi/2$.

Notice that if N is replaced by $4N$ in the rational function of (1.5) then the $2j$th interpolation point in the "$4N$"-rational is the same as the jth point of the "N"-rational, and that the "$4N$"-rational has roughly twice as many correct significant figures of accuracy as the "N"-rational. This result is of practical value, particularly when the user is unable to determine a region $\mathbf{D}$ of analyticity.

Let us now briefly describe the layout of the paper.

In Section 2 we give precise statements and proofs of the results (a), (b) and (c) stated at the beginning of this section. These proofs would ordinarily be lengthy, and for this reason some of the details are carried out in Appendices A and B.

In Section 3 we illustrate connections of the results with the well-known approximation algorithms, the Thiele, or ρ-algorithm, the Shanks-Wynn, or ε-algorithm, and the Padé method. In view of the results of Section 2, we are able to determine *a priori* when we can expect these algorithms to work.

In Section 4, we prove Theorem 1.1 above. While the exact optimal rate of convergence of rational approximation is not known, we conjecture that, in the notation of (1.1),

$$(1.13)\qquad \sup_{g\in \mathbf{H}_p(\mathbf{U}),\|g\|_p=1}\left\{\inf_{\mu\in \mathbf{P}_N,\sigma\in \mathbf{P}_N}\left[\sup_{-1<x<1}\left|f(x)-\frac{\mu(x)}{\sigma(x)}\right|^{(p'/N)^{1/2}}\right]\right\}\to e^{-\pi}$$

as $N\to\infty$ (compare [5]).

In Appendix A we study bounds on rational functions related to (1.5). The Jacobi theta functions turn out to be most convenient for this purpose, since, while it is possible to obtain similar results via the approximate integration of the function $F(z,t)=t^{-1}\log|(z+t)/(z-t)|$ over $0\leqslant t\leqslant\infty$ via the trapezoidal (resp. midordinate) rule evaluated at the points e^{jh} (resp. $e^{(j-1/2)h}$), $j=0,\pm1,\pm2,\ldots$, and using the concavity of this function (for fixed $z\in(0,\infty)$) as a function of t, it is possible to achieve more accurate error bounds via the theta functions, since it is possible to get exact bounds via known properties of the theta functions. However, while we use elliptic functions to obtain our results, the final results are independent of elliptic functions.

In Appendix B we obtain accurate bounds on contour integrals encountered in the proofs of Section 2.

We close this section with a few historical remarks. Stieltjes [27] seems to have been the first to identify classes of functions which may be represented exactly by infinite continued fraction expansions, and which may therefore be accurately approximated via truncated forms of continued fractions, i.e., rational functions. These functions are expressible in the form

$$(1.14)\qquad F(z)=\int_a^b\frac{d\mu(t)}{t-z},$$

and the continued fraction expression obtainable via this representation converges uniformly in any closed region of the complex plane which does not contain the interval $[a,b]$ (see [10]). Unfortunately, given a function F, it is not possible to easily check in applications whether or not F has a representation of the form (1.14).

In [8] Gautschi gives an excellent summary of the use of rational functions in numerical analysis.

It has long been suspected and verified in *ad hoc* cases that rational functions can do a better job of approximation than polynomials. That this is *not* in general the case for approximation on $[-1,1]$ of functions that are analytic in the ellipse $\mathbf{E}_R$ defined above was shown by Szabados [28]. Newman's result [15] (see also [16]) on the approximation of $|x|$ on $[-1,1]$ showed that rational functions are much better for approximating $|x|$ than polynomials. The error bounds of the present paper all have the $O(\exp[-cn^{1/2}])$ rate of convergence when used to approximate functions with singularities; this rate which is typical of rational and SINC function [23] approximation, was originally found by Newman in his rational approximation of $|x|$. Also of interest is the idea of Ganelius [7] for using the Green's function of a region of analyticity to obtain rational approximations; indeed, the rational functions of this paper have this property. For the case of rational approximation on a finite or semi-infinite interval, the poles of the rational functions of this paper lie on the real line outside of the interval, as is the case for best approximation of Stieltjes transforms—see Borwein [4].

2. Rational Approximation with Error Bounds. This section contains the main approximations theorems of the paper. While the proofs are complete, we shall use results derived in the appendices in order to shorten the proofs.

As mentioned in the introduction, the rationals of this paper and SINC approximations [23] share many similarities. We shall therefore use the notation of [23] in order to emphasize these similarities and also in order to facilitate the understanding of this paper. Let us therefore briefly review the notation of [23].

Let d be a positive number in the range $0 < d < \pi$, let $\mathbf{C}$ denote the complex plane, and let regions $\mathbf{D}_d^i$, $i = 1, 2, 3, 4$, and $\mathbf{D}_d$ be defined by

$$\mathbf{D}_d^1 = \{ z \in \mathbf{C}\colon |\arg(z)| < d \} \quad \text{(see Figure 2.1)}; \tag{2.1}$$

$$\mathbf{D}_d^2 = \{ z \in \mathbf{C}\colon |\arg[(1+z)/(1-z)]| < d \} \quad \text{(see Figure 2.2)}; \tag{2.2}$$

$$\mathbf{D}_d^3 = \{ z \in \mathbf{C}\colon |\arg[\sinh(z)]| < d \} \quad \text{(see Figure 2.3)}; \tag{2.3}$$

$$\mathbf{D}_d^4 = \{ z = x + iy \in \mathbf{C}\colon y^2/\sin^2(d) - x^2/\cos^2(d) \leqslant 1 \} \quad \text{(see Figure 2.4)}; \tag{2.4}$$

$$\mathbf{D}_d = \{ z \in \mathbf{C}\colon |\mathrm{Im}(z)| < d \} \quad \text{(see Figure 2.5)}. \tag{2.5}$$

Definition 2.1. *Let* $\mathbf{D}$ *be a simply connected region in the complex plane* $\mathbf{C}$, *let* $\partial\mathbf{D}$ *denote the boundary of* $\mathbf{D}$, *let* a *and* b $(b \neq a)$ *be points of* $\partial\mathbf{D}$, *and let* $\mathbf{D}_d$ *be defined as in* (2.5). *Let* ϕ *be a conformal map of* $\mathbf{D}$ *onto* $\mathbf{D}_d$, *such that* $\phi(a) = -\infty$, *and* $\phi(b) = \infty$. *Let* $\psi = \phi^{-1}$ *denote the inverse map, and set*

$$\Gamma = \{ \psi(x)\colon -\infty \leqslant x \leqslant \infty \}. \tag{2.6}$$

Given ϕ, ψ, *a positive number* h, *and a number* σ *which is either* 0 *or* $1/2$, *we denote by* $z_k = z_k(h)$ *the set of points*

$$z_k = \psi((k+\sigma)h), \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{2.7}$$

Let us also define ρ *by*

$$\rho(z) = e^{\phi(z)}. \tag{2.8}$$

Let $\mathbf{A}(\mathbf{D})$ *denote the family of all functions* F *that are analytic in* $\mathbf{D}$, *and let* $\mathbf{B}(\mathbf{D})$ *denote the family of all* F *in* $\mathbf{A}(\mathbf{D})$ *such that*

$$N(F, \mathbf{D}) = \int_{\partial\mathbf{D}} |F(z)\, dz| < \infty, \tag{2.9}$$

where the contour integral is defined by the limit

$$\int_{\partial\mathbf{D}} |F(z)\, dz| = \inf_{C \to \partial\mathbf{D}, C \subset \mathbf{D}} \int_C |F(z)\, dz|. \tag{2.10}$$

Given constants α, β and K, such that $0 < \alpha < 1$, $0 < \beta < 1$, and $K > 0$, let us define classes of functions $\mathbf{B}_{\alpha,\beta}(\mathbf{D})$ and $\mathbf{B}_\alpha(\mathbf{D})$ by

$$\text{(2.11)}\quad \mathbf{B}_{\alpha,\beta}(\mathbf{D}) = \left\{ F \in \mathbf{A}(\mathbf{D}) : |F(z)| \leqslant K|\rho(z)|^{\alpha}|1 + \rho(z)|^{-\alpha-\beta},\ z \in \mathbf{D} \right\}$$

and

$$\text{(2.12)}\quad \mathbf{B}_\alpha(\mathbf{D}) = \mathbf{B}_{\alpha,\alpha}(\mathbf{D}).$$

In view of the above definition, we shall construct rational functions in the variable $\rho(z)$, in order to carry out rational approximation of functions $F \in \mathbf{B}_\alpha(\mathbf{D})$, or $F \in \mathbf{B}_{\alpha,\beta}(\mathbf{D})$ on Γ.

Although it is most convenient to derive the SINC approximations [23] by first deriving them for the interval $\mathbf{R} = [-\infty, \infty]$, it turns out that it is simplest to derive the rationals of this paper for the interval $[0, \infty]$. The other cases then follow similarly as for the case of SINC approximation, via the use of conformal maps.

2.1. *Rational Approximation in the Variable z on* $[0, \infty]$. Let us house some important concepts for this case in an example, in order to achieve consistency with results to follow involving approximation on other intervals.

Example 2.1. If $\mathbf{D} = \mathbf{D}_d^1$ (see Eq. (2.1)), then

$$\text{(2.13)}\quad \begin{aligned} &\phi(z) = \log(z), \qquad \rho(z) = z, \\ &\psi(w) = e^w, \quad \Gamma = [0, \infty], \quad z_k = e^{(k+\sigma)h}, \\ &\mathbf{B}_{\alpha,\beta}(\mathbf{D}_d^1) = \left\{ F \in \mathbf{A}(\mathbf{D}_d^1) : |F(z)| < K|z|^{\alpha}|1 + z|^{-\alpha-\beta},\ z \in \mathbf{D}_d^1 \right\}. \end{aligned}$$

We set

$$\text{(2.14)}\quad \beta(z) = \prod_{j=-M}^{N-2\sigma} \frac{z - e^{(j+\sigma)h}}{z + e^{(j+\sigma)h}}, \qquad B(z) = \frac{z}{1+z}\beta(z).$$

The difference between F and its rational approximant takes the form

$$\text{(2.15)}\quad \eta(x) = F(x) - \sum_{j=-M}^{N-2\sigma} \frac{F(z_j)B(x)}{(x - z_j)B'(z_j)}, \qquad x \in (0, \infty).$$

By this method of approximation we can approximate functions such as $x^{1/4}(1 + x^2)^{-1/3}\log(x)$.

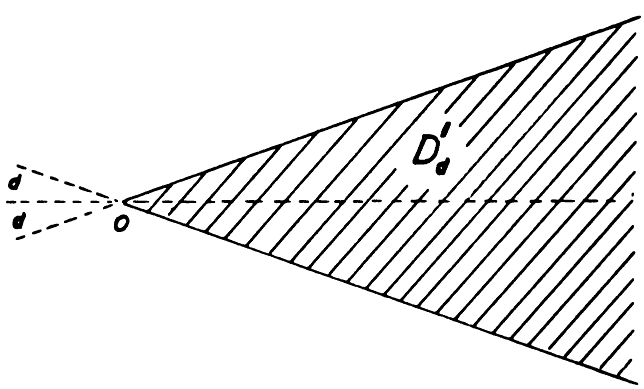

FIGURE 2.1. *Equation* (2.1)

We shall describe two typical situations of rational approximation in the variable z on the interval $[0, \infty]$. The conditions in the first case (Assumption 2.1a below) are of theoretical interest, particularly when $d = \pi/2$, that is, when $\mathbf{D}_d^1$ (see Eq. (2.1)) is the right halfplane, while the conditions in the second case are more amenable to applications.

Assumption 2.1a. *Given F analytic in $\mathbf{D}_d^1$, let $f \in \mathbf{B}(\mathbf{D}_d^1)$, where for some $p \in (1, \infty)$*

$$f(z) = \left[F(z)(1+z)^2/z\right]^p/(1+z)^2 \tag{2.16}$$

and let $\|F\|_p < \infty$, where the norm is defined by

$$\|F\|_p = \lim_{\delta \to d^-} \left[\frac{1}{2\pi}\int_{\partial \mathbf{D}_\delta^1} |f(z)\,dz|\right]^{1/p}. \tag{2.17}$$

Let $M = N$, and let $B(z)$ and $\eta(x)$ be defined by (2.14) *and* (2.15), *respectively.*

THEOREM 2.1a. *Let Assumption* 2.1a *be satisfied, let $0 < d \leqslant \pi/2$, and define p' and h by*

$$p' = p/(p-1), \qquad h = \pi/\left[p'/(2N)\right]^{1/2}. \tag{2.18}$$

Then there exists a constant C depending only on p', such that for $0 < x < \infty$,

$$|\eta(x)| \leqslant CN^{1/(2p')}\exp\left\{-d\left[2N/p'\right]^{1/2}\right\}\|F\|_p. \tag{2.19}$$

Proof. It is readily seen that $\eta(x)$ in (2.15) also has the representation

$$\eta(x) = \frac{B(x)}{2\pi i}\int_{\partial \mathbf{D}_d^1} \frac{F(z)\,dz}{(z-x)B(z)}. \tag{2.20}$$

Now, by Lemma A.2, the function $\beta(z)$ defined in (2.14) satisfies

$$\max_{z \in \partial \mathbf{D}_d^1} |\beta(z)|^{-1} \leqslant \exp\left[\frac{\pi(\pi/2 - d)}{h} + \varepsilon\right], \tag{2.21}$$

where ε is defined in Eq. (A.20) of Appendix A. Hence, setting

$$A = \exp[\varepsilon], \tag{2.22}$$

taking absolute values of each term in (2.20), applying Hölder's inequality using (2.17), and then using (2.21) and (2.22), we get

$$|\eta(x)| \leqslant A\frac{x}{1+x}|\beta(x)|\exp\left[\frac{\pi(\pi/2-d)}{h}\right]G(p', d, x)\|F\|_p, \tag{2.23}$$

where $G(p', d, x)$ is defined in Eq. (B.1) of Appendix B. Hence, using Eq. (B.6), we get

$$\frac{x}{1+x}|\beta(x)|G(p', d, x) \leqslant A(p', d)|\beta(x)|\begin{cases} x^{1/p'} & \text{if } 0 \leqslant x \leqslant 1, \\ x^{-1/p'} & \text{if } x \geqslant 1. \end{cases} \tag{2.24}$$

Next, using (A.35), and the fact that $|(x - z_j)/(x + z_j)| \leqslant 1$, we get

$$|\beta(x)| \leqslant \begin{cases} 2\exp\{-\pi^2/(2h)[1 - N^{-1/2}]\} & \text{if } N^{1/2}z_{-N} \leqslant x \leqslant N^{-1/2}z_{N-2\sigma}, \\ 1 \quad \text{if } x \leqslant N^{1/2}z_{-N} \text{ or if } x \geqslant N^{-1/2}z_{N-2\sigma}. \end{cases} \tag{2.25}$$

Hence, substituting (2.25) into (2.24), we get, for $0 < x < \infty$,

$$\frac{x}{1+x}|\boldsymbol{\beta}(x)|G(p', d, x)$$

$$(2.26) \qquad \leqslant A(p', d)\begin{cases} 2\exp\{-\pi^2/(2h)[1 - N^{-1/2}]\} & \text{if } N^{1/2}z_{-N} \leqslant x \leqslant N^{-1/2}z_{N-2\sigma}, \\ [N^{1/2}z_{-N}]^{1/p'} & \text{otherwise.} \end{cases}$$

Noting that $[N^{1/2}z_{-N}]^{1/p'} \leqslant N^{1/(2p')}\exp\{-(N-\sigma)h\}$, using (2.18) and combining with (2.23) yields (2.19).

Assumption 2.1b. *Let* $0 < d \leqslant \pi/2$, *and for some constants* α *and* β, *with* $0 < \alpha < 1$, $0 < \beta < 1$, *let* F *belong to* $\mathbf{B}_{\alpha,\beta}(\mathbf{D}_d^1)$.

THEOREM 2.1b. *Let Assumption* 2.1b *be satisfied, let* $\tau = \min(\alpha, \beta)$, $\delta = \max(\alpha, \beta)$, *and corresponding to some positive integer* n, *let* h *be defined by*

$$(2.27) \qquad h = \pi/(2\tau n)^{1/2}.$$

If $\tau = \alpha$, *let* M *and* N *be defined by*

$$(2.28) \qquad M = n, \qquad N = [(\alpha/\beta)n],$$

where $[u]$ *denotes the greatest integer in* u, *while if* $\tau = \beta$, *let* M *and* N *be defined by*

$$(2.29) \qquad M = [(\beta/\alpha)n], \qquad N = n.$$

Let z_j, $\boldsymbol{\beta}(z)$, *and* $B(z)$ *be defined as in* (2.14), *and let* $\eta(x)$ *be defined as in* (2.15). *Then there exists a constant* C *such that for all* x *on* $(0, \infty)$

$$(2.30) \qquad |\eta(x)| \leqslant Cn^{\delta/2}\exp\{-d(2\tau n)^{1/2}\}.$$

Proof. By our notation, η is now also defined by (2.20). Taking absolute values, replacing $|F(z)|$ by its bound as given in (2.13), and using Lemma A.3, we get

$$(2.31) \qquad |\eta(x)| \leqslant A_1C_1\frac{x}{1+x}|\boldsymbol{\beta}(x)|H(\alpha, \beta, d, x)\exp\left\{\frac{\pi(\pi/2 - d)}{h}\right\},$$

where the constant A_1 depends only on d, and where $H(\alpha, \beta, d, x)$ is defined by (B.2). By Theorem B.2, we get

$$(2.32) \qquad \frac{x}{1+x}H(\alpha, \beta, d, x) \leqslant A(\alpha, \beta, d)\begin{cases} x^{\alpha} & \text{if } 0 \leqslant x \leqslant 1, \\ x^{-\beta} & \text{if } x \geqslant 1. \end{cases}$$

We can now use Lemma A.3 to get a uniform bound on $|\boldsymbol{\beta}(x)|[\min(x^{\alpha}, x^{-\beta}]$ on $[0, \infty]$. Taking $M^{1/2}z_{-M} \leqslant x \leqslant N^{-1/2}z_{N-2\sigma}$, we have $|\boldsymbol{\beta}(x)|$ bounded by the right-hand side of (A.34) and $\max[\min(x^{\alpha}, x^{-\beta})] = 1$. On the other hand, if $x \leqslant M^{1/2}z_{-M}$ or else if $x \geqslant N^{-1/2}z_{N-2\sigma}$, then $|\boldsymbol{\beta}(x)|$ is bounded by 1, and

$$(2.33) \qquad \max[\min(x^{\alpha}, x^{-\beta})] \leqslant n^{\delta/2}e^{-nh}e^{\sigma h}.$$

Substituting h defined by (2.27), the result (2.30) follows.

2.2. *Rational Approximation in the Variable* $\rho(z)$ *on* Γ. We now let z_j be defined as in (2.7), and for some positive integers M and N, and $\sigma = 0$ or $1/2$, we set

$$(2.34) \qquad \boldsymbol{\beta}(w) = \prod_{j=-M}^{N-2\sigma}\frac{\rho(w) - e^{(j+\sigma)h}}{\rho(w) + e^{(j+\sigma)h}},$$

$$(2.35) \qquad B(w) = \frac{\rho(w)}{1+\rho(w)}\boldsymbol{\beta}(w).$$

Assumption 2.2a. *Let Definition* 2.1 *be valid, let g be analytic in* $\mathbf{D}$, *and for some p in the range* $1 < p < \infty$, *let G be defined by*

$$G(w) = \left[\frac{g(w)[1+\rho(w)]^2}{\rho(w)}\right]^p. \tag{2.36}$$

Let G satisfy the inequality $\|g\| < \infty$, *where*

$$\|g\| \equiv \inf_{C\subset \mathbf{D}, C\to\partial\mathbf{D}} \left[\frac{1}{2\pi}\int_C |G(w)|\,|\rho(w)\phi'(w)\,dw|\right]^{1/p}. \tag{2.37}$$

Assumption 2.2b. *Let g be analytic in* $\mathbf{D}$, *and for all w in* $\mathbf{D}$, *let*

$$|g(w)| \leqslant C_1|\rho(w)|^{\alpha}|1+\rho(w)|^{-\alpha-\beta}, \tag{2.38}$$

where C_1, α, *and* β *are positive constants, with* $0 < \alpha < 1$ *and* $0 < \beta < 1$.

Proceeding as in the preceding section, we let $B(w)$ be defined by (2.35), and we set

$$\eta(u) = \frac{B(u)}{2\pi i}\int_{\partial\mathbf{D}} \frac{g(w)\rho(w)\phi'(w)\,dw}{[\rho(w)-\rho(u)]B(w)} \tag{2.39}$$

to get

$$\eta(u) = g(u) - \sum_{j=-M}^{N-2\sigma} \frac{g(z_j)e^{(j+\sigma)h}\phi'(z_j)B(u)}{\left[\rho(u)-e^{(j+\sigma)h}\right]B'(z_j)}. \tag{2.40}$$

THEOREM 2.2a. *Let Assumption* 2.2a *be satisfied, let* $0 < d \leqslant \pi/2$, *let* $M = N$ *in* (2.40), *and let* z_j (*for* $\sigma = 0$ *or* $1/2$) *and* Γ *be defined by Definition* 2.1. *If h is selected by the expression*

$$h = \pi\left[p'/(2N)\right]^{1/2}, \tag{2.41}$$

where $p' = p/(p-1)$, *then there exists a constant C depending only on* p', *such that for all x on* Γ,

$$|\eta(x)| \leqslant CN^{1/(2p')}\exp\left\{-d(2N/p')^{1/2}\right\}\|g\|. \tag{2.42}$$

Proof. If we set $w = \psi(z)$ and $u = \psi(x)$ in (2.39), we get (2.20), with $g(\psi(z)) = f(z)$. The right-hand side of (2.40) reduces similarly to the right-hand side of (2.15). Hence the proof is identical to the proof of Theorem 2.1a.

The proof of the following theorem is also similar to the proof of Theorem 2.1b, in view of the above remarks.

THEOREM 2.2b. *Let Assumption* 2.2b *be satisfied, and let* $0 < d \leqslant \pi/2$. *Let* $\tau = \min(\alpha,\beta)$, $\delta = \max(\alpha,\beta)$, *and corresponding to a positive integer n, let h be selected by the formula*

$$h = \pi/(2\tau n)^{1/2}. \tag{2.43}$$

If $\tau = \alpha$, *let M and N be defined by*

$$M = n, \qquad N = [(\alpha/\beta)n], \tag{2.44}$$

while if $\tau = \beta$, *let M and N be defined by*

$$M = [(\beta/\alpha)n], \qquad N = n. \tag{2.45}$$

Let $\eta(u)$ *be defined as in* (2.40). *Then there exists a constant* C *depending only on* α, β, *and* d, *such that for all* u *on* Γ,

$$|\eta(u)| \leqslant Cn^{\delta/2}\exp\{-d(2\tau n)^{1/2}\}. \tag{2.46}$$

Example 2.2: *SINC Approximation* [23]. Let the assumptions of Theorem 2.2b be satisfied, and take $\beta = \alpha$, $\sigma = 0$. Set

$$h = [\pi d/(\alpha N)]^{1/2}, \qquad S(j,h)\circ(x) = \frac{\sin\{(\pi/h)[x - jh]\}}{(\pi/h)[x - jh]}. \tag{2.47}$$

Then there exists a constant A, depending only on α and C_1, such that for all $u \in \Gamma$,

$$\left| g(u) - \sum_{j=-N}^{N} g(z_j)S(j,h)\circ\phi(u) \right| \leqslant AN^{1/2}\exp\{-(\pi d\alpha N)^{1/2}\}. \tag{2.48}$$

Notice that the bound (2.48) is sharper than the corresponding one in (2.46) above, for $0 < d < \pi/2$, but it reduces essentially to the same one for the case when $d = \pi/2$.

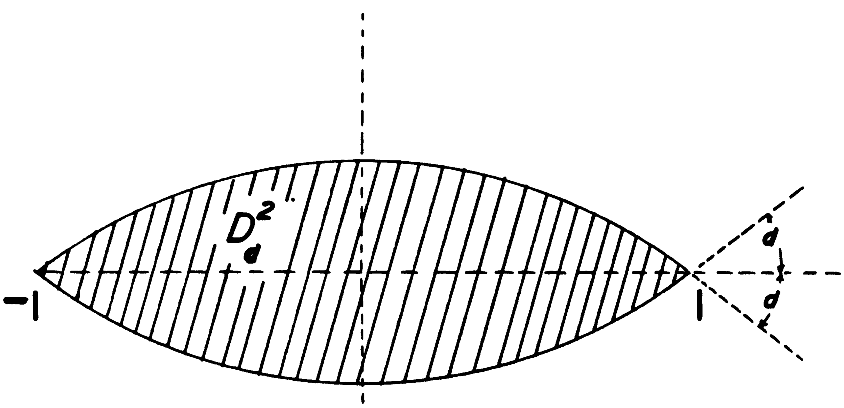

FIGURE 2.2. *Equation* (2.2)

Example 2.3: *Approximation on* $[-1,1]$. If $\mathbf{D} = \mathbf{D}_d^2$ (see Eq. (2.2)), then

$$\phi(z) = \log\frac{1+z}{1-z}, \qquad \rho(z) = \frac{1+z}{1-z},$$

$$\psi(w) = \frac{e^w - 1}{e^w + 1}, \quad \Gamma = [-1,1], \quad z_k = \frac{e^{(k+\sigma)h} - 1}{e^{(k+\sigma)h} + 1}, \tag{2.49}$$

$$\mathbf{B}_{\alpha,\beta}(\mathbf{D}_d^2) = \left\{F \in \mathbf{A}(\mathbf{D}_d^2) : |F(z)| < K|1+z|^{\alpha}|1-z|^{\beta},\ z \in \mathbf{D}_d^2\right\}.$$

In this case,

$$B(w) = (1 - w^2)\prod_{j=-M}^{N-2\sigma}\frac{w - z_j}{1 - wz_j}, \tag{2.50}$$

and the difference between g and its rational approximation takes the form

$$\eta(u) = g(u) - \sum_{j=-M}^{N-2\sigma}\frac{g(z_j)B(u)}{(u - z_j)B'(z_j)}; \tag{2.51}$$

(i) If $g \in \mathbf{B}_{\alpha,\beta}(\mathbf{D}_d^2)$, if h is selected according to (2.43), and if M and N are selected according to (2.44)–(2.45), then η defined in (2.51) is bounded by the right-hand side of (2.46).

(ii) Let $\mathbf{U} = \mathbf{D}_{\pi/2}^2$ denote the unit disc in the complex plane, let $p \in (0, \infty)$, let g be analytic in $\mathbf{U}$, so that G defined by $G(w) = g(w)/(1 - w^2)$ is in the Hardy space $\mathbf{H}_p(\mathbf{U})$, i.e., such that

$$\|g\| = \lim_{r \to 1^-} \left[\frac{1}{2\pi} \int_0^{2\pi} \left| G(re^{i\theta}) \right|^p d\theta \right]^{1/p} < \infty, \tag{2.52}$$

and let $p' = p/(p-1)$, $h = \pi[p'/(2N)]^{1/2}$, let z_j be defined by (2.49) and $B(w)$ by (2.50), in which we take $M = N$. Then η defined by (2.51), in which we also take $M = N$, satisfies the inequality

$$|\eta(u)| \leqslant C N^{1/(2p')} \exp\left\{ -\pi [N/(2p')]^{1/2} \right\} \|g\|, \tag{2.53}$$

where C is a constant depending only on p'. Notice that in this case the rationals in $\rho(u)$ are just rationals in u. Some typical examples include $g(u) = (1 - u^2)^{2/3}(1 + u^2)^{-1/2}$, $(1 + u)^{-\alpha}(1 - u)^{-\beta} \log(1 - u)$, etc.

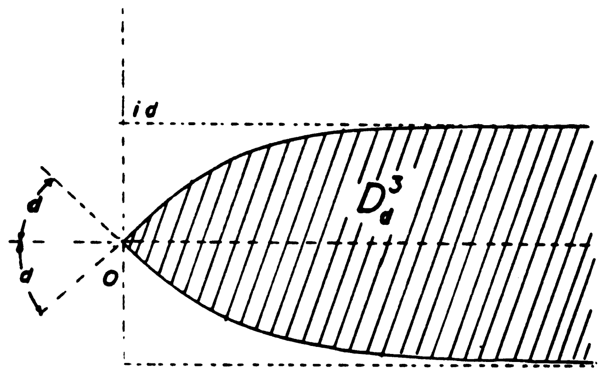

FIGURE 2.3. *Equation* (2.3)

Example 2.4: *Approximation of Decaying Oscillatory Functions on* $[0, \infty]$. Whereas the approximation scheme of Example 2.1 is well suited for functions that are analytic in a sector and have algebraic decay to zero at 0 and at ∞, the scheme of this example is suited for functions that are analytic and bounded only in the strip $\mathbf{D}_d^3$. This situation occurs frequently for the case of Fourier transforms, which may decay to zero at an algebraic rate at 0 and (ideally) at an exponential and possibly also oscillatory rate at ∞. In this case,

$$\begin{aligned}
&\phi(z) = \log[\sinh(z)]([12]), \qquad \rho(z) = \sinh(z), \\
&\psi(w) = \log\left[e^w + \sqrt{(1 + e^{2w})}\right], \\
&z_k = \log\left[e^{(k+\sigma)h} + \sqrt{(1 + e^{2(k+\sigma)h})}\right], \\
&\mathbf{B}_{\alpha,\beta}(\mathbf{D}_d^3) = \left\{ f \in \mathbf{A}(\mathbf{D}_d^3) : |F(z)| \leqslant K|z|^\alpha \quad \text{if } \operatorname{Re}(z) \leqslant 1; \right. \\
&\qquad\qquad\qquad \left. |F(z)| \leqslant K e^{-\beta|z|} \quad \text{if } \operatorname{Re}(z) \geqslant 1 \right\}.
\end{aligned} \tag{2.54}$$

We set

$$(2.55)\qquad B(w)=\frac{\sinh(w)}{1+\sinh(w)}\prod_{j=-M}^{N-2\sigma}\frac{\sinh(w)-e^{(j+\sigma)h}}{\sinh(w)+e^{(j+\sigma)h}}.$$

If $g\in\mathbf{B}_{\alpha,\beta}(\mathbf{D}_d^3)$, and if h is selected by the formula (2.43), then the difference

$$(2.56)\qquad \eta(u)=g(u)-\sum_{j=-M}^{N-2\sigma}\frac{g(z_j)\sqrt{(1+e^{2(j+\sigma)h})}\,B(u)}{\left[\sinh(u)-e^{(j+\sigma)h}\right]B'(z_j)}$$

is bounded on $[0,\infty]$ by the right-hand side of (2.46). In this case, the rationals in the variable $\rho(u)=\sinh(u)$ are just rationals in e^u. A typical example of a function g is $g(u)=u^{1/4}[(u-100)^2+1]^{1/2}\cos(2u)e^{-u}$.

Example 2.5: *Rational Approximation on* $[-\infty,-\infty]$; *The Algebraically Decaying Case*. If $\mathbf{D}=\mathbf{D}_d^4$ (see Eq. (2.4)) then

$$(2.57)\qquad \begin{aligned}
&\phi(z)=\log\left[z+\sqrt{(1+z^2)}\right],\qquad \rho(z)=z+\sqrt{(1+z^2)},\\
&\psi(w)=\sinh(w),\quad z_j=\sinh[(j+\sigma)h],\quad \Gamma=[-\infty,\infty],\\
&\mathbf{B}_{\alpha,\beta}(\mathbf{D}_d^4)=\left\{g \text{ analytic in } \mathbf{D}_d^4\colon |g(w)|\leqslant C_1|1-w|^{-\alpha}\quad \text{if } \mathrm{Re}(w)\leqslant 0;\right.\\
&\qquad\qquad\qquad \left.|g(w)|\leqslant C_1|1+w|^{-\beta}\quad \text{if } \mathrm{Re}(w)\geqslant 0\right\},\\
&B(w)=\frac{\sqrt{(1+w^2)}+w}{1+\sqrt{(1+w^2)}+w}\prod_{j=-M}^{N-2\sigma}\frac{w+\sqrt{(1+w^2)}-e^{(j+\sigma)h}}{w+\sqrt{(1+w^2)}+e^{(j+\sigma)h}},\\
&\eta(u)=g(u)-\sum_{j=-M}^{N-2\sigma}\frac{g(z_j)e^{(j+\sigma)h}\operatorname{sech}\{(j+\sigma)h\}B(u)}{\left[u+\sqrt{(1+u^2)}-e^{(j+\sigma)h}\right]B'(z_j)}.
\end{aligned}$$

If $g\in\mathbf{B}_{\alpha,\beta}(\mathbf{D}_d^4)$, and if h is selected by (2.43) then $\eta(u)$ is bounded on $[-\infty,\infty]$ by the right-hand side of (2.46). Notice that we now have rationals in the variable $u+\sqrt{(1+u^2)}$.

Some important cases from applications include $g(u)=(1+u^2)^{-1/3}\log(1+u^2)$, $\exp\{-u^2\}$, $\exp\{ia\sqrt{(u^2+b^2)}\}/\sqrt{(u^2+b^2)}$ ($\mathrm{Im}(a)>0$, $b>0$), $\operatorname{sech}^\alpha(u)$. We remark also, that by a slightly different choice of h than that described in Eq. (2.43) (e.g., take $M=N$, $h=(1/N)\log(N)$ in Theorem 2.2b) we can achieve an

$$O\left[\exp\{-cN/\log(N)\}\right]\qquad (c>0)$$

rate of convergence to zero of the error of $(2N+1)$-point rational approximation for the case of the last three examples. For example, after replacing $\tanh(u)$ by x in $\operatorname{sech}^\alpha(u)$, we get an even better approximation to $(1-x^2)^\alpha$ on $[-1,1]$ than by polynomial approximation having error $O(N^{-2\alpha})$, or by the rational approximation method of Example 2.2 having an $O(N^{\alpha/2}\exp\{-(\pi/2)N^{1/2}\}$ error.

Example 2.6: *Rational Approximation on* $[-\infty,\infty]$; *Exponentially Decaying Case.* If $\mathbf{D}=\mathbf{D}_d$ (see Eq. (2.5)), then

$$(2.58)\qquad \begin{aligned}
&\phi(z)=z,\qquad \rho(z)=e^z,\\
&\psi(w)=w,\quad \Gamma=[-\infty,\infty],\quad z_k=(k+\sigma)h,\\
&\mathbf{B}_{\alpha,\beta}(\mathbf{D}_d)=\left\{F\in\mathbf{A}(\mathbf{D}_d)\colon |F(z)|<Ke^{-\alpha|z|},\ z\in\mathbf{D}_d\right\}.
\end{aligned}$$

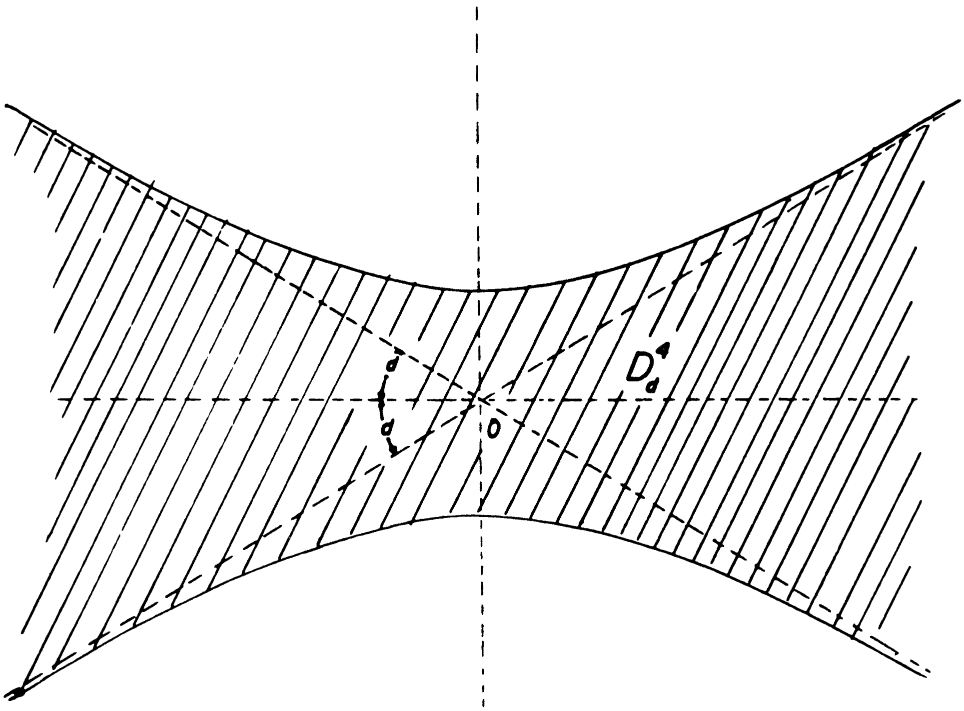

FIGURE 2.4. *Equation* (2.4)

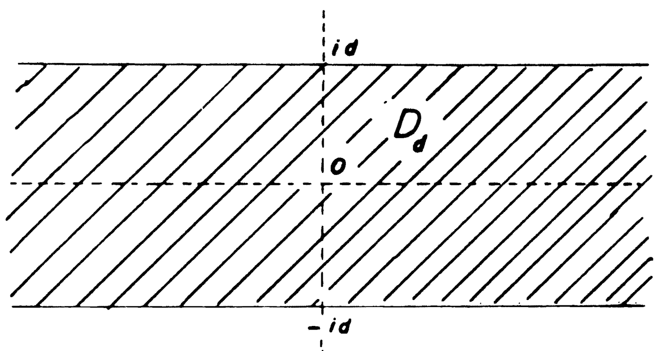

FIGURE 2.5. *Equation* (2.5)

The functions $B(w)$ and $\eta(u)$ take the form

$$(2.59)\qquad \begin{aligned} B(w) &= \frac{e^w}{1+e^w} \prod_{j=-M}^{N-2\sigma} \frac{e^w - e^{(j+\sigma)h}}{e^w + e^{(j+\sigma)h}}, \\ \eta(u) &= g(u) - \sum_{j=-M}^{N-2\sigma} \frac{g(z_j) e^{(j+\sigma)h} B(u)}{\left(e^u - e^{(j+\sigma)h}\right) B'(z_j)}. \end{aligned}$$

If $g \in \mathbf{B}_{\alpha,\beta}(\mathbf{D}_d)$, and if h is selected by the formula (2.43), then η is bounded on $[-\infty, \infty]$ by the right-hand side of (2.46). Notice that the rationals are now rationals in e^u. Examples include $g(u) = \exp\{-u^2\}$, $\operatorname{sech}\{u\} \log(1 + 2u^2)(2u + 1)/(1 + u^2)$, etc.

3. Implications and Applications. In this section, we study the connection of the results of the previous section with the Thiele algorithm, the epsilon algorithm, and the Padé method.

3.1. *The Thiele Algorithm.* The Thiele, or ρ-algorithm for interpolating f at $m + 1$ distinct points $x_0, x_1, \ldots, x_m$ is described as follows. Define ρ_i^j by

$$\begin{aligned} &\rho_0^j = f(x_j), \qquad j = 0, 1, \ldots, m, \\ &\rho_1^j = \frac{x_{j+1} - x_j}{\rho_0^{j+1} - \rho_0^j}, \qquad j = 0, 1, \ldots, m-1, \\ &\rho_i^j = \frac{x_{i+j} - x_j}{\rho_{i-1}^{j+1} - \rho_{i-1}^j} + \rho_{i-2}^{j+1}, \qquad j = 0, 1, \ldots, m-i, \quad i = 2, 3, \ldots, m. \end{aligned} \tag{3.1}$$

Then the rational function $r(x)$ which interpolates the data $\{x_j, f(x_j)\}_{j=0}^m$ is given by the continued fraction representation

$$r(x) = \rho_0^0 + \frac{x - x_0|}{|\ \rho_1^0} + \frac{x - x_1|}{|\rho_2^0 - \rho_0^0} + \cdots + \frac{x - x_{m-1}|}{|\rho_m^0 - \rho_{m-2}^0}. \tag{3.2}$$

The function $r(x)$ has the form

$$r(x) = p_n(x)/q_n(x) \tag{3.3}$$

if $m = 2n$, where p_n and q_n are polynomials of degree n in x, and it has the form

$$r(x) = p_{n+1}(x)/q_n^*(x) \tag{3.4}$$

if $m = 2n + 1$. Furthermore, if $m = 2n$, then

$$r(x) = \frac{\rho_{2n}^0 x^n + c_1 x^{n-1} + \cdots + c_n}{x^n + d_1 x^{n-1} + \cdots + d_n}. \tag{3.5}$$

That is,

$$\rho_{2n}^0 = \lim_{x \to \infty} r(x), \tag{3.6}$$

so that the algorithm provides an excellent method carrying out analytic continuation.

For example, if f is analytic and bounded in the region $\mathbf{D}_d^1$ of Eq. (2.1) and if f is of class Lip_α $(\alpha > 0)$ on $[x_0, \infty]$, where $x_0 \geqslant 0$, then we may effectively use the Thiele algorithm to approximate $f(\infty)$ via the use of a few values of $f(x)$, for finite x. Indeed, this has been done recently in an ultrasonic tomography algorithm [26].

3.2. *Evaluation of the Rationals of the Previous Section via the Thiele Algorithm.* Let $\mathbf{P}_n$ denote the family of all polynomials of degree $\leqslant n$, and consider the evaluation of the rational function [29]

$$r(x) = p_n(x)/q_{n+\sigma}(x) \tag{3.7}$$

for $p_n \in \mathbf{P}_n$, $q_{n+\sigma} \in \mathbf{P}_{n+\sigma}$, where $\sigma = 0$ or 1, and such that

$$\begin{aligned} &r(x_{2k}) = f(x_{2k}), \qquad k = 0, 1, \ldots, n, \\ &r(x_{2k-1}) = \infty, \qquad k = 1, 2, \ldots, n + \sigma. \end{aligned} \tag{3.8}$$

Then

$$\rho(x) = 1/r(x) \tag{3.9}$$

can be evaluated via the Thiele algorithm, using the $2n + 1 + \sigma$ values $\rho(x_k) = 1/r(x_k)$, $k = 0, 1, \ldots, 2n + 1 + \sigma$. Equation (3.9) then yields $r(x) = 1/\rho(x)$. In general, there seems to be no guarantee that the ρ-algorithm will always work (see, however, the excellent altered algorithm of Graves-Morris [9]); however, interlacing the zero and nonzero values of ρ in the above fashion has worked, in our experience. Since all of the poles of $r(x)$ have been predetermined, there are no unwanted poles.

For example, let us consider the evaluation of the rational function in (2.51). Since $z_0 = 0$, $B(x) = p_{2N+3}(x)/q_{2N}(x)$, where $p_{2N+3} \in \mathbf{P}_{2N+3}$, $q_{2N} \in \mathbf{P}_{2N}$. Hence $r(x) = p_{2N+2}(x)/q_{2N}(x)$, and it has $(1 - x^2)$ as a factor. Hence

$$\rho(x) = \frac{1 - x^2}{r(x)} = \frac{p_{2N}(x)}{\pi_{2N}(x)}, \qquad p_{2N}, \pi_{2N} \in \mathbf{P}_{2N}, \tag{3.10}$$

is completely determined by the $4N + 1$ values

$$\begin{aligned} (x_{2k}, \rho(x_{2k})) &= \left(z_{-N+k}, \frac{1 - z_{N+k}^2}{f(z_{-N+k})} \right), && k = 0, 1, \ldots, 2N, \\ (x_{2k-1}, \rho(x_{2k-1})) &= \begin{cases} (1/z_{-N+k-1}, 0), & k = 1, 2, \ldots, N, \\ (1/z_{-N+k}, 0), & k = N+1, \ldots, 2N, \end{cases} \end{aligned} \tag{3.11}$$

and may thus be evaluated via the Thiele algorithm, as above. The rational function $r(x)$ may then be computed via (3.10), i.e.,

$$r(x) = (1 - x^2)/\rho(x).$$

3.3. *The ε-Algorithm and the Padé Approximation*. The ε-algorithm [21], [32] is described as follows. Given a sequence of $m + 1$ numbers S_j, $j = 0, 1, \ldots, m$, define ε_i^j by

$$\begin{aligned} \varepsilon_0^j &= S_j, && j = 0, 1, \ldots, m, \\ \varepsilon_1^j &= \frac{1}{\varepsilon_0^{j+1} - \varepsilon_0^j}, && j = 0, 1, \ldots, m - 1, \\ \varepsilon_i^j &= \frac{1}{\varepsilon_{i-1}^{j+1} - \varepsilon_{i-1}^j} + \varepsilon_{i-2}^{j+1}, && j = 0, 1, \ldots, m - i, \\ & && i = 2, 3, \ldots, m. \end{aligned} \tag{3.12}$$

The numbers ε_i^j may be used to either predict the limiting value of a function, or to evaluate Padé approximations [31].

For example, if

$$S(x) = L + \sum_{k=0}^{\mu} e^{\alpha_k x} \sum_{l=0}^{\mu_k - 1} d_{k,l} x^l, \tag{3.13}$$

if $S_j = S(jh)$, $h \neq 0$, and if

$$M = \sum_{k=0}^{\mu} \mu_k, \tag{3.14}$$

then

$$\varepsilon_M^0 = L. \tag{3.15}$$

On the other hand, if S_j is defined by

$$S_j = \sum_{i=0}^{j} c_i \tau^i, \tag{3.16}$$

then ε_{2k}^n yields [17] the Padé approximation

$$\varepsilon_{2k}^n = p_{n+k-1}(\tau)/q_k(\tau). \tag{3.17}$$

The results of Section 2 of this paper together with the representations (3.13) or (3.16) tell us when we may expect the approximations (3.15) or (3.17) to be accurate, when applied to a function f.

For example, $f(x) - L$ (e.g., $L = f(\infty)$) may be assumed to satisfy the conditions of Theorem 2.1a, provided, e.g., that

$$\left|c_k x^k\right| = O(e^{-\alpha k}), \qquad \alpha > 0. \tag{3.18}$$

Hence in the case when (3.18) is satisfied, we may expect (3.17) to converge rapidly to $f(\infty) = \sum_{k=0}^{\infty} c_k x^k$ as $n \to \infty$.

4. A Lower Bound on the Error of Best Rational Approximation. Let $\mathbf{U}$ denote the unit disc in the complex plane, i.e., $\mathbf{U} = \mathbf{D}_{\pi/2}^2$ in the notation of Eq. (2.2). Let g satisfy the conditions of Theorem 2.1a. Then G defined as in (2.52) is in $\mathbf{H}_p(\mathbf{U})$, so that (2.52) is satisfied. Let $\mathbf{S}$ denote the family of all functions G such that the integral on the right-hand side of (2.52) is at most 1. Then by Andersson [2] we have

$$C_1 \exp\left\{-\pi\left(\frac{2N}{p'}\right)^{1/2}\right\} \leqslant \inf_{p_{2N}, q_{2N+1}} \sup_{G \in \mathbf{S}} \left|\int_{-1}^{1} \left[G(u) - \frac{p_{2N}(u)}{q_{2N+1}(u)}\right] du\right|, \tag{4.1}$$

where $p' = p/(p-1)$, where C_1 is a positive constant depending only on p, and where p_{2N} and q_{2N+1} denote polynomials of degree $2N$ and $2N+1$, respectively.

Now if $\eta(u)$ is the quantity in (2.51), then the expression in square brackets on the right-hand side of (4.1) is just $\eta(u)/(1-u^2)$. Hence

$$C_1 \exp\left\{-\pi\left(\frac{2N}{p'}\right)^{1/2}\right\} \leqslant \left|\int_{-1}^{1} \frac{\eta(u)\,du}{(1-u^2)}\right|. \tag{4.2}$$

We now split the integral on the right-hand side into an integral over $(-\delta, \delta)$, $0 < \delta < 1$, plus an integral over $[-1,1] - [-\delta, \delta]$. Then

$$\left|\int_{-\delta}^{\delta} \frac{\eta(u)\,du}{(1-u^2)}\right| \leqslant \max_{-1 \leqslant u \leqslant 1} |\eta(u)| \int_{-\delta}^{\delta} \frac{dt}{(1-t^2)} = \|\eta\|_\infty 2 \log\left[\frac{1+\delta}{1-\delta}\right], \tag{4.3}$$

while from (2.24), after transformation from $(0, \infty)$ to $(-1, 1)$, to get $|\eta(u)| \leqslant C_3(1-u^2)^{1/p}$, we have

$$\left|\int_{[-1,1]-(-\delta,\delta)} \frac{\eta(u)\,du}{(1-u^2)}\right| \leqslant C_2 \int_{\delta}^{1} (1-u^2)^{-1/p'}\,du \leqslant C_2 p'(1-\delta)^{1/p'}, \tag{4.4}$$

where C_2 is a constant depending only on p'.

Hence, by (4.2), (4.3), and (4.4), we get

$$\|\eta\|_\infty \geq \frac{1}{2\log\{(1+\delta)/(1-\delta)\}}\left[C_1 \exp\left\{-\pi\left(\frac{2N}{p'}\right)^{1/2}\right\} - C_2 p'(1-\delta)^{1/p'}\right]. \tag{4.5}$$

Taking $1 - \delta = e^{-N}$, and combining with the results of Example 2.3, we find that there exists an integer N_0, such that if $N > N_0$, then there are constants C_3 and C_4 such that

$$\begin{aligned} C_3 N^{-1}\exp\left\{-\pi\left(\frac{2N}{p'}\right)^{1/2}\right\} &\leq \sup_{G\in \mathbf{S}} \inf_{p_{2N+2}, q_{2N+1}} \sup_{-1<u<1} \left| g(u) - \frac{p_{2N+2}(u)}{q_{2N+1}(u)}\right| \\ &\leq C_4 N^{1/(2p')}\exp\left\{-\pi\left[\frac{N}{2p'}\right]^{1/2}\right\}. \end{aligned} \tag{4.6}$$

These inequalities show that while the exact lower bound on the left-hand side of (4.6) is not known, the results of this paper are in the right "ballpark" with respect to their accuracy. We mention also, that the approximations of this paper are linear, and as was shown in a recent very interesting paper [5], the bounds obtained in this paper are in fact optimal, in the sense that the constants multiplying $N^{1/2}$ in the exponents of the bounds cannot be replaced by a smaller number.

In view of the above, we conclude with a problem: Given g analytic in $\mathbf{U}$, $G \in \mathbf{H}_p(\mathbf{U})$, where $G(w) = g(w)/(1 - w^2)$, and given N, can a rational approximation p_N/q_N which is linear in g be as accurate as the best rational approximation to g of the form p_N/q_N? Here p_N and q_N are polynomials of degree at most N.

Appendix A: Blaschke Product Estimates. We consider rational interpolation at the points e^{-jh}, $j = 0, \pm 1, \ldots, \pm N$. It is then natural to start with the product

$$\Phi_N^*(z) = \prod_{j=-N}^{N} \frac{(z - q^{2j})}{(z + q^{2j})}, \qquad q = e^{-h/2}. \tag{A.1}$$

Unfortunately, this product does not have a limit as $N \to \infty$, since the product changes sign with N as N increases. However, the alternate form

$$\Phi_N(z) = \frac{z-1}{z+1}\prod_{j=1}^{N} \frac{1 - q^{2j}(z + 1/z) + q^{4j}}{1 + q^{2j}(z+1/z) + q^{4j}} \tag{A.2}$$

has the same zeros and poles as Φ_N^*, and moreover, Φ_N converges as $N \to \infty$, to

$$\Phi(z, q^2) = \frac{z-1}{z+1}\prod_{j=1}^{\infty} \frac{1 - q^{2j}(z + 1/z) + q^{4j}}{1 + q^{2j}(z+1/z) + q^{4j}}. \tag{A.3}$$

For purposes of interpolating at the points $q^{2j-1} = e^{-(2j-1)h/2}$ we shall also require the function

$$\Psi(z, q^2) = \prod_{j=1}^{\infty} \frac{1 - q^{2j-1}(z + 1/z) + q^{4j-2}}{1 + q^{2j-1}(z+1/z) + q^{4j-2}}. \tag{A.4}$$

Let us now relate the functions Φ and Ψ to the Jacobi theta functions, using the definitions given in [14, Eqs. (16.37.1) to (16.37.4)]. To this end, we let $0 < k < 1$, and set

$$\text{(A.5)}\qquad \begin{aligned} &u = u(w) = \int_0^w [(1-t^2)(1-k^2t^2)]^{-1/2}\,dt \Leftrightarrow w = \operatorname{sn}[u;k],\\ &K = K(k) = u(1) \Leftrightarrow \operatorname{sn}[K;k] = 1, \qquad v = \pi u/(2K),\\ &\operatorname{cn}[u;k] = (1-\operatorname{sn}^2[u;k])^{1/2}, \qquad -K \leqslant u \leqslant K,\\ &k_1 = (1-k^2)^{1/2}, \qquad K' = K(k_1),\\ &q = e^{-h/2} = e^{-\pi K'/K}, \qquad q_1 = e^{-\pi K/K'}. \end{aligned}$$

Then, we have

$$\text{(A.6)}\qquad \begin{aligned} &\sin(v)\prod_{j=1}^{\infty}\left[1 - 2q^{2j}\cos(2v) + q^{4j}\right] = \left[\frac{k^2k_1^2}{16q}\right]^{1/6}\theta_s(u),\\ &\cos(v)\prod_{j=1}^{\infty}\left[1 + 2q^{2j}\cos(2v) + q^{4j}\right] = \left[\frac{k^2}{16q_1}\right]^{1/6}\theta_c(u),\\ &\prod_{j=1}^{\infty}\left[1 + 2q^{2j-1}\cos(2v) + q^{4j-2}\right] = \left[\frac{16q}{k^2k_1^2}\right]^{1/12}\theta_d(u),\\ &\prod_{j=1}^{\infty}\left[1 - 2q^{2j-1}\cos(2v) + q^{4j-2}\right] = \left[\frac{16qk_1^4}{k^2}\right]^{1/12}\theta_n(u), \end{aligned}$$

where [14, Eq. (16.36.3)]

$$\text{(A.7)}\qquad \operatorname{pq}[u;k] = \theta_p(u)/\theta_q(u).$$

Hence, if we set

$$\text{(A.8)}\qquad u = -(iK/\pi)\log(z), \qquad v = -(i/2)\log(z),$$

then (A.2) and the first two equations of (A.6) (resp. (A.3) and the second two equations of (A.6)) yield

$$\text{(A.9)}\qquad \Phi(z,q^2) = -ik_1^{1/2}\operatorname{sc}[-i(K/\pi)\log(z);k],$$

$$\text{(A.10)}\qquad \Psi(z,q^2) = k_1^{1/2}\operatorname{nd}[-i(K/\pi)\log(z);k],$$

and finally, via the imaginary transformation of [14, Eq. (16.20)], we get

$$\text{(A.11)}\qquad \Phi(z,q^2) = k_1^{1/2}\operatorname{sn}[(K/\pi)\log(z);k_1],$$

$$\text{(A.12)}\qquad \Psi(z,q^2) = k_1^{1/2}\operatorname{cd}[(K/\pi)\log(z);k_1].$$

Both functions $\operatorname{sn}[u;k]$ and $\operatorname{cd}[u;k]$ map $\mathbf{R}$ onto $[-1,1]$, and therefore

$$\text{(A.13)}\qquad \sup_{0<z<\infty}\left|\Phi(z,q^2)\right| = \sup_{0<z<\infty}\left|\Psi(z,q^2)\right| = k_1^{1/2}.$$

LEMMA A.1. *Let $\Phi(z,q^2)$ and $\Psi(z,q^2)$ be defined by* (A.3) *and* (A.4), *respectively. Then, for all $z > 0$,*

$$|\Phi(z,q^2)| \leqslant k_1^{1/2} \leqslant 2\exp[-\pi^2/(2h)], \tag{A.14}$$

$$|\Psi(z,q^2)| \leqslant k_1^{1/2} \leqslant 2\exp[-\pi^2/(2h)]. \tag{A.15}$$

Proof. The first inequalities (A.14) and (A.15) follow from (A.13). Next, from (A.6) and (A.7) we have

$$\operatorname{cd}[0;k] = 1 = \frac{2q^{1/4}}{k^{1/2}} \prod_{j=1}^{\infty} \left[\frac{1+q^{2j}}{1+q^{2j-1}}\right]^2. \tag{A.16}$$

But since $0 < q < 1$, we have $[1 + q^{2j}]/[1 + q^{2j-1}] \leqslant 1$, and hence (A.16) implies that

$$1 \leqslant 2q^{1/4}/k^{1/2}. \tag{A.17}$$

Replacing k by k_1 in (A.17), and using the identity $\log(q)\log(q_1) = \pi^2$, we get the right-hand sides of (A.14) and (A.15).

LEMMA A.2. *Let $0 < d < \pi$, let $z = te^{i\theta}$, where $t > 0$, and where $|\theta| = d$. Then, with σ either 0 or $1/2$, $\tau = te^{-\sigma h}$,*

$$\sum_{j=-\infty}^{\infty} \log\left|\frac{e^{(j-\sigma)h} + z}{e^{(j-\sigma)h} - z}\right| = \frac{\pi}{h}\left[\frac{\pi}{2} - d\right] + \delta \tag{A.18}$$

where

$$\delta = \sum_{j=1}^{\infty} \frac{\sinh\{(\pi^2 j/h)(1-2d/\pi)\}}{\cosh\{\pi^2 j/h\}} \cos\{2\pi j\log(\tau)\}. \tag{A.19}$$

In particular,

$$|\delta| \leqslant \varepsilon \equiv \left|\frac{e^{-2\pi d/h}}{1-e^{-2\pi d/h}} - \frac{e^{-2\pi(\pi-d)/h}}{1-e^{-2\pi(\pi-d)/h}}\right|. \tag{A.20}$$

Proof. Replacing z by $ze^{-\sigma h}$ in (A.18) shows that we may consider the case of $\sigma = 0$ without loss of generality. Let us define r_n, a contour C and arcs C_j, $j = 1, \ldots, 4$, by

$$C = \bigcup_{j=1}^{4} C_j, \qquad r_n = e^{(n+1/2)h}, \qquad n = 1,2,\ldots, \tag{A.21}$$

where

$$\begin{aligned} C_1 &= \{w \in \mathbf{C}: w = r_n^{-1}e^{iv},\ d-\pi \leqslant v \leqslant d\},\\ C_2 &= \{w \in \mathbf{C}: w = ue^{i(d-\pi)},\ r_n^{-1} \leqslant u \leqslant r_n\},\\ C_3 &= \{w \in \mathbf{C}: w = r_n e^{iv},\ d-\pi \leqslant v \leqslant d\},\\ C_4 &= \{w \in \mathbf{C}: w = ue^{id},\ r_n^{-1} \leqslant u \leqslant r_n\}. \end{aligned} \tag{A.22}$$

We also set

$$
\text{(A.23)}\qquad
\begin{aligned}
&F(z,w) = \log\left[\frac{w+z}{w-z}\right],\\
&T_n^j = \frac{1}{2\pi i}\int_{C_j} \frac{F(z,w)(1/w)\,dw}{\tan\{(\pi/h)\log(w)\}},\\
&R_n^j = \operatorname{Re}\{T_n^j\},\\
&T_n = \sum_{j=1}^{4} T_n^j, \qquad R_n = \operatorname{Re}\{T_n\},\\
&R^j = \lim_{n\to\infty} R_n^j, \qquad R = \lim_{n\to\infty} R_n = \sum_{j=1}^{4} R^j.
\end{aligned}
$$

We remark that while T_n does not converge as $n \to \infty$, R_n does.

The denominator, $\tan\{(\pi/h)\log(w)\}$ can be handled just as in [25]. We thus find that

$$
\text{(A.24)}\left\{
\begin{aligned}
T_n^1 &= \frac{1}{2\pi i}\int_d^{d-\pi}\left\{i\pi + \log\left[\frac{z+w}{z-w}\right]\bigg|_{|w|=r_n^{-1}}\right\}(-i)\tanh\left\{\frac{\pi v}{h}\right\}dv,\\
T^1 &= R^1 = -\frac{1}{2}\int_{d-\pi}^{d}\tanh\left\{\frac{\pi v}{h}\right\}dv\\
&= \frac{\pi}{2} - d - \frac{h}{2\pi}\log\left[\frac{1+e^{-2\pi d/h}}{1+e^{-2\pi(\pi-d)/h}}\right],\\
T_n^2 &= \frac{1}{2\pi i}\int_{r_n^{-1}}^{t}(-\pi)\left[1+\frac{2}{e^{2\pi[\pi-d+i\log(t)]/h}-1}\right]\frac{du}{u}\\
&\quad+\frac{1}{2\pi i}\int_{r_n^{-1}}^{r_n}(i)\log\left|\frac{u-t}{u+t}\right|\left[1+\frac{2}{e^{2\pi[\pi-d+i\log(u)]/h}-1}\right]\frac{du}{u},
\end{aligned}\right.
$$

$$
\text{(A.25)}\left\{
\begin{aligned}
R^2 &= \lim_{n\to\infty}\left[-\operatorname{Im}\int_{r_n^{-1}}^{t}\frac{(1/u)\,du}{e^{2\pi[\pi-d+i\log(u)]/h}-1}\right]-\frac{1}{2\pi}\int_0^\infty\log\left|\frac{u+t}{u-t}\right|\frac{du}{u}\\
&\quad-\operatorname{Re}\left[\frac{1}{\pi}\int_0^\infty\log\left|\frac{u+t}{u-t}\right|\frac{(1/u)\,du}{e^{2\pi[\pi-d+i\log(u)]/h}-1}\right],
\end{aligned}\right.
$$

$$
\text{(A.26)}\left\{
\begin{aligned}
T_n^3 &= \frac{1}{2\pi i}\int_{d-\pi}^{d}\log\left[\frac{1+z/w}{1-z/w}\right]\bigg|_{|w|=r_n}(-i)\tan\left\{\frac{\pi v}{h}\right\}(i\,dv)\\
&\to 0 \quad \text{as } n\to\infty,\\
R^3 &= 0,\\
T_n^4 &= \frac{1}{2\pi i}\int_t^{r_n^{-1}}(\pi i)(-i)\left[1+\frac{2}{e^{2\pi[d-i\log(u)]/h}-1}\right]\frac{du}{u}\\
&\quad+\frac{1}{2\pi i}\int_{r_n}^{r_n^{-1}}\log\left|\frac{u+t}{u-t}\right|(-i)\left[1+\frac{2}{e^{2\pi[d-i\log(u)]/h}-1}\right]\frac{du}{u},
\end{aligned}\right.
$$

$$
\text{(A.27)}\left\{
\begin{aligned}
R^4 &= \lim_{n\to\infty}\left[\frac{-\operatorname{Im}}{2\pi}\int_{r_n^{-1}}^{t}\frac{(1/u)\,du}{e^{2\pi[d-i\log(u)]/h}-1}\right]+\frac{1}{2\pi}\int_0^\infty\log\left|\frac{u+t}{u-t}\right|\frac{du}{u}\\
&\quad+\operatorname{Re}\left[\frac{1}{\pi}\int_0^\infty\log\left|\frac{u+t}{u-t}\right|\frac{(1/u)\,du}{e^{2\pi[d-i\log(u)]/h}-1}\right].
\end{aligned}\right.
$$

Next, taking residues of the contour integral T_n at the zeros of $\tan\{(\pi/h)\log(w)\}$, then taking the real part, and letting $n \to \infty$, we get

$$R = \frac{h}{\pi} \sum_{j=-\infty}^{\infty} \log\left|\frac{e^{jh} + z}{e^{jh} - z}\right|. \tag{A.28}$$

Similarly, summing R^1 to R^4 in (A.24) to (A.27), we get

$$\begin{aligned} R = \frac{\pi}{2} - d - \frac{h}{2\pi}\operatorname{Re}\log\left[\frac{1 - e^{-2\pi[\pi - d + i\log(t)]/h}}{1 - e^{-2\pi[d - i\log(t)]/h}}\right] \\ + \operatorname{Re}\frac{1}{\pi}\int_0^\infty \log\left|\frac{u+t}{u-t}\right|\left[\frac{1}{e^{2\pi[d - i\log(u)]/h} - 1}\right. \\ \left. - \frac{1}{e^{2\pi[\pi - d + i\log(u)]/h} - 1}\right]\frac{du}{u}. \end{aligned} \tag{A.29}$$

The last integral in (A.29) may be evaluated via term-by-term integration of the expansion of the terms in brackets in powers of $\exp\{\pm(2\pi i/h)\log(u)\}$, using the formula

$$\int_0^\infty \log\left|\frac{u+t}{u-t}\right| e^{ic\log(u)}\frac{du}{u} = \frac{\pi^2}{2}\frac{\tanh\{\pi c/2\}}{(\pi c/2)}, \tag{A.30}$$

which is valid for c real, and which can in turn be obtained by expansion of $\log|(u+t)/(u-t)|$ in powers of u/t for $u < t$, and in powers of t/u for $u > t$, and then carrying out termwise integration. Finally, the first log term on the right-hand side of (A.29) can also be expanded using the expansion of $\log\{1 - w\}$ in powers of w for $|w| < 1$. Combining these expansions, we get (A.18).

The inequality (A.20) follows if we note that each coefficient of $\cos\{2\pi j\log(\tau)\}$ in (A.19) has the same sign. Taking absolute values, replacing the cosine term by 1, and then replacing $\cosh\{\pi^2 j/h\}$ by $\frac{1}{2}\exp\{\pi^2 j/h\}$, we can sum the result explicitly, to get (A.20).

LEMMA A.3. *Let $z = te^{i\theta}$, where $t = |z|$, $|\theta| = d$, and let ε be defined as in* (A.20). *Let M and N be positive integers, and set*

$$P = \prod_{j=-M}^{N-2\sigma}\left|\frac{z + e^{(j+\sigma)h}}{z - e^{(j+\sigma)h}}\right|. \tag{A.31}$$

(i) *If $0 < d < \pi/2$, then*

$$P \leqslant \exp\{\pi(\pi/2 - d)/h + \varepsilon\}. \tag{A.32}$$

(ii) *If $\pi/2 \leqslant d < \pi$, if*

$$M^{1/2}e^{-Mh} \leqslant t \leqslant N^{1/2}e^{Nh}, \tag{A.33}$$

and if $Q = \exp\{\varepsilon + [\pi^2/(4h)][M^{-1/2} + N^{-1/2}]\}$, then

$$Q^{-1}\exp\{\pi(\pi/2 - d)/h\} \leqslant P \leqslant Q\exp\{\pi(\pi/2 - d)/h\}. \tag{A.34}$$

(iii) *If $d = \pi$, and if $M^{1/2}e^{-Mh} \leqslant t \leqslant N^{-1/2}e^{Nh}$, then*

$$P \leqslant 2\exp\{-[\pi^2/(2h)][1 - (M^{-1/2} + N^{-1/2})/2]\}. \tag{A.35}$$

Proof. Let us consider the case of $\sigma = 0$; the case of $\sigma = 1/2$ is similar. In this case

$$P = |\Phi(z, e^{-h})|W(-\infty, -N-1)W(M+1, \infty), \tag{A.36}$$

where

$$W(m,n) = \prod_{j=m}^{n} |z - e^{jh}|/|z + e^{jh}|. \tag{A.37}$$

(i) If $d \leqslant \pi/2$, then for $a > 0$, we have $|z - a|/|z + a| \leqslant 1$; hence Lemma A.3 follows from (A.18) and (A.20).

(ii) If $\pi/2 \leqslant d \leqslant \pi$, then

$$\begin{aligned}
\log\{W(M+1,\infty)\} &= \operatorname{Re} \sum_{m=0}^{\infty} \frac{2}{2m+1} \sum_{j=M+1}^{\infty} \left(e^{-jh}/z\right)^{2m+1} \\
&\leqslant \sum_{m=0}^{\infty} \frac{2}{2m+1} \sum_{j=M+1}^{\infty} \left(e^{-jh}/t\right)^{2m+1} \\
&= \sum_{m=0}^{\infty} \frac{2}{2m+1} \left(e^{-(M+1/2)h}/t\right)^{2m+1} e^{-(m+1/2)h}/[1 - e^{-(2m+1)h}] \\
&\leqslant \sum_{m=0}^{\infty} \frac{2}{(2m+1)^2} \left[e^{-(M+1/2)h}/t\right]^{2m+1}/h \\
&\leqslant \sum_{m=0}^{\infty} \frac{2}{(2m+1)^2} \left(M^{-1/2}\right)^{2m+1}/h \\
&\leqslant (\pi^2/4) M^{-1/2}/h,
\end{aligned} \tag{A.38}$$

where, in the first inequality, we replaced $\operatorname{Re} z^{-2m-1}$ by t^{-2m-1}; in the second, we replaced $\exp\{-(m+1/2)h\}/[1 - \exp\{-(2m+1)h\}]$ by $1/[(2m+1)h]$; in the third we replaced t by $M^{1/2}e^{-(M+1/2)h}$; and finally, we used the fact that $M \geqslant 1$ and the identity $\sum_{m=0}^{\infty} 2/(2m+1)^2 = \pi^2/4$.

Similarly, we also have

$$\log\{W(N+1,\infty)\} \leqslant (\pi^2/4) N^{-1/2}/h. \tag{A.39}$$

(iii) In this case we use (A.14) and then proceed as in the proof of (ii) above.

Appendix B: Estimates of Integrals. We shall estimate the integral

$$G(p',d,x) = \left[\frac{1}{2\pi} \int_{\partial \mathbf{D}_d^1} \left|\frac{z+1}{z-x}\right|^{p'} \frac{|dz|}{|z+1|^2}\right]^{1/p'} \tag{B.1}$$

for $1 < p' < \infty$, $0 < x < \infty$, and $0 < d \leqslant \pi/2$, and where $\mathbf{D}_d^1$ is defined in Eq. (2.1) and where $\partial \mathbf{D}_d^1$ denotes the boundary of $\mathbf{D}_d^1$. We shall also estimate the integral

$$H(\alpha,\beta,d,x) = \frac{1}{2\pi} \int_{\partial \mathbf{D}_d^1} |z|^{\alpha-1} |1+z|^{1-\alpha-\beta} |z-x|^{-1} |dz| \tag{B.2}$$

for $0 < \alpha < 1$, $0 < \beta < 1$, $0 < x < \infty$, and $0 < d \leqslant \pi/2$.

Let $F(a,b;c;z)$ denote the usual hypergeometric function, which is defined for $|z| < 1$ by the series

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \tag{B.3}$$

and for $\operatorname{Re} c > \operatorname{Re} b > 0$ by the integral

$$\begin{aligned}
&F(a,b;c;z) \\
&\quad = \frac{2\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^{\infty} w^{2b-1}(1+w^2)^{a-c}\left[1 + (1-z)w^2\right]^{-a} dw.
\end{aligned} \tag{B.4}$$

THEOREM B.1. *Let $G(p', d, x)$ be defined by* (B.1), *and set*

$$A(p', d) = 2^{-1/p'}\cot(d/2)\begin{cases} 2^{-1/2}\sec(d/2)\left[\Gamma((p'-1)/2)\pi^{-1/2}/\Gamma(p'/2)\right]^{1/p'} & \text{if } 1 < p' \leqslant 2, \\ 2^{-1/p'}\sec^{2/p'}(d/2) & \text{if } p' \geqslant 2. \end{cases} \tag{B.5}$$

Then

$$G(p', d, x) \leqslant A(p', d)\begin{cases} x^{1/p'-1} & \text{if } 0 < x \leqslant 1, \\ x^{-1/p'} & \text{if } 1 \leqslant x. \end{cases} \tag{B.6}$$

Proof. (i) If $d = \pi/2$, we set

$$G(x) = \left[G(p', \pi/2, x)\right]^{p'}. \tag{B.7}$$

Then by Eqs. (B.1) and (2.1) we have

$$\begin{aligned} G(x) &= \frac{1}{\pi}\int_0^\infty (t^2+1)^{p'/2-1}(t^2+x^2)^{-p'/2}\,dt \\ &= \frac{x^{-p'}}{\pi}\int_0^\infty (t^2+1)^{p'/2-1}(1+t^2/x^2)^{-p'/2}\,dt, \end{aligned} \tag{B.8}$$

i.e.,

$$G(x) = (x^{-p'}/2)F(p'/2, 1/2; 1; 1 - 1/x^2), \tag{B.9}$$

where we have used (B.4) and the identity $\Gamma(1/2) = \pi^{1/2}$. Next, using [14, Eqs. (15.3.3) and (15.3.5)], we get the expressions

$$G(x) = 1/(2x)F(1 - p'/2, 1/2; 1; 1 - 1/x^2), \tag{B.10}$$

$$G(x) = (x^{1-p'}/2)F(1 - p'/2, 1/2; 1; 1 - x^2), \tag{B.11}$$

where the last equality could also have been obtained upon replacing t by xt in (B.8), yielding

$$G(x) = \frac{x^{1-p'}}{\pi}\int_0^\infty \left[\frac{1+x^2t^2}{1+t^2}\right]^{p'/2-1}\frac{dt}{1+t^2}. \tag{B.12}$$

Now for $0 \leqslant x \leqslant 1$,

$$\left[\frac{1+x^2t^2}{1+t^2}\right]^{p'/2-1} \leqslant \begin{cases} (1+t^2)^{1-p'/2} & \text{if } 1 < p' \leqslant 2, \\ 1 & \text{if } p' \geqslant 2, \end{cases} \tag{B.13}$$

so that, for this range of x, we get

$$\begin{aligned} &F(1 - p'/2, 1/2; 1; 1 - x^2) \\ &\quad \leqslant \begin{cases} \pi^{-1/2}\Gamma((p'-1)/2)/\Gamma(p'/2) & \text{if } 1 < p' \leqslant 2, \\ 1 & \text{if } p' \geqslant 2. \end{cases} \end{aligned} \tag{B.14}$$

Hence, combining this inequality with (B.11) we get (B.6) for $0 < x \leqslant 1$. Equation (B.10) may be bounded similarly, to yield (B.6) for $x \geqslant 1$.

(ii) If $0 < d \leqslant \pi/2$, we have, by (B.1) and the definition (2.1) of $\mathbf{D}_d^1$, that

$$\left[G(p', d, x)\right]^{p'} = \frac{1}{\pi}\int_0^\infty |t + e^{-id}|^{p'-2}|t - xe^{-id}|^{-p'}\,dt. \tag{B.15}$$

Since t and x are both positive, we have

$$\text{(B.16)}\quad \begin{aligned} \left|t - xe^{-id}\right|^2 &= t^2 + x^2 - 2xt\cos(d) \\ &\geqslant t^2 + x^2 - (t^2 + x^2)\cos(d) = (t^2 + x^2)2\sin^2(d/2), \end{aligned}$$

and also, since $0 < d \leqslant \pi/2$,

$$\text{(B.17)}\quad \begin{aligned} t^2 + 1 &\leqslant t^2 + 1 + 2t\cos(d) = \left|t + e^{-id}\right|^2 \\ &\leqslant t^2 + 1 + (t^2 + 1)\cos(d) = (t^2 + 1)2\cos^2(d/2). \end{aligned}$$

Hence, if $1 < p' \leqslant 2$, then by (B.8)

$$\text{(B.18)}\quad \begin{aligned} [G(p', d, x)]^{p'} &\leqslant \frac{1}{\pi\left[2^{1/2}\sin(d/2)\right]^{p'}} \int_0^\infty (t^2 + 1)^{p'/2-1}(t^2 + x^2)^{-p'/2}\,dt \\ &= \frac{G(x)}{\left[2^{1/2}\sin(d/2)\right]^{p'}}, \end{aligned}$$

whereas, if $p' \geqslant 2$, then

$$\text{(B.19)}\quad [G(p', d, x)]^{p'} \leqslant \frac{\left[2^{1/2}\cos(d/2)\right]^{p'-2}}{\left[2^{1/2}\sin(d/2)\right]^{p'}} G(x),$$

where $G(x)$ is defined in (B.7). The inequality (B.6) for arbitrary d in the range $0 < d \leqslant \pi/2$ now follows from (B.18), (B.19), and (B.14).

THEOREM B.2. *Let $H(\alpha, \beta, d, x)$ be defined by* (B.2), *and set*

$$\text{(B.20)}\quad \begin{aligned} A(\alpha, \beta, d) &= \frac{\Gamma(\alpha/2)}{2^{3/2}\pi\sin(d/2)} \\ &\times \begin{cases} \Gamma(\beta/2)/\Gamma((\alpha+\beta)/2) & \text{if } \alpha + \beta \leqslant 1, \\ \Gamma((1-\alpha)/2)\pi^{-1/2}\left[2^{1/2}\cos(d/2)\right]^{1-\alpha-\beta} & \text{if } \alpha + \beta \geqslant 1. \end{cases} \end{aligned}$$

Then

$$\text{(B.21)}\quad H(\alpha, \beta, d, x) \leqslant A(\alpha, \beta, d) \begin{cases} x^{\alpha-1} & \text{if } 0 < x \leqslant 1, \\ x^{-\beta} & \text{if } x \geqslant 1. \end{cases}$$

Proof. It is convenient to set

$$\text{(B.22)}\quad H(x) = H(\alpha, \beta, \pi/2, x).$$

(i) If $d = \pi/2$, then by Eqs. (B.2) and (2.1), we have

$$\text{(B.23)}\quad \begin{aligned} H(x) &= \frac{1}{\pi}\int_0^\infty t^{\alpha-1}(1 + t^2)^{(1-\alpha-\beta)/2}(t^2 + x^2)^{-1/2}\,dt \\ &= \frac{x^{-1}}{\pi}\int_0^\infty t^{\alpha-1}(1 + t^2)^{(1-\alpha-\beta)/2}(1 + t^2/x^2)^{-1/2}\,dt, \end{aligned}$$

i.e., by (B.4),

$$\text{(B.24)}\quad H(x) = cx^{-1}F(1/2, \alpha/2; (\alpha + \beta)/2; 1 - 1/x^2),$$

where

$$\text{(B.25)}\quad c = \frac{\Gamma(\alpha/2)\Gamma(\beta/2)}{2\pi\Gamma((\alpha + \beta)/2)}.$$

Next, applying Eqs. (15.3.5) of [14] to (B.24), we get the two expressions

$$H(x) = cx^{-\beta}F\big(\beta/2, (\alpha+\beta-1)/2; (\alpha+\beta)/2; 1 - 1/x^2\big) \tag{B.26}$$

and

$$H(x) = cx^{\alpha-1}F\big(\alpha/2, (\alpha+\beta-1)/2; (\alpha+\beta)/2; 1 - x^2\big). \tag{B.27}$$

Using (B.27) and (B.4) we may also write (B.27) in the form

$$H(x) = \frac{x^{\alpha-1}}{\pi}\int_0^\infty \left[\frac{1+w^2x^2}{1+w^2}\right]^{(1-\alpha-\beta)/2} w^{\alpha-1}(1+w^2)^{(\alpha+\beta)/2}\,dw. \tag{B.28}$$

Now, if $0 \leqslant x \leqslant 1$, we have

$$\left[\frac{1+w^2x^2}{1+w^2}\right]^{(1-\alpha-\beta)/2} \leqslant \begin{cases} 1 & \text{if } \alpha+\beta \leqslant 1, \\ (1+w^2)^{(\alpha+\beta-1)/2} & \text{if } \alpha+\beta \geqslant 1. \end{cases} \tag{B.29}$$

Hence, for this range of x, we substitute the right-hand side of (B.29) into (B.28), to get

$$H(x) \leqslant \frac{x^{\alpha-1}}{2\pi}\Gamma(\alpha/2)\begin{cases} \Gamma(\beta/2)/\Gamma((\alpha+\beta)/2) & \text{if } \alpha+\beta \leqslant 1, \\ \Gamma((1-\alpha)/2)/\pi^{1/2} & \text{if } \alpha+\beta \geqslant 1. \end{cases} \tag{B.30}$$

This inequality is just (B.21) for the case of $d = \pi/2$, $0 \leqslant x \leqslant 1$. The case of $d = \pi/2$ and $x \geqslant 1$ follows by bounding the hypergeometric function in (B.26), which is just the same as that in (B.27), after interchanging α and β and replacing x by $1/x$.

(ii) In the case of $0 < d \leqslant \pi/2$, we have

$$\begin{aligned} H(\alpha, \beta, d, x) = \frac{1}{\pi}\int_0^\infty & t^{\alpha-1}\left[t^2 + 2t\cos(d) + 1\right]^{(1-\alpha-\beta)/2} \\ & \times\left[t^2 + 2xt\cos(d) + x^2\right]^{-1/2} dt, \end{aligned} \tag{B.31}$$

and we can now use the estimates (B.16) and (B.17) to get (B.20)–(B.21).

This completes the proof.

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