MATHEMATICS OF COMPUTATION
VOLUME 47, NUMBER 17
JULY 1986, PAGES 347-350

An Asymptotic Expansion for the First Derivative of the Generalized Riemann Zeta Function

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Abstract. An asymptotic expansion for the partial derivative $\partial\zeta(z,q)/\partial z$ of the generalized Riemann zeta function $\zeta(z,q)$, for all negative integer values of z, is obtained.

The generalized Riemann zeta function $\zeta(z,q)$ is defined by

$$\zeta(z,q) = \sum_{n=0}^{\infty} (n+q)^{-z}, \qquad \operatorname{Re} z > 1,\ q \neq 0, -1, -2, \ldots. \tag{1}$$

For the particular values $z = -m$, $m = 0, 1, 2, \ldots,$ this function reduces to

$$\zeta(-m,q) = -\frac{B_{m+1}(q)}{m+1}, \tag{2}$$

where $B_k(q)$ are the Bernoulli polynomials. The only derivatives of $\zeta(z,q)$ which can be found in the usual tables are the following [5, pp. 22–25], [3, pp. 24–27], [4, pp. 1072–1074]

$$\begin{aligned} &\frac{\partial}{\partial q}\zeta(z,q) = -z\zeta(z+1,q), \\ &\frac{\partial}{\partial z}\zeta(z,q)\Big|_{z=0} = \log\Gamma(q) - \frac{1}{2}\log(2\pi). \end{aligned} \tag{3}$$

There is a useful integral representation of $\zeta(z,q)$, valid for $\operatorname{Re} q > 0$ and $z \neq 1$,

$$\zeta(z,q) = \frac{1}{2}q^{-z} + \frac{q^{1-z}}{z-1} + 2\int_0^{\infty} (q^2+t^2)^{-z/2} \sin\left[z\tan^{-1}\left(\frac{t}{q}\right)\right]\frac{dt}{e^{2\pi t}-1}. \tag{4}$$

In what follows, we shall denote

$$\zeta'(z,q) \equiv \frac{\partial}{\partial z}\zeta(z,q), \tag{5}$$

and our purpose is to obtain an asymptotic expansion for $\zeta'(z,q)$ valid for all negative integer values of z.

Received May 9, 1984; revised April 2, 1985.
1980 *Mathematics Subject Classification*. Primary 33A15, 41A60, 30E15.

From the integral representation (4), one immediately gets

$$\zeta'(z,q) = -\frac{1}{2}q^{-z}\log q - \frac{q^{1-z}}{z-1}\log q - \frac{q^{1-z}}{(z-1)^2}$$
$$(6)\qquad +2\int_0^\infty (q^2+t^2)^{-z/2}\cos\left[z\tan^{-1}\left(\frac{t}{q}\right)\right]\tan^{-1}\left(\frac{t}{q}\right)\frac{dt}{e^{2\pi t}-1}$$
$$-\int_0^\infty (q^2+t^2)^{-z/2}\sin\left[z\tan^{-1}\left(\frac{t}{q}\right)\right]\log(q^2+t^2)\frac{dt}{e^{2\pi t}-1},$$

and putting $z = -m$, $m = 0, 1, 2, \dots$,

$$(7)\qquad \zeta'(-m,q) = -\frac{1}{2}q^m\log q + \frac{1}{m+1}q^{m+1}\log q - \frac{1}{(m+1)^2}q^{m+1} + I_m(q),$$

where

$$I_m(q) \equiv 2\int_0^\infty (q^2+t^2)^{m/2}\cos\left[m\tan^{-1}\left(\frac{t}{q}\right)\right]\tan^{-1}\left(\frac{t}{q}\right)\frac{dt}{e^{2\pi t}-1}$$
$$(8)\qquad +\int_0^\infty (q^2+t^2)^{m/2}\sin\left[m\tan^{-1}\left(\frac{t}{q}\right)\right]\log(q^2+t^2)\frac{dt}{e^{2\pi t}-1}.$$

In order to obtain an asymptotic expansion for $I_m(q)$, we shall use a procedure similar to the ordinary one derived from Watson's lemma and Laplace's method, i.e., replacement of $\tan^{-1}(t/q)$ and $\log(q^2+t^2)$ in the integrand by their power series expansion near $t = 0$,

$$\tan^{-1}\left(\frac{t}{q}\right) = \sum_{h=0}^\infty \frac{(-1)^h}{2h+1}\left(\frac{t}{q}\right)^{2h+1},$$
$$(9)\qquad \log(q^2+t^2) = 2\log q + \sum_{h=1}^\infty \frac{(-1)^{h-1}}{h}\left(\frac{t}{q}\right)^{2h}.$$

The proof that the procedure yields also an asymptotic expansion in our slightly modified case (we have $(e^{2\pi t}-1)^{-1}$ instead of $e^{-2\pi t}$ in the integrand) is essentially the same used for the derivation of the general methods that we have just mentioned [6, pp. 71–72, 80–84], [1, pp. 261–265]. Moreover, an alternative procedure can be employed which leads to exactly the same results that we are going to obtain, namely integration by parts. Of course, this procedure is much more lengthy.

It is easy to see that Eq. (8) can be written in the form

$$(10)\qquad I_m(q) = 2\,\mathrm{Re}\int_0^\infty \left[\tan^{-1}\left(\frac{t}{q}\right) - \frac{i}{2}\log(q^2+t^2)\right](q+it)^m\frac{dt}{e^{2\pi t}-1}.$$

Substituting the expansions (9) and making an obvious change of variables, we get

$$I_m(q) = 2\sum_{\substack{k=0\\k\text{ even}}}^m \sum_{h=0}^\infty \binom{m}{k}\frac{(-1)^h i^k}{2h+1}\frac{q^{m-k-2h-1}}{(2\pi)^{2h+k+2}}(2h+k+1)!\zeta(2h+k+2)$$
$$(11)\qquad -2\sum_{\substack{k=0\\k\text{ odd}}}^m \binom{m}{k} i^{k+1}\frac{q^{m-k}\log q}{(2\pi)^{k+1}}k!\zeta(k+1)$$
$$-\sum_{\substack{k=0\\k\text{ odd}}}^m \sum_{h=1}^\infty \binom{m}{k}\frac{(-1)^{h-1}i^{k+1}}{h}\frac{q^{m-k-2h}}{(2\pi)^{2h+k+1}}(2h+k)!\zeta(2h+k+1),$$

where the following identity has been employed

$$\int_0^\infty \frac{t^{z-1}\,dt}{e^t - 1} = \Gamma(z)\zeta(z), \qquad \operatorname{Re} z > 1. \tag{12}$$

Using now

$$\zeta(2k) = (-1)^{k+1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}, \qquad k = 1, 2, 3, \ldots, \tag{13}$$

B_{2k} being the Bernoulli numbers $B_{2k} = B_{2k}(0)$, we obtain

$$\begin{aligned} I_m(q) = {} & \sum_{\substack{k=0 \\ k \text{ even}}}^{m} \sum_{h=0}^{\infty} \binom{m}{k} \frac{B_{2h+k+2}}{(2h+1)(2h+k+2)} q^{-(2h+k-m+1)} \\ & + \sum_{\substack{k=1 \\ k \text{ odd}}}^{m} \binom{m}{k} \frac{B_{k+1}}{k+1} q^{-(k-m)} \log q \\ & - \sum_{\substack{k=1 \\ k \text{ odd}}}^{m} \sum_{h=1}^{\infty} \binom{m}{k} \frac{B_{2h+k+1}}{2h(2h+k+1)} q^{-(2h+k-m)}. \end{aligned} \tag{14}$$

Finally, after some calculations, this equation can be brought to the form

$$I_m(q) = \frac{1}{12}(1 + m\log q)q^{m-1} + \sum_{k=1}^{\infty} a_{2k} q^{-(2k-m+1)}, \tag{15}$$

where the coefficients a_{2k} are given by

$$a_{2k} = \begin{cases} \dfrac{B_{2k+2}}{2k+2}\left[\dbinom{m}{2k+1}\log q + \displaystyle\sum_{h=0}^{2k} \dbinom{m}{h} \frac{(-1)^h}{2k-h+1}\right], & 2k \leqslant m-1, \\ \dfrac{B_{2k+2}}{2k+2} \displaystyle\sum_{h=0}^{m} \dbinom{m}{h} \frac{(-1)^h}{2k-h+1}, & 2k \geqslant m. \end{cases} \tag{16}$$

Summing up, we have found the following asymptotic expansion of $\zeta'(z, q)$, valid for any negative integer value of z,

$$\begin{aligned} \zeta'(-m, q) = {} & \frac{1}{m+1} q^{m+1} \log q - \frac{1}{(m+1)^2} q^{m+1} - \frac{1}{2} q^m \log q \\ & + \frac{m}{12} q^{m-1} \log q + \frac{1}{12} q^{m-1} + \sum_{k=1}^{\infty} a_{2k} q^{-(2k-m+1)}, \end{aligned} \tag{17}$$

with the a_{2k} given by (16). As has already been pointed out, we have obtained the same result (Eqs. (17), (16)) by repeated partial integration of Eq. (8), but this alternative procedure is much more involved and will not be described here.

It is worthwhile to write down the particular expansions that one gets from (17) for the values $m = 1$ and $m = 2$. They are

$$\begin{aligned} \zeta'(-1, q) = {} & \frac{1}{2} q^2 \log q - \frac{1}{4} q^2 - \frac{1}{2} q \log q + \frac{1}{12} \log q + \frac{1}{12} \\ & - \sum_{k=1}^{\infty} \frac{B_{2k+2}}{(2k+2)(2k+1)2k} q^{-2k}, \end{aligned} \tag{18}$$

and

$$\zeta'(-2,q) = \frac{1}{3}q^3 \log q - \frac{1}{9}q^3 - \frac{1}{2}q^2 \log q + \frac{1}{6}q \log q + \frac{1}{12}q + 2\sum_{k=1}^{\infty} \frac{B_{2k+2}}{(2k+2)(2k+1)2k(2k-1)} q^{-(2k-1)}. \tag{19}$$

These asymptotic expansions turn out to be very useful in the effective Lagrangian theory of quark confinement [2].

Acknowledgment. This investigation has been partially supported by CAICYT, Project No. 0435. The suggestions of the referee concerning a preliminary version of this paper are gratefully acknowledged.

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