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Difference Schemes for Degenerate Parabolic Equations

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Abstract. Diagonal dominant implicit-difference schemes approximating a porous media type class of multidimensional nonlinear equations are shown to generate semigroups in an approximate L^1-space, and the rate of convergence to the semigroup solution in L^1 is given. The numerical schemes proposed by Berger et al. in [4] are described and a proof of convergence for the fully discrete algorithms is outlined. Numerical experiments are discussed.

1. Introduction. In this paper we consider difference schemes for the problem

$$\text{(P)}\qquad \begin{aligned} &u_t - \Delta f(u) + g(u) - au = 0, && t > 0,\ x \in \Omega,\\ &u(t,x) = 0, && x \in \partial\Omega,\\ &u(0,x) = u_0(x), && x \in \Omega. \end{aligned}$$

Here, f is continuous monotone increasing, g is monotone nondecreasing, $a \geqslant 0$, and $f(0) = g(0) = 0$. We also assume that f^{-1} and g are Hölder continuous with exponents α and β, respectively. (P) is a parabolic problem that degenerates when $u = 0$. It is found, for instance, in the diffusion through a porous medium and in some population models [17]. In the porous medium case, $g(u) = a = 0$ with $f(u) = u^m$, $m > 1$, and for Lotka's population model, $g(u) = Ku^2$, $a > 0$, with $f(u) = u^m$, $m \geqslant 2$.

Our difference schemes are patterned on the nonlinear semigroup theory for (P) [1], [10]. We carry the analysis for one particular scheme and sketch its extension to schemes obtained from other diagonal dominant discretizations of the Laplacian. We concentrate on the case $a = 0$, which involves the essential techniques, and outline the case $a > 0$. The details of the case $a > 0$ and the nonhomogeneous equation are considered in [25, Chapter I]. We discretize in space and prove that the resulting operator generates a nonlinear semigroup $S^h(t)$ in a discrete L^1-space. $S^h(t)$ is a discrete analog of $S(t)$, the semigroup of operators in $L^1(\Omega)$ generated by the space operator in (P) and, in a sense to be made precise, we show that for any $u_0 \in L^1(\Omega)$, "$S^h(t)u_0 \to S(t)u_0$ as $h \to 0$".

There are a number of papers discussing difference schemes for problems like (P) when $g \equiv 0$ and $a = 0$, e.g., [4], [11], [13], [18], [21]. In Section 5 we consider the schemes proposed in [4], which also arise from a semigroup approach to (P). The

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analytical algorithm consists of discretizing in time and approximating the generator equation. Its convergence was established in [4]. Here we outline a proof based on our results, for the fully discrete schemes.

An important manifestation of the degeneracy of (P) is that if u_0 is positive of compact support, so is the solution u of (P) for all $t > 0$ [12]. Our method also has finite speed of propagation, and in Section 6 we discuss numerical experiments conducted with both methods for a known exact solution of compact support [2], [23]. In this case, our chosen scheme was more accurate, especially in determining the interface, but not as accurate as the one-dimensional schemes in [18], [21] which are specifically designed to track the interface.

The scheme in [11] is essentially the same as our sample scheme. It is proven there that the limit of the difference solutions yields a weak solution of (P). As indicated in [11], those proofs hold for Ω one-dimensional or a rectangle. In contrast, our proofs are valid for more general regions and grids. We also show that the rate of convergence to the semigroup solution of (P) for smooth initial data is $O(\Delta t^{1/2}) + O(h^{\alpha\beta/(3+2\alpha)})$. The computed rates were $O(\Delta t)$.

Finite element discretizations can be found in [24] for a problem similar to (P), and in [19], [25] for a Lagrangian formulation of the porous media problem.

2. Preliminary Results. Let $\Omega \subset \mathbf{R}^d$ be a bounded domain with $C^{2,\alpha\beta}$ boundary, and $\|v\|_p$ denote the $L^p(\Omega)$-norm of v. An operator A with domain $D(A)$ belongs to the class $\mathscr{A}(a)$ for $a > 0$ if for each $0 < \lambda < a^{-1}$ and any $x, y \in D(A)$:

$$\|(x + \lambda Ax) - (y + \lambda Ay)\| \geqslant (1 - \lambda a)\|x - y\|.$$

Let us now define the operator A by $Au := -\Delta f(u) + g(u) - au$ with domain $D(A) = \{u \in L^1(\Omega)\colon f(u) \in W_0^{1,1}(\Omega),\ g(u) \in L^1(\Omega) \text{ and } -\Delta f(u) \in L^1(\Omega) \text{ in weak sense}\}$. Let γ_λ be defined by $\gamma_\lambda(x) := f^{-1}(x) + \lambda g \circ f^{-1}(x)$; then the assumptions made on f and g imply that for any $\lambda > 0$, γ_λ has a maximal monotone graph. It follows that A has the following properties:

LEMMA 2.1. (i) *$A \in \mathscr{A}(a)$ in $L^1(\Omega)$, and* Range$(I + \lambda A) = L^1(\Omega)$, *i.e., for $0 < \lambda < a^{-1}$ the resolvent operator $J_\lambda := (I + \lambda A)^{-1}$ is Lipschitz continuous with constant $(1 - \lambda a)^{-1}$ and $D(J_\lambda) = L^1(\Omega)$.*

(ii) *$D(A)$ is dense in $L^1(\Omega)$.*

(iii) *A generates a nonlinear semigroup $S(t)$ on $L^1(\Omega)$ and $u(t) := S(t)u_0$ is a solution of* (P) *in the semigroup sense.*

(iv) *Let $v \in L^p(\Omega)$, $1 \leqslant p \leqslant \infty$; then $\|J_\lambda v\|_p \leqslant (1 - \lambda a)^{-1}\|v\|_p$.*

(v) *Let $v, \bar{v} \in L^1(\Omega)$; also let $u := J_\lambda(v)$ and $\bar{u} := J_\lambda \bar{v}$; then $\|[u - \bar{u}]^+\|_1 \leqslant \|[v - \bar{v}]^+\|_1$. Consequently, if $v \leqslant \bar{v}$ a.e., then $u \leqslant \bar{u}$ and $f(u) \leqslant f(\bar{u})$ a.e.*

Proof. First consider the case where $g \equiv 0$ and $a = 0$. Brezis and Strauss established statements (i), (iv), and (v) in [7, Theorem 1, Propositions 4 and 5]. Benilan also established (i) in [3, Chap. II, Théorème 2.1]. Assertion (ii) is a known result and can be found for instance in [15, Proposition 1]. Statement (iii) follows from the Generation Theorem of Crandall and Liggett [8] and the general theory of nonlinear semigroups, see [1]. Statement (i) can also be found in [10, Theorem 4.12].

Most of these statements were established by considering the problem $\gamma(w) - \lambda \Delta w = v$ for $w \in D(-\Delta) := \{u \in W_0^{1,1}(\Omega)\colon -\Delta u \in L^1(\Omega) \text{ in weak sense}\}$ and defining $J_\lambda v := \gamma(w)$. In the case $g \not\equiv 0$ and $a = 0$, J_λ is defined as $J_\lambda v := f^{-1}(w)$, and the statements easily follow from the monotonicity of g and $g(0) = 0$. Finally, the case $a > 0$ is established by a simple fixed-point argument (see [25]).

3. Approximate L^1-Spaces and Difference Schemes. We construct either a full or a uniform degree-zero grid approximation and denote the gridpoints by x_i, i being their corresponding index. We recall that in a full grid the nodes are given by the intersection of lines parallel to the axis with themselves or the boundary. In a degree-zero grid the boundary nodes are the interior intersection points closest to the boundary. Grid functions are indicated with a subscript h, and v_i denotes the value of v_h at x_i.

In one dimension we let $h_i := x_i - x_{i-1}$ and $h := \max_i h_i$. Q_i is the interval around x_i given by

$$Q_i := \left\{x\colon -\tfrac{1}{2}h_i \leqslant x - x_i < \tfrac{1}{2}h_{i+1}\right\}$$

with volume $w_i = \frac{1}{2}(h_i + h_{i+1})$. In the $\mathbf{R}^d$ case, i and h_i are vectors, $Q_i = Q_{i_1} \times Q_{i_2} \times \cdots \times Q_{i_d}$ with $w_i = w_{i_1} w_{i_2} \cdots w_{i_d}$.

We denote by l_h^1 the space of grid functions defined over the interior points of the grid normed by $\|v_h\|_h := \sum w_i |v_i|$. A restriction operator $r_h\colon C(\Omega) \to l_h^1$ is defined by $(r_h v)_i := v(x_i)$, and a prolongation operator $p_h\colon l_h^1 \to L^1(\Omega)$ by $p_h v_h := \sum_i v_i \chi_{Q_i}$, where χ_{Q_i} denotes the characteristic function of Q_i. It is clear that for any Riemann integrable v, $\|v - p_h r_h v\|_1 \to 0$ as $h \to 0$; also as $\{Q_i\}$ are disjoint,

$$\int_\Omega \left|\sum_i v_i \chi_{Q_i}\right| = \sum_i |v_i|\,|Q_i \cap \Omega| \leqslant \sum_i w_i |v_i| \quad \text{or} \quad \|p_h v_h\|_1 \leqslant \|v_h\|_h .$$

Our finite-difference analog of (P) is

$$(\mathrm{P}_h) \qquad \begin{aligned} &\left(u_i^n - u_i^{n-1}\right)/\Delta t - \Delta_h f\left(u_i^n\right) + g\left(u_i^n\right) - a u_i^n = 0 \\ &\qquad\qquad\qquad \text{for } x_i \text{ an interior point of the grid,} \\ &u_h^0 = r_h v_0, \qquad v_0 \in C^{0,\alpha\beta}(\overline{\Omega}) \text{ approximates } u_0. \end{aligned}$$

Here the superscript n denotes the discrete time level and Δ_h is a diagonally dominant scheme approximating the Laplacian. We describe this family of schemes below.

Let L_h be a scheme consistent with the linear differential operator L, and let I denote the family of indices of gridpoints at which we apply L_h. Also let v_h be a grid function satisfying $v_i = 0$ for $i \notin I$. We have

$$\sum_{i \in I} w_i L_h v_i = \sum_{i \in I} a_i v_i, \qquad a_i = \sum_{k \in I} w_k b_{ik},$$

where b_{ik} is the coefficient of u_i in the term $L_h v_k$. We say that L_h is diagonally dominant if

$$w_i b_{ii} \geqslant \sum_{k \neq i} w_k |b_{ik}|.$$

For simplicity we shall consider the case when Δ_h is the usual difference approximation of the Laplacian, which in one space dimension is given by

$$-\Delta_h v_i := \frac{2}{(h_i + h_{i+1})}\left\{\left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right) v_i - \frac{1}{h_i} v_{i-1} - \frac{1}{h_{i+1}} v_{i+1}\right\},$$

where, whenever $i-1$ or $i+1$ indicate boundary nodes, we substitute v_{i-1} or v_{i+1} by the boundary value at the node, which in this case is zero. In $\mathbf{R}^d$, $\Delta_h := \sum_{j=1}^d \Delta_{h,j}$, where $\Delta_{h,j}$ is the above discrete Laplacian taken in the jth direction.

Let $k = \Delta t$. We can rewrite (P_h) as

$$(I + kA_h)u_h^n = u_h^{n-1}, \qquad u_h^0 = r_h v_0, \tag{3.1}$$

with the operator $A_h \subset l_h^1 \times l_h^1$ defined by

$$(A_h v_h)_i := -\Delta_h f(v_i) + g(v_i) - av_i.$$

THEOREM 3.1. *$A_h \in \mathscr{A}(a)$ and $\operatorname{Range}(I + \lambda A_h) = l_h^1$, i.e., for $0 < \lambda < a^{-1}$ the resolvent operator $J_\lambda^h := (I + \lambda A_h)^{-1}$ is Lipschitz continuous with constant $(1 - \lambda a)^{-1}$, and its domain is all of l_h^1.*

Proof. First consider the case $a = 0$, i.e., we show that A_h is m-accretive. Let $u_h, v_h \in l_h^1$, and define the sets of indices I^+ and I^- by

$$I^+ := \{i\colon u_i \geqslant v_i \text{ and } x_i \text{ an interior point of the grid}\},$$
$$I^- := \{i\colon u_i < v_i \text{ and } x_i \text{ an interior point of the grid}\}.$$

Let $\lambda > 0$; then as g is monotone nondecreasing,

$$\begin{aligned} &\|(I + \lambda A_h)u_h - (I + \lambda A_h)v_h\|_h \\ &\quad = \sum_{i \in I^+ \cup I^-} w_i \left| u_i - v_i - \lambda \Delta_h [f(u_i) - f(v_i)] + g(u_i) - g(v_i) \right| \\ &\quad \geqslant \sum_{i \in I^+} w_i \left(u_i - v_i - \lambda \Delta_h [f(u_i) - f(v_i)] \right) \\ &\qquad + \sum_{i \in I^-} w_i \left(v_i - u_i - \lambda \Delta_h [f(v_i) - f(u_i)] \right). \end{aligned} \tag{3.2}$$

As f is monotone increasing, $[f(u_i) - f(v_i)] \geqslant 0$ when $i \in I^+$, and $[f(v_i) - f(u_i)] \geqslant 0$ when $i \in I^-$. Hence, in one space dimension with a uniform grid we obtain

$$\begin{aligned} &\sum_{i \in I^+} w_i(-\Delta_h)[f(u_i) - f(v_i)] + \sum_{i \in I^-} w_i(-\Delta_h)[f(v_i) - f(u_i)] \\ &\quad = \sum_{i \in I^+} \frac{1}{h} \{2[f(u_i) - f(v_i)] - [f(u_{i-1}) - f(v_{i-1})] - [f(u_{i+1}) - f(v_{i+1})]\} \\ &\qquad + \sum_{i \in I^-} \frac{1}{h} \{2[f(v_i) - f(u_i)] - [f(v_{i-1}) - f(u_{i-1})] - [f(v_{i+1}) - f(u_{i+1})]\} \\ &\quad = \sum_{i \in I^+ \cup I^-} \frac{1}{h} [2 \pm 1 \pm 1] |f(u_i) - f(v_i)| \geqslant 0. \end{aligned}$$

For a nonuniform grid the coefficient above is replaced by

$$\left[\frac{1}{h_i} + \frac{1}{h_{i+1}} \pm \frac{1}{h_i} \pm \frac{1}{h_{i+1}} \right] \geqslant 0.$$

In the $\mathbf{R}^d$ case, for each $\Delta_{h,j}$ we sum over the jth index first to obtain the same expression multiplied by $(\prod_{l \neq j} w_{i_l})$ and summed over all other indices. In general, for any diagonal dominant scheme the coefficient of $|f(u_i) - f(v_i)|$ is not less than $w_i b_{ii} - \sum_{k \neq i} w_k |b_{ik}| \geqslant 0$. Finally, substituting the above in (3.2) yields accretiveness;

m-accretiveness follows from the fact that for fixed h, A_h is continuous [1], [20]. The case $a > 0$ is established by a simple fixed-point argument (see [25]).

Remark 3.2. From (3.1) it follows that the solution of (P_h) is given by

$$u_h^n = \left(J_k^h\right)^n r_h v_0, \tag{3.3}$$

and Theorem 3.1 shows that (3.3) has a unique solution and is stable. □

The next lemma is a maximum principle type result for (P_h) when Δ_h satisfies:

$$\text{(L)} \qquad \begin{array}{ll} \text{For} \quad v_{i_0} = \min_i v_i, & \Delta_h v_{i_0} \geqslant 0 \quad \text{and} \\ \text{for } v_{i_0} = \max v_i, & \Delta_h v_{i_0} \leqslant 0. \end{array}$$

This property holds for all schemes such that $b_{ii} = \sum_{k \neq i} |b_{ki}|$ and $b_{ki} < 0$, since $\Delta_h v_i = \sum_k b_{ki} v_k = \sum_{k \neq i} |b_{ki}|(v_i - v_k)$. In particular, for the usual one-dimensional discrete Laplacian,

$$b_{i,i-1} = -2/(h_i + h_{i+1})h_i, \qquad b_{i,i+1} = -2/(h_i + h_{i+1})h_{i+1} \quad \text{and}$$
$$b_{i,i} = -b_{i,i-1} - b_{i,i+1}.$$

LEMMA 3.2. *Let $\{u_h^n\}_{n \geqslant 1}$ be the solution of (P_h), where Δ_h satisfies property* (L). *Also let $b = (1 - ak)^{-1}$, $\bar{u}^m = \max_i u_i^m$ and $\underline{u}^m = \min_i u_i^m$. Then*

(a) *For $\underline{u}^{n-1} \geqslant 0$, $0 \leqslant \underline{u}^n \leqslant \bar{u}^n \leqslant b\bar{u}^{n-1}$;*

(b) *For $\underline{u}^{n-1} \leqslant 0$, $b\underline{u}^{n-1} \leqslant \underline{u}^n \leqslant \bar{u}^n \leqslant 0$;*

(c) *For $\underline{u}^{n-1} < 0$ and $\bar{u}^{n-1} > 0$, $b\underline{u}^{n-1} \leqslant \underline{u}^n \leqslant \bar{u}^n \leqslant b\bar{u}^{n-1}$.*

Proof. Let $u_{i_0}^n = \bar{u}^n$; then as f is monotone nondecreasing, $f(u_{i_0}^n) \geqslant f(u_i^n)$, hence $\Delta_h f(u_{i_0}^n) \leqslant 0$, and it follows that

$$(1 - ak)u_{i_0}^n + kg\left(u_{i_0}^n\right) = u_{i_0}^{n-1} + k\Delta_h f\left(u_{i_0}^n\right) \leqslant \bar{u}^{n-1}.$$

Similarly, letting i_0 be such that $u_{i_0}^n = \underline{u}^n$, we obtain $\underline{u}^{n-1} \leqslant (1 - ak)\underline{u}^n + kg(\underline{u}^n)$. As g is monotone with $g(0) = 0$, we have sign $g(x) = \operatorname{sign}(x)$; hence (a), (b), and (c) follow by considering the different cases. □

4. Convergence Results. Our main convergence result is for smooth initial data u_0. The density of $D(A)$ and the fact that the operators J_λ, J_λ^h, and $S(t)$ are Lipschitz continuous immediately yield convergence for any $u_0 \in L^1(\Omega)$.

THEOREM 4.1. *Let $u_0 \in C^{0,\alpha\beta}(\overline{\Omega}) \cap D(A)$, $f(u_0) \in C^2(\overline{\Omega})$, and $\Delta t = t/m$, t fixed. Then for u_h^m computed with scheme* (P_h),

$$\left\| S(t)u_0 - p_h u_h^m \right\|_1 = O\left(\Delta t^{1/2}\right) + O\left(h^{\alpha\beta/(3+2\alpha)}\right). \tag{4.1}$$

Remark. As will be apparent from the proof, we can eliminate the assumption $f(u_0) \in C^2(\overline{\Omega})$ and improve the constants in (4.1) by taking $\Delta t = h^{2\alpha\beta/(3+2\alpha)}$. Here, Δ_h is any diagonal dominant scheme consistent with the Laplacian in the sense that for $v \in C^{2,\alpha\beta}(\overline{\Omega})$ and x in Ω, $|\Delta_h v(x) - \Delta v(x)| \leqslant \text{const} \cdot \|v\|_{C^{2,\alpha\beta}(\overline{\Omega})} h^{\alpha\beta}$.

Proof of Theorem 4.1. Assume $a = 0$. We recall that if $B \in \mathscr{A}(a)$ and $R(I + \lambda B) \supseteq \overline{D(B)}$ for any $0 < \lambda < a^{-1}$, then for any $x \in D(B)$ (cf. (1.10) in [8]),

$$\left\| S(t)x - J_{t/n}^n x \right\| \leqslant 2tn^{-1/2}e^{4at}\| Bx \|,$$
$$\left\| J_{t/m}^m x - J_{t/n}^n x \right\| \leqslant 2te^{4at}\left|1/m - 1/n\right|^{1/2}\| Bx \|.$$

From the above, and the fact that $S(t)$, J_λ, and p_h are contractions, we obtain

$$
(4.2)\quad
\begin{aligned}
&\|S(t)u_0 - p_h u_h^m\|_1 \\
&\leqslant \left\|S(t)u_0 - J_{t/m}^m u_0\right\|_1 + \left\|J_{t/m}^m u_0 - J_{t/n}^n u_0\right\|_1 \\
&\quad + \left\|J_{t/n}^n u_0 - p_h\left(J_{t/n}^h\right)^n r_h u_0\right\|_1 + \left\|\left(J_{t/n}^h\right)^n r_h u_0 - \left(J_{t/m}^h\right)^m r_h u_0\right\|_h \\
&\leqslant 2t\left((1/m)^{1/2} + |1/m - 1/n|^{1/2}\right)\|Au_0\|_1 + 2t|1/n - 1/m|^{1/2}\|A_h r_h u_0\|_h \\
&\quad + \left\|J_{t/n}^n u_0 - p_h r_h J_{t/n}^n u_0\right\|_1 + \left\|r_h J_{t/n}^n u_0 - \left(J_{t/n}^h\right)^n r_h u_0\right\|_h.
\end{aligned}
$$

For notational convenience we drop the t/n subscript. Repeatedly using that J^h is a contraction, for the last term in (4.2) we obtain

$$
(4.3)\quad
\begin{aligned}
&\left\|r_h J^n u_0 - (J^h)^n r_h u_0\right\|_h \\
&\leqslant \left\|r_h J\left(J^{n-1}u_0\right) - J^h r_h\left(J^{n-1}u_0\right)\right\|_h + \left\|r_h J^{n-1}u_0 - (J^h)^{n-1} r_h u_0\right\|_h \\
&\leqslant n \max_{0\leqslant s\leqslant n-1} \left\|r_h J(J^s u_0) - J^h r_h(J^s u_0)\right\|_h.
\end{aligned}
$$

By definition, $v := J(J^s u_0)$ and $v_h := J^h r_h(J^s u_0)$ satisfy

$$
(4.4)\quad
\begin{aligned}
&v_h + (t/n)\left[-\Delta_h f(v_h) + g(v_h)\right] = r_h J^s u_0, \\
&r_h v + (t/n)\left[-\Delta_h f(r_h v) + g(r_h v)\right] \\
&\qquad = r_h J^s u_0 + (t/n)\left[r_h \Delta f(v) - \Delta_h f(r_h v)\right].
\end{aligned}
$$

Here we used that since r_h is a pointwise evaluation, $r_h f(v) = f(r_h v)$ and $r_h g(v) = g(r_h v)$. Given that A_h is accretive, from (4.4) we obtain

$$
(4.5)\quad \left\|r_h J(J^s u_0) - J^h r_h(J^s u_0)\right\|_h \leqslant (t/n)\left\|\Delta_h r_h f(v) - r_h \Delta f(v)\right\|_h.
$$

To complete our estimates, we need some regularity results. Given any s, $0 \leqslant s \leqslant n-1$, let w_s be the solution of

$$
\gamma(w_s) - (t/n)\Delta w_s = J^s u_0, \qquad w_s \in D(-\Delta).
$$

By Lemma 2.1, we have $\|\gamma(w_s)\|_q \leqslant \|J^s u_0\|_q \leqslant \|u_0\|_q$ for $1 \leqslant q \leqslant \infty$. First let $q > d$ and put $\varepsilon = 1 - d/q$ (> 0). Since w_s is a solution of the equation $-\Delta w_s = (n/t)(J^s u_0 - \gamma(w_s))$ and $\Delta w_s \in L^q(\Omega)$, it follows that $w_s \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$. But $W^{2,q}(\Omega)$ is embedded in $C^{1,\varepsilon}(\overline{\Omega})$, and so there is a positive number $K = K(\Omega, d, q)$, such that

$$
\begin{aligned}
\|w_s\|_{C^{1,\varepsilon}(\overline{\Omega})} &\leqslant K\|\Delta w_s\|_q = K(n/t)\|J^s u_0 - \gamma(w_s)\|_q \\
&\leqslant K \cdot (n/t) \cdot 2\|u_0\|_q = \mathrm{const}(n/t).
\end{aligned}
$$

Thus, in particular, $\|w_s\|_{C^1(\overline{\Omega})} \leqslant \mathrm{const} \cdot (n/t)$. Next, we show that $\|w_s\|_{C^{2,\alpha\beta}(\overline{\Omega})} \leqslant \mathrm{const} \cdot (n/t)^{1+\alpha}$. Since $w_s \in C^1(\overline{\Omega})$ for s with $0 \leqslant s \leqslant n-1$, we have

$$
J^s u_0 \equiv f^{-1}(w_{s-1}) \in C^{0,\alpha}(\overline{\Omega}) \quad \text{for } 1 \leqslant s \leqslant n
$$

and

$$
\gamma(w_s) \equiv f^{-1}(w_s) + (t/n)g\left(f^{-1}(w_s)\right) \in C^{0,\alpha\beta}(\overline{\Omega}) \text{ for } 1 \leqslant s \leqslant n.
$$

Hence,

$$
-\Delta w_s = (n/t)\left(J^s u_0 - \gamma(w_s)\right) \in C^{0,\alpha\beta}(\overline{\Omega}).
$$

Noting that Δ is considered in the bounded domain Ω under the Dirichlet condition, we apply the usual Hölder estimates to obtain

$$\begin{aligned}\|w_s\|_{C^{2,\alpha\beta}(\bar\Omega)} &\leqslant \text{const}\cdot\|\Delta w_s\|_{C^{0,\alpha\beta}(\bar\Omega)}\\ &\leqslant \text{const}\cdot(n/t)\left[\|J^s u_0\|_{C^{0,\alpha\beta}(\bar\Omega)}+\|\gamma(w_s)\|_{C^{0,\alpha\beta}(\bar\Omega)}\right].\end{aligned}$$

We then demonstrate that both $\|J^s v_0\|$ and $\|\gamma(w_s)\|$ are bounded by const $\cdot$ $(n/t)^\alpha$. Since

$$\begin{aligned}|f^{-1}(w_{s-1}(x))| &\leqslant \text{const}\cdot|w_{s-1}(x)|^\alpha \leqslant \text{const}\cdot\|w_{s-1}\|^\alpha_{L^\infty(\Omega)}\\ &\leqslant \text{const}\cdot\|w_{s-1}\|^\alpha_{C^1(\bar\Omega)} \leqslant \text{const}\cdot(n/t)^\alpha,\end{aligned}$$

and

$$\begin{aligned}|f^{-1}(w_{s-1}(x))-f^{-1}(w_{s-1}(y))| &\leqslant \|f^{-1}\|_{C^{0,\alpha}(\bar\Omega)}\cdot|w_{s-1}(x)-w_{s-1}(y)|^\alpha\\ &\leqslant \|f^{-1}\|_{C^{0,\alpha}(\bar\Omega)}\left(\|w_{s-1}\|_{C^1(\bar\Omega)}\cdot|x-y|\right)^\alpha \leqslant \text{const}\cdot(n/t)^\alpha\cdot|x-y|^\alpha,\end{aligned}$$

we have

$$\|J^s v_0\|_{C^{0,\alpha\beta}(\bar\Omega)}=\|f^{-1}(w_{s-1})\|_{C^{0,\alpha\beta}(\bar\Omega)}\leqslant \text{const}\cdot\|f^{-1}(w_{s-1})\|_{C^{0,\alpha}(\bar\Omega)}\leqslant \text{const}\cdot(n/t)^\alpha.$$

Similarly, we obtain

$$\begin{aligned}\|\gamma(w_s)\|_{C^{0,\alpha\beta}(\bar\Omega)} &\leqslant \|f^{-1}(w_s)\|_{C^{0,\alpha\beta}(\bar\Omega)}+(t/n)\|g(f^{-1}(w_s))\|_{C^{0,\alpha\beta}(\bar\Omega)}\\ &\leqslant \text{const}\cdot(n/t)^\alpha+\text{const}\cdot(t/n)\cdot(n/t)^\alpha.\end{aligned}$$

Noting that t is a fixed positive number and n is sufficiently large, we conclude that $n/t > 1$ and $\|\gamma(w_s)\|_{C^{0,\alpha\beta}(\bar\Omega)} \leqslant$ const $\cdot$ $(n/t)^\alpha$. Consequently, it follows that

$$\|w_s\|_{C^{2,\alpha\beta}(\bar\Omega)}\leqslant \text{const}\cdot(n/t)^{1+\alpha} \tag{4.6}$$

holds for n sufficiently large.

In (4.2) and (4.5) we have $J^n u_0 \equiv f^{-1}(w_{n-1})$ and $f(v) \equiv w_s$; consequently from (4.6), taking $(t/n) = h^r$, we obtain

$$\|\Delta_h r_h w_s - r_h\Delta w_s\|_h \leqslant \text{const}\cdot\|w_s\|_{C^{2,\alpha\beta}(\bar\Omega)}h^{\alpha\beta}\leqslant \text{const}\cdot h^{\alpha\beta-r(1+\alpha)},$$

$$\|J^n u_0 - p_h r_h J^n u_0\|_1 \leqslant \text{const}\cdot\|f^{-1}(w_{n-1})\|_{C^{0,\alpha}(\bar\Omega)}h^\alpha\leqslant \text{const}\cdot h^{\alpha-r\alpha}.$$

Substituting back in (4.5), (4.3), and (4.2), since

$$h^{r/2}=(t/n)^{1/2}\leqslant(t/m)^{1/2}+|(t/n)-(t/m)|^{1/2}\leqslant \Delta t^{1/2}+h^{r/2},$$

it is easy to see that the best choice of r is $r = 2\alpha\beta/(3 + 2\alpha)$, thus yielding (4.1). For the case $a > 0$, only some t-dependent coefficients would have to be included in the inequalities (see [25]).

5. Berger et al. Algorithms. Let $R(t)$ denote the semigroup generated by $-\Delta$ in $L^1(\Omega)$. Berger et al. [4] approximate (P) for $g(u) \equiv 0$ and $a = 0$ by

$$(u^{n+1}-u^n)/k+\frac{1}{\sigma}(I-R(\sigma))f(u^n)=0,\qquad u^0=u_0,$$

where σ is a positive function of k satisfying $\lim_{k\to 0}\sigma = 0$. From it we obtain the "analytical" form of the algorithm,

$$u^{n+1}=u^n+\frac{k}{\sigma}[R(\sigma)f(u^n)-f(u^n)],\qquad u^0=u_0,$$

which involves no spatial discretization. For $k = t/n$, t fixed, convergence of u^n to $S(t)u_0$ was established in [4], using approximation results in [6], under the assumptions that $u_0 \in L^\infty(\Omega)$, f is locally Lipschitz continuous with constant μ on $[-\|u_0\|_\infty, \|u_0\|_\infty]$, and $\mu k \leqslant \sigma$.

The numerical schemes are obtained by discretizing the linear heat problem $R(\sigma)f(u^n)$, and are given by

$$(5.1) \qquad u_h^{n+1} = T(k)u_h^n, \qquad u_h^0 = r_h v_0,$$

where $T(k)$ is defined by

$$(5.2) \qquad T(k)v_h := v_h + \frac{k}{\sigma}\left[K_\sigma^h f(v_h) - f(v_h)\right],$$

and K_σ^h is one of the usual implicit finite-difference operators for the heat equation, i.e.,

$$K_\sigma^h = (I - \theta\sigma\Delta_h)^{-1}(I + (1-\theta)\sigma\Delta_h)$$

for some $0 < \theta \leqslant 1$.

Our convergence results allow us to prove convergence of the fully discretized (5.1)–(5.2) schemes under the above or other similar conditions. It is enough to consider the fully implicit scheme, and the proof consists of (a) establishing that $(I - T(k))/k$ is m-accretive in l_h^1, (b) showing that $T(k)$ is Lipschitz continuous, (c) using arguments like those in Section 4, and (d) applying Lemma 4′ in [22] together with (4.1). See [25] for details.

6. Numerical Experiments. We consider the one-dimensional problem with $f(u) \equiv u^2$ and $g \equiv 0$. Barenblatt [2] and Pattle [23] established the exact weak solution when $f(u) \equiv u^m$ and the initial data is the Dirac delta; in this case it is given by

$$u(x,t) = \begin{cases} \left(1 - \left[x/(9t)^{1/3}\right]^2\right)/\tfrac{4}{3}(9t)^{1/3} & \text{for } |x| \leqslant (9t)^{1/3}, \\ 0 & \text{elsewhere,} \end{cases}$$

which shows that the interfaces are at $x = \pm(9t)^{1/3}$.

The initial data for the numerical computations were taken to be the values at gridpoints of $u_0(x)$, the above solution at $t = 1$. The grids were uniform with $\Delta x = L/N$, where $L = (9)^{1/3}$, and $N + 1$ is the number of points in $[0, L]$, since we exploited the problem's symmetry. We took $k = \sqrt{\Delta x}$, to be able to neglect the effect of the space discretization, and consider the error E in l_h^1 as $E = ck^p$.

From the algorithms of [4] we chose the totally implicit one with $k/\sigma = \max\{f'(z)\colon z \in [-\|u_0\|_\infty, \|u_0\|_\infty]\} = (3/8)^{1/3}$ and computed over the fixed interval $[0, 2L]$; we shall refer to it as BBR.

The nonlinear equations of (P_h) were solved with Newton's method, taking the solution in the previous time step as initial guess; we shall refer to this algorithm as NLN. Since the support of the numerical solution advances one gridpoint per Newton iteration, our initial set of equations corresponded to the gridpoints in $[0, L]$ and we added one equation per iteration. Also, we only iterated over those equations for which the relative error in the previous iteration m was greater than $[.01(\Delta x)^2/\max u_i^{(m)}]$. For the following time step, only the new gridpoints at which the solution was greater than 10^{-25} were added to the support, thus determining the location of the numerical interface. With the above implementation, NLN was slightly faster than BBR, offsetting its advantage of only solving a linear system. Table 1 contains the errors and numerical rates of convergence to the exact solution at $t = 4.2250$.

TABLE 1

N	Δx	k	BBR E	BBR RATE	NLN E	NLN RATE
20	.1040	.3225	.1574		.00880	
				.5160		1.0697
80	.0260	.1612	.1100		.00419	
				.6492		1.0136
320	.0065	.0806	.07018		.00207	
				.7579		1.0031
1280	.00162	.0403	.04150		.00103	

For this example, NLN was more accurate than BBR, particularly around the interface which at $t = 4.225$ is at 3.3627. Table 2 contains, for both algorithms, the location of the smallest value greater than 10^{-20}, and for NLN the first point at which the solution is less than 0.00005, which computationally might be considered as zero, since the values at the flat part of the solution are close to 0.2.

TABLE 2

N	BBR x	BBR VALUE	NLN x	NLN VALUE	NLN x	NLN VALUE
20	4.160	1.30(−3)	3.7441	3.07(−11)	3.6401	1.25(−6)
80	4.160	4.52(−4)	3.5101	4.80(−11)	3.4581	4.60(−5)
320	4.160	1.18(−4)	3.4321	1.40(−14)	3.4126	4.39(−5)
1280	4.160	2.39(−5)	3.3899	6.07(−5)	3.3899	6.07(−5)

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