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An Estimate of Goodness of Cubatures for the Unit Circle in $\mathbf{R}^2$

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Abstract. The Sarma-Eberlein estimate s_E is an estimate of goodness of cubature formulae for n-cubes defined as the integral of the square of the formula truncation error, over a function space provided with a measure. In this paper, cubature formulae for the unit circle in $\mathbf{R}^2$ are considered and an estimate of the above type is constructed with the desirable property of being compatible with the symmetry group of the circle.

1. Isometries and Two-dimensional Cubature Formulae. Let

$$S_2 = \{(x, y) \in \mathbf{R}^2\colon x^2 + y^2 \leqslant 1\} \tag{1.1}$$

be the unit circle in the two-dimensional Euclidean space $\mathbf{R}^2$ and let $\mathscr{U}(S_2)$ denote the symmetry group of S_2. This group consists of all linear bijective maps u: $\mathbf{R}^2 \to \mathbf{R}^2$ which preserve the Euclidean distance (that is, isometries of $\mathbf{R}^2$ leaving the origin invariant). Each element of $\mathscr{U}(S_2)$ can be identified with a 2×2 real orthogonal matrix and therefore

$$\mathscr{U}(S_2) = \{u_\alpha, u_\alpha \circ v;\ \alpha \in [0, 2\pi)\}, \tag{1.2}$$

where u_α denotes the rotation of α radians around the origin and v is the reflection about any fixed straight line passing through the origin; thus

$$\begin{aligned} u_\alpha(x, y) &= (x\cos\alpha - y\sin\alpha,\ x\sin\alpha + y\cos\alpha),\\ v(x, y) &= (x, -y). \end{aligned} \tag{1.3}$$

Let $w(x, y)$ be a normalized weight function compatible with $\mathscr{U}(S_2)$, that is, a real positive continuous function in the interior of S_2 such that

$$\iint_{S_2} w(x, y)\,dx\,dy = 1 \quad \text{and} \quad w \circ u = w \quad \text{for all } u \in \mathscr{U}(S_2). \tag{1.4}$$

A cubature formula for the w-weighted circle S_2 has the form

$$I(f) = Q_N(f) + E(f), \tag{1.5}$$

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where

$$I(f) = \iint_{S_2} w(x, y) f(x, y)\, dx\, dy,$$
(1.6)
$$Q_N(f) = \sum_{i=1}^{N} A_i f(x_i, y_i), \qquad (x_i, y_i) \in S_2,$$

and the constants A_i are independent of f.

Let us consider a symmetry $u \in \mathcal{U}(S_2)$ acting on (1.5). Since $I(f \circ u) = I(f)$, it leads to another cubature formula

$$I(f) = Q'_N(f) + E'(f), \tag{1.7}$$

where

$$Q'_N(f) = Q_N(f \circ u) = \sum_{i=1}^{N} A_i f(u(x_i, y_i)),$$
(1.8)
$$E'(f) = E(f \circ u).$$

Definition 1. For every $u \in \mathcal{U}(S_2)$, the cubature formulae (1.5) and (1.7) are said to be $\mathcal{U}(S_2)$-equivalent or equivalent with respect to the symmetry group of S_2.

The integration of a function on the w-weighted circle S_2 is independent of the pair of orthogonal axis OX, OY whose origin O lies in the center of the circle. Therefore, all $\mathcal{U}(S_2)$-equivalent formulae have identical characteristics when they are considered as approximations of the operator I.

Therefore, any estimate of goodness for cubature formulae (1.5) should be compatible with the $\mathcal{U}(S_2)$-equivalence relation, that is, all $\mathcal{U}(S_2)$-equivalent formulae should have the same estimate of goodness. For instance, the degree of precision of a cubature formula (1.5) is an estimate compatible with $\mathcal{U}(S_2)$, because the space of polynomials of degree at most n is invariant under all the symmetries in (1.2).

The aim of this paper is to construct an $\mathcal{U}(S_2)$-compatible estimate of goodness of cubature formulae for S_2 similar to that defined by V. L. N. Sarma in [3] for cubatures for the square.

The next section is devoted to recalling briefly some characteristics of the Sarma-Eberlein estimate that are useful for our purpose. A detailed exposition of the construction of this estimate can be found in [3], [4] and [5] and an excellent summary of these results in [6, pp. 188–192].

2. The Sarma-Eberlein Estimate of Goodness s_E. Let us consider the square

$$C_2 = \{(x, y) \in \mathbf{R}^2\colon |x| \leqslant 1, |y| \leqslant 1\}$$

and cubature formulae

$$I(f) = Q_N(f) + E(f), \tag{2.1}$$

where

$$I(f) = \frac{1}{4} \iint_{C_2} f(x, y)\, dx\, dy,$$
(2.2)
$$Q_N(f) = \sum_{i=1}^{N} A_i f(x_i, y_i), \qquad (x_i, y_i) \in C_2.$$

Sarma in [3], [4] defines the estimate of goodness of the cubature formula (2.1) as

$$s_E^2 = \int_{FS_\infty} E(f)^2\, df, \tag{2.3}$$

where the integral is defined over the unit sphere of a normed space of functions provided with a measure defined as follows:

Let l_1 be the space of real sequences

$$f = \{f_{nk};\ n = 0,1,\ldots;\ k = 0,1,\ldots,n\} \tag{2.4}$$

such that

$$\|f\|_1 = \sum_{n,k} |f_{nk}| < \infty; \qquad n = 0,1,\ldots;\ k = 0,1,\ldots,n. \tag{2.5}$$

The unit sphere $S_\infty = \{f \in l_1\colon\ \|f\|_1 \leqslant 1\}$ is compact in the weak*-topology of l_1, and an elementary integral defined for the weak*-continuous real functions on S_∞ can be extended by the Daniell process inducing a countably additive measure on S_∞.

Among the properties of this measure, let us recall that

$$\int_{S_\infty} f_{nk} f_{ml}\, df = 0 \quad \text{if } (n,k) \neq (m,l), \tag{2.6}$$

$$\int_{S_\infty} f_{nk}^2\, df = \frac{2^{n+2}}{(n+2)!(n+3)!} = q_n^2. \tag{2.7}$$

Real two-dimensional power series

$$f(x,y) = \sum_{n,k} f_{nk} x^{n-k} y^k; \qquad n = 0,1,\ldots;\ k = 0,1,\ldots,n, \tag{2.8}$$

whose coefficients satisfy the condition

$$\|f\|_1 = \sum_{n,k} |f_{nk}| < \infty \tag{2.9}$$

converge uniformly and absolutely for all points $(x,y) \in C_2$.

The space Fl_1 of all functions defined by (2.8) and (2.9) can be identified with the sequence space l_1 and is dense in the space $\mathscr{C}(C_2)$ of all real continuous functions on C_2 with the uniform norm. This identification allows us to consider the above integral as an integral over the unit sphere FS_∞ of the function space Fl_1.

The truncation error $E(f)$ of the cubature formula (2.1) is a continuous linear form over $\mathscr{C}(C_2)$ with the uniform norm and therefore also over Fl_1 with the $\|\cdot\|_1$-norm. Using (2.6) and (2.7), it follows that the estimate s_E defined by (2.3) can be written as

$$s_E^2 = \sum_{n=0}^{\infty} q_n^2 e_n^2, \tag{2.10}$$

where q_n is defined in (2.7) and

$$e_n^2 = \sum_{k=0}^{n} E\left(x^{n-k} y^k\right)^2. \tag{2.11}$$

It should be noted that the identification of l_1 and Fl_1 is made through the monomials $x^{n-k}y^k$ and the use of these particular functions makes s_E compatible with $\mathscr{U}(C_2)$, the symmetry group of C_2, in the sense described in the previous section. In effect, $\mathscr{U}(C_2)$ consists of the eight symmetries

$$(2.12)\qquad (x, y) \to (\pm x, \pm y); \qquad (x, y) \to (\pm y, \pm x)$$

and the equalities

$$(2.13)\qquad \begin{aligned} e_n^2 &= \sum_{k=0}^{n} E(x^{n-k}y^k)^2 = \sum_{k=0}^{n} E((\pm x)^{n-k}(\pm y)^k)^2 \\ &= \sum_{k=0}^{n} E((\pm y)^{n-k}(\pm x)^k)^2 \end{aligned}$$

imply that $\mathscr{U}(C_2)$-equivalent cubature formulae have the same estimate of goodness s_E. Unfortunately, this estimate of goodness is not useful for cubature formulae (1.5), (1.6) for the unit circle S_2, because it is not compatible with $\mathscr{U}(S_2)$, as can be computationally checked. For instance, taking $w(x, y) = 1/\pi$, the cubature formula (degree 3, 4 points) given by

$$(2.14)\quad Q_4(f) = \tfrac{1}{4}\left[f(\sqrt{2}/2, 0) + f(-\sqrt{2}/2, 0) + f(0, \sqrt{2}/2) + f(0, -\sqrt{2}/2)\right]$$

has an estimate of goodness $s_E = (-4)1.75032$, whereas the $\mathscr{U}(S_2)$-equivalent formula (use a rotation of $\pi/4$ radians) given by

$$(2.15)\qquad \begin{aligned} Q_4(f) = \tfrac{1}{4}[&f(1/2, 1/2) + f(-1/2, 1/2) \\ &+ f(1/2, -1/2) + f(-1/2, -1/2)] \end{aligned}$$

has an estimate of goodness $s_E = (-4)3.81547$.

3. An Estimate of Goodness of Cubatures for the Unit Circle. In the previous section, the sequence space l_1 was identified with the space of functions Fl_1 by using the family of monomials $\{x^{n-k}y^k;\ n = 0, 1, \ldots;\ k = 0, 1, \ldots, n\}$, but we can also identify l_1 with other subspaces of $\mathscr{C}(C_2)$ or $\mathscr{C}(S_2)$ by using other families of polynomials. For each n, let us denote

$$(3.1)\qquad M_n = \left\{a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n;\ a_i \in \mathbf{R}\right\}$$

and let

$$(3.2)\qquad \Phi_n = \{\varphi_{n0}, \ldots, \varphi_{nn}\} \subset M_n$$

be a basis of M_n, i.e., $M_n = \operatorname{span}\Phi_n$.

If the polynomials φ_{nk} satisfy

$$(3.3)\quad \|\varphi_{nk}\|_\infty = \max_{(x,y)\in S_2} |\varphi_{nk}(x, y)| \leqslant c; \qquad n = 0, 1, \ldots;\ k = 0, 1, \ldots, n,$$

then the series

$$(3.4)\qquad f(x, y) = \sum_{n,k} f_{nk}\varphi_{nk}(x, y)$$

whose coefficients satisfy (2.9) converge uniformly and absolutely for all points $(x, y) \in S_2$. If we denote $\Phi = \{\Phi_1, \Phi_2, \ldots\}$, the space $Fl_1(\Phi)$ of all functions defined by (3.4) and (2.9) can be identified with the sequence space l_1. Let us note that $Fl_1(\Phi)$ contains all real polynomials in two variables and therefore is dense in $\mathscr{C}(S_2)$ with the uniform norm.

This identification allows us to define, in a natural way, an estimate of goodness for cubatures (1.5) by

$$(3.5)\qquad s_E^2(\Phi) = \int_{FS_\infty(\Phi)} E(f)^2\, df,$$

where

$$FS_\infty(\Phi) = \left\{ f \in Fl_1(\Phi):\ \sum_{n,k} |f_{nk}| \leqslant 1 \right\}. \tag{3.6}$$

It is straightforward to deduce that this estimate can be expressed by

$$s_E^2(\Phi) = \sum_{n=0}^{\infty} q_n^2 e_n^2(\Phi_n), \tag{3.7}$$

where q_n^2 is given in (2.7) and

$$e_n^2(\Phi_n) = \sum_{k=0}^{n} E(\varphi_{nk})^2. \tag{3.8}$$

Our problem at this stage is to choose suitable families Φ_n satisfying (3.3), such that the estimate $s_E^2(\Phi)$ is compatible with the symmetry group $\mathscr{U}(S_2)$ in the sense described in Section 1.

As the matrix

$$\begin{pmatrix} \cos\alpha, & -\sin\alpha \\ \sin\alpha, & \cos\alpha \end{pmatrix} \tag{3.9}$$

associated with the rotation $u_\alpha \in \mathscr{U}(S_2)$ has eigenvalues $e^{i\alpha}$, $e^{-i\alpha}$ and eigenvectors $(1, i)^T$, $(1, -i)^T$, the use of complex arithmetic will simplify the calculations. Let us denote

$$M_n^* = \left\{ a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n;\ a_\iota \in \mathbf{C} \right\}, \tag{3.10}$$

and let

$$\Phi_n^* = \left\{ \varphi_{n0}^*, \ldots, \varphi_{nn}^* \right\} \tag{3.11}$$

be a basis of M_n^*, i.e., $M_n^* = \operatorname{span}^*(\Phi_n^*)$.

Considering the natural complexification of linear operators

$$E(f + ig) = E(f) + iE(g) \tag{3.12}$$

with the standard complex notation

$$|E(f + ig)|^2 = \overline{E(f + ig)}\, E(f + ig) = E(f)^2 + E(g)^2, \tag{3.13}$$

we can define

$$e_n^2(\Phi_n^*) = \sum_{k=0}^{n} \left| E\left(\varphi_{nk}^*\right) \right|^2. \tag{3.14}$$

THEOREM 1. *For every n, let $\Phi_n^* = \{\varphi_{n0}^*, \ldots, \varphi_{nn}^*\}$ and $\Phi_n = \{\varphi_{n0}, \ldots, \varphi_{nn}\}$ be bases of M_n^* and M_n, respectively, satisfying*

(i) *$(\varphi_{n0}, \ldots, \varphi_{nn})^T = A_n(\varphi_{n0}^*, \ldots, \varphi_{nn}^*)^T$ where A_n is an $n \times n$ complex unitary matrix, i.e., $A^H = A^{-1}$;*

(ii) *$\sum_{k=0}^n |E(\varphi_{nk}^*)|^2 = \sum_{k=0}^n |E(\varphi_{nk}^* \circ u_\alpha)|^2 = \sum_{k=0}^n |E(\varphi_{nk}^* \circ u_\alpha \circ v)|^2$ for all $\alpha \in [0, 2\pi)$;*

(iii) *there exists a $c \in \mathbf{R}$ such that $\|\varphi_{nk}\|_\infty \leqslant c$ for all n, k.*

Then, the estimate $s_E(\Phi)$ associated with the family $\Phi = \{\Phi_0, \Phi_1, \ldots\}$ is compatible with the symmetry group $\mathscr{U}(S_2)$.

Proof. Let us remark that the operators

$$f^* \in M_n^* \to E(f^*) \in \mathbf{C},$$
$$f^* \in M_n^* \to f^* \circ u_\alpha \in M_n^*,$$
$$f^* \in M_n^* \to f^* \circ u_\alpha \circ v \in M_n^*$$

are linear and therefore "pass through the matrix A_n".

Moreover, $E(\varphi_{nk})$ and $E(\varphi_{nk} \circ u)$ are real and therefore

$$\begin{aligned}
\sum_{k=0}^{n} & E(\varphi_{nk} \circ u_\alpha)^2 \\
&= \left(\overline{E(\varphi_{n0} \circ u_\alpha)}, \dots, \overline{E(\varphi_{nn} \circ u_\alpha)}\right)\left(E(\varphi_{n0} \circ u_\alpha), \dots, E(\varphi_{nn} \circ u_\alpha)\right)^T \\
&= \left(\overline{E(\varphi_{n0}^* \circ u_\alpha)}, \dots, \overline{E(\varphi_{nn}^* \circ u_\alpha)}\right) A_n^H A_n \left(E(\varphi_{n0}^* \circ u_\alpha), \dots, E(\varphi_{nn}^* \circ u_\alpha)\right)^T \\
&= \sum_{k=0}^{n} \left|E(\varphi_{nk}^* \circ u_\alpha)\right|^2 = \sum_{k=0}^{n} \left|E(\varphi_{nk}^*)\right|^2 \\
&= \left(\overline{E(\varphi_{n0}^*)}, \dots, \overline{E(\varphi_{nn}^*)}\right)\left(E(\varphi_{n0}^*), \dots, E(\varphi_{nn}^*)\right)^T \\
&= \left(\overline{E(\varphi_{n0})}, \dots, \overline{E(\varphi_{nn})}\right) A_n A_n^H \left(E(\varphi_{n0}), \dots, E(\varphi_{nn})\right)^T = \sum_{k=0}^{n} E(\varphi_{nk})^2,
\end{aligned}$$

given that A_n is unitary. Similarly, it can be shown that

$$\sum_{k=0}^{n} E(\varphi_{nk} \circ u_\alpha \circ v)^2 = \sum_{k=0}^{n} E(\varphi_{nk})^2,$$

and therefore it follows in a straightforward way that $s_E(\Phi)$ is compatible with $\mathscr{U}(S_2)$. □

Now let us consider the complex polynomials

$$\varphi_{nk}^* = (x + iy)^{n-k}(x - iy)^k \in M_n^* \tag{3.15}$$

obtained from the monomials $x^{n-k}y^k$ by a linear transformation with Jacobian

$$J = \begin{vmatrix} 1 & i \\ 1 & -i \end{vmatrix} = -2i,$$

so that $\varphi_{n0}^*, \dots, \varphi_{nn}^*$ are linearly independent in M_n^*.

Also,

$$\begin{aligned}
(\varphi_{nk}^* \circ u_\alpha)(x, y) &= \varphi_{nk}^*(x\cos\alpha - y\sin\alpha,\ x\sin\alpha + y\cos\alpha) \\
&= e^{i(n-k)\alpha}(x + iy)^{n-k} e^{-ik\alpha}(x - iy)^k = e^{i(n-2k)\alpha}\varphi_{nk}^*(x, y),
\end{aligned}$$

thus

$$\sum_{k=0}^{n} \left|E(\varphi_{nk}^* \circ u_\alpha)\right|^2 = \sum_{k=0}^{n} \left|E(\varphi_{nk}^*)\right|^2. \tag{3.16}$$

Similarly,

$$\begin{aligned}
(\varphi_{nk}^* \circ u_\alpha \circ v)(x, y) = (\varphi_{nk}^* \circ u_\alpha)(x, -y) &= e^{i(n-2k)\alpha}(x - iy)^{n-k}(x + iy)^k \\
&= e^{i(n-2k)\alpha}\varphi_{n,n-k}^*(x, y),
\end{aligned}$$

and then

$$\sum_{k=0}^{n} \left| E\left(\varphi_{nk}^{*} \circ u_{\alpha} \circ v\right) \right|^{2} = \sum_{k=0}^{n} \left| E\left(\varphi_{nk}^{*}\right) \right|^{2}. \tag{3.17}$$

Therefore, for each n, the family $\Phi_n^* = \{\varphi_{n0}^*, \ldots, \varphi_{nn}^*\}$ is a basis of M_n^* which satisfies the hypothesis (ii) of Theorem 1.

For $k < n/2$ let us define

$$\begin{aligned} \varphi_{nk} &= \frac{1}{\sqrt{2}}\left(\varphi_{nk}^{*} + \varphi_{n,n-k}^{*}\right) \\ &= \frac{1}{\sqrt{2}}\left(x^2 + y^2\right)^{k}\left[(x+iy)^{n-2k} + (x-iy)^{n-2k}\right], \end{aligned} \tag{3.18}$$

$$\begin{aligned} \varphi_{n,n-k} &= \frac{1}{\sqrt{2}\,i}\left(\varphi_{nk}^{*} - \varphi_{n,n-k}^{*}\right) \\ &= \frac{1}{\sqrt{2}\,i}\left(x^2 + y^2\right)^{k}\left[(x+iy)^{n-2k} - (x-iy)^{n-2k}\right], \end{aligned} \tag{3.19}$$

and if n is even,

$$\varphi_{n,n/2} = \varphi_{n,n/2}^{*} = \left(x^2 + y^2\right)^{n/2}. \tag{3.20}$$

Then, $\Phi_n = \{\varphi_{n0}, \ldots, \varphi_{nn}\}$ is formed by polynomials with real coefficients and is a basis of M_n. Also the matrix A_n of Theorem 1 that relates the elements of Φ_n and Φ_n^* is a unitary matrix, because the matrices

$$\begin{pmatrix} \dfrac{1}{\sqrt{2}}, & \dfrac{1}{\sqrt{2}} \\[2ex] \dfrac{1}{\sqrt{2}\,i}, & \dfrac{-1}{\sqrt{2}\,i} \end{pmatrix}$$

that relate the pairs $(\varphi_{nk}, \varphi_{n,n-k})$ and $(\varphi_{nk}^*, \varphi_{n,n-k}^*)$ are unitary. Moreover, it can easily be shown that

$$\|\varphi_{nk}\|_\infty = \|\varphi_{n,n-k}\|_\infty = \sqrt{2}, \qquad k < n/2,$$

and $\|\varphi_{n,n/2}\|_\infty = 1$ for n even.

Using the results above, and applying Theorem 1, we deduce the following

THEOREM 2. *Let* $\Phi = \{\Phi_0, \Phi_1, \ldots\}$ *where, for each* n, $\Phi_n = \{\varphi_{n0}, \ldots, \varphi_{nn}\}$ *is the basis of* M_n *defined by* (3.18), (3.19) *and* (3.20). *Then the estimate* $s_E(\Phi)$ *defined by* (3.5) *is an estimate of goodness of cubature formulae for the unit circle that is compatible with the symmetry group* $\mathcal{U}(S_2)$.

Following the proof of Theorem 1, we can also deduce that

$$s_E^2(\Phi) = \sum_{n=0}^{\infty} q_n^2 \sum_{k=0}^{n} E(\varphi_{nk})^2 = \sum_{n=0}^{\infty} q_n^2 \sum_{k=0}^{n} \left| E\left(\varphi_{nk}^{*}\right) \right|^2, \tag{3.21}$$

and therefore the estimate $s_E(\Phi)$ can be calculated using any of these two expressions.

TABLE 1

Formula	D	N	$s_E(\Phi)$
Centroid	1	1	(−2)3.72941
$S_2: 3-1$	3	4	(−3)1.52574
$S_2: 5-1$	5	7	(−5)4.17361
$S_2: 5-2$	5	9	(−5)1.56155
$S_2: 7-1$	7	12	(−7)7.32827
$S_2: 7-2$	7	16	(−7)7.31334
$S_2: 9-1$	9	19	(−10)2.86050
$S_2: 9-3$	9	21	(−9)8.64763
$S_2: 9-5$	9	28	(−10)5.70093
$S_2: 11-1$	11	28	(−11)7.64307
$S_2: 11-2$	11	28	(−12)2.00147
$S_2: 11-3$	11	28	(−11)4.55280
$S_2: 11-4$	11	32	(−11)7.64002
$S_2: 13-1$	13	37	(−14)3.05146
$S_2: 13-2$	13	41	(−14)1.03972
$S_2: 15-1$	15	44	(−15)2.92306
$S_2: 15-2$	15	48	(−15)2.88250
$S_2: 17-1$	17	61	(−20)4.97655

Table 1 shows the values of $s_E(\Phi)$ for some cubature formulae (1.5) for the unit circle with $w(x, y) = 1/\pi$. The nomenclature of these formulae corresponds to the one in [6, pp. 277–289]. N stands for the number of nodes and D for the degree of precision.

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