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Properties of the Polynomials Associated with the Jacobi Polynomials

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Abstract. Power forms and Jacobi polynomial forms are found for the polynomials $W_n^{(\alpha,\beta)}$ associated with Jacobi polynomials. Also, some differential-difference equations and evaluations of certain integrals involving $W_n^{(\alpha,\beta)}$ are given.

1. Introduction. Let w be a nonnegative function defined on an interval $[a, b]$ for which all moments

$$m_n := \int_a^b x^n w(x)\,dx, \qquad n = 0, 1, \ldots,$$

exist and are finite, $m_0 > 0$. Let $\{p_n\}$ be the monic polynomials orthogonal on $[a, b]$ with respect to w. The polynomials

$$q_n(x) := \int_a^b \frac{p_n(x) - p_n(t)}{x - t} w(t)\,dt, \qquad n = 0, 1, \ldots, \tag{1.1}$$

are called the polynomials *associated* with the p_n. As is well known, $\{p_n(x)\}$, $\{q_n(x)\}$ are linearly independent solutions of the recurrence formula

$$y_{n+1} - (x - a_n) y_n + b_n y_{n-1} = 0, \qquad n = 0, 1, \ldots, \tag{1.2}$$

where

$$\begin{aligned} &a_n := (xp_n, p_n)/(p_n, p_n), && n \geqslant 0, \\ &b_0 := m_0, \quad b_n := (p_n, p_n)/(p_{n-1}, p_{n-1}), && n \geqslant 1, \end{aligned} \tag{1.3}$$

and

$$(f, g) := \int_a^b f(x) g(x) w(x)\,dx.$$

The initial conditions are

$$p_{-1}(x) := 0, \qquad p_0(x) := 1$$

and

$$q_{-1}(x) := -1, \qquad q_0(x) := 0,$$

respectively.

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Let us consider the continued fraction

$$\frac{b_0}{x-a_0-}\,\frac{b_1}{x-a_1-}\cdots, \tag{1.4}$$

where a_n, b_n are the coefficients of (1.2) given in (1.3). The nth convergent of (1.4) is

$$\frac{b_0}{x-a_0-}\,\frac{b_1}{x-a_1-}\cdots\frac{b_{n-1}}{x-a_{n-1}}=\frac{q_n(x)}{p_n(x)}, \qquad n\geqslant 1.$$

If $[a,b]$ is a finite interval, then

$$\lim_{n\to\infty}\frac{q_n(x)}{p_n(x)}=F(x), \qquad x\notin[a,b], \tag{1.5}$$

where

$$F(x):=\int_a^b\frac{w(t)\,dt}{x-t}, \tag{1.6}$$

holds by Markov's theorem [18, p. 56]. In this case, the *functions of the second kind*,

$$f_n(x):=F(x)p_n(x)-q_n(x), \qquad n\geqslant 0, \tag{1.7}$$

represent the *minimal solution* of (1.2), normalized by $f_{-1}(x):=1$ (see [6]; or [19, pp. 53 ff.]).

Note that the set of polynomials $\{q_n\}$ belongs to a family of the *generalized associated polynomials* $\{p_n(\cdot;\,c)\}$ which are defined by

$$p_{n+1}(x;\,c)-(x-a_{n+c})p_n(x;\,c)+b_{n+c}p_{n-1}(x;\,c)=0, \qquad n=0,1,\ldots,$$
$$p_{-1}(x;\,c)=0, \qquad p_0(x;\,c)=1.$$

Here c is a fixed nonnegative integer, although often it can be taken to be an arbitrary real positive number. Obviously, q_n is a constant multiple of $p_{n-1}(\cdot;\,1)$, $n=0,1,\ldots$.

According to Favard's theorem, the set $\{q_n\}$ is orthogonal with respect to some weight function u. Nevai [14] gave a formula for the weight function of the polynomials associated with polynomials belonging to a large class which included the Jacobi polynomials. The generalized associated Legendre polynomials have been studied by Barrucand and Dickinson [3]; their weight for these polynomials was contained in Pollaczek's earlier results for his set of orthogonal polynomials with four free parameters, which includes the associated Gegenbauer polynomials (see [5, Vol. 2, Section 10.21]; or [4]). Bustoz and Ismail [4] have found the orthogonality relation for the generalized associated q-ultraspherical (or q-Gegenbauer) polynomials. Askey and Wimp [2] determined the weight function and found an explicit formula for the generalized associated Laguerre and Hermite polynomials. A theory developed by Grosjean [7], [8], [9] permits us to deduce an explicit formula for the weight function u and a procedure for obtaining the basic interval $[c,d]\subset[a,b]$ associated with the orthogonality property of the sequence $\{q_n\}$. These results are obtained for an arbitrary weight w being piecewise continuous as well as containing discrete mass points. For instance, if w is piecewise continuous, then $[c, d] = [a, b]$ and

$$u(x) = \frac{w(x)}{\left[\dfrac{1}{m_0} ⨍_a^b \dfrac{w(t)\,dt}{t-x}\right]^2 + \left[\dfrac{\pi w(x)}{m_0}\right]^2}, \qquad a \leqslant x \leqslant b, \tag{1.8}$$

with $⨍$ meaning the Cauchy principal value integral.

In this paper, we give some properties of the polynomials $W_n^{(\alpha,\beta)}$ associated with the classical Jacobi polynomials. More specifically, we show that $W_n^{(\alpha,\beta)}$, $n \geqslant 0$, satisfies a linear nonhomogeneous differential equation of second order. Further, we obtain explicit formulae for $W_n^{(\alpha,\beta)}$. Also, we give some differential-difference equations and evaluate certain integrals involving $W_n^{(\alpha,\beta)}$. All these results can be found in Section 2. Similar (neater looking, however) results for the polynomials associated with Gegenbauer polynomials are given in Section 3.

2. Polynomials Associated with the Jacobi Polynomials. We use the standard notation $P_n^{(\alpha,\beta)}$ for the Jacobi polynomials,

$$P_n^{(\alpha,\beta)}(x) := \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, n+\lambda \\ \alpha+1 \end{matrix}\,\middle|\, \frac{1-x}{2}\right), \tag{2.1}$$
$$\alpha > -1,\ \beta > -1,\ n \geqslant 0,$$

where $\lambda := \alpha + \beta + 1$, and $(c)_n := \Gamma(c+n)/\Gamma(c)$; they are orthogonal on $[-1,1]$ with the weight function

$$w^{(\alpha,\beta)}(x) := (1-x)^\alpha (1+x)^\beta.$$

The monic Jacobi polynomials are defined by

$$\overline{P}_n^{(\alpha,\beta)} := 2^n \frac{n!}{(n+\lambda)_n} P_n^{(\alpha,\beta)}, \qquad n \geqslant 0. \tag{2.2}$$

In the sequel, we give results related to the polynomials (2.1), as they seem to be in much wider use than (2.2).

Slightly modifying Grosjean's notation [8], [9], we define the polynomials associated with $P_n^{(\alpha,\beta)}$ by

$$W_n^{(\alpha,\beta)}(x) := \frac{1}{m_0} \int_{-1}^1 \frac{P_n^{(\alpha,\beta)}(x) - P_n^{(\alpha,\beta)}(t)}{x-t} w^{(\alpha,\beta)}(t)\,dt, \qquad n \geqslant 0, \tag{2.3}$$

where

$$m_0 := \int_{-1}^1 w^{(\alpha,\beta)}(t)\,dt = 2^\lambda B(\alpha+1, \beta+1),$$

so that

$$W_0^{(\alpha,\beta)}(x) = 0, \qquad W_1^{(\alpha,\beta)}(x) = \frac{\lambda+1}{2},$$
$$W_2^{(\alpha,\beta)}(x) = \frac{(\lambda+2)}{8(\lambda+1)}\left[(\lambda+1)(\lambda+3)x + \alpha^2 - \beta^2\right],$$

etc.

In (2.3), the factor $1/m_0$ is included in order to simplify the formulae presented below.

Jacobi's functions of the second kind are defined for $x \notin [-1,1]$ by ([5, Vol. 2, Section 10.8]; or [18, Section 4.6])

$$(2.4)\quad Q_n^{(\alpha,\beta)}(x) := \frac{1}{2m_0}(x-1)^{-\alpha}(x+1)^{-\beta}\int_{-1}^{1}\frac{P_n^{(\alpha,\beta)}(t)w^{(\alpha,\beta)}(t)}{x-t}\,dt, \qquad n \geqslant 0.$$

(We added the factor $1/m_0$ for convenience.) For $x \in [-1,1]$, we put

$$Q_n^{(\alpha,\beta)}(x) := \tfrac{1}{2}\left[Q_n^{(\alpha,\beta)}(x+i0) + Q_n^{(\alpha,\beta)}(x-i0)\right], \qquad n \geqslant 0.$$

Functions (2.1), (2.3), and (2.4) are related in the following way:

$$(2.5)\quad Q_n^{(\alpha,\beta)}(x) = Q_0^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x) - \tfrac{1}{2}(x-1)^{-\alpha}(x+1)^{-\beta}W_n^{(\alpha,\beta)}(x).$$

They satisfy the same recurrence relation

$$(2.6)\qquad y_{n+1} - (A_n x + B_n)y_n + C_n y_{n-1} = 0, \qquad n \geqslant 1,$$

where

$$A_n := \frac{(2n+\lambda)_2}{2(n+1)(n+\lambda)},$$

$$B_n := \frac{(\alpha^2-\beta^2)(2n+\lambda)}{2(n+1)(n+\lambda)(2n+\lambda-1)},$$

$$C_n := \frac{(n+\alpha)(n+\beta)(2n+\lambda+1)}{(n+1)(n+\lambda)(2n+\lambda-1)}.$$

Grosjean's theory [9] yields

$$(2.7)\quad \int_{-1}^{1} W_n^{(\alpha,\beta)}(x)W_p^{(\alpha,\beta)}(x)u^{(\alpha,\beta)}(x)\,dx = \frac{2^\lambda\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\lambda)\Gamma(n+\lambda)}\delta_{np}, \qquad n,p = 1,2,\ldots,$$

where

$$(2.8)\qquad u^{(\alpha,\beta)}(x) := \frac{1/w^{(\alpha,\beta)}(x)}{\left[2Q_0^{(\alpha,\beta)}(x)\right]^2 + [\pi/m_0]^2}, \qquad -1 \leqslant x \leqslant 1,$$

and

$$Q_0^{(\alpha,\beta)}(x) = \frac{1}{2m_0 w^{(\alpha,\beta)}(x)}\int_{-1}^{1}\frac{w^{(\alpha,\beta)}(t)}{x-t}\,dt$$
$$= \frac{\pi\cot(\alpha\pi)}{2m_0} - \frac{\lambda}{4\alpha w^{(\alpha,\beta)}(x)}\;{}_2F_1\!\left(\begin{matrix}1-\lambda,1\\ 1-\alpha\end{matrix}\,\middle|\,\frac{1-x}{2}\right)$$

(see (1.8) and [5, Vol. 2, Section 10.8]).

2.1. *Differential Equation.* We show that the polynomial $W_n^{(\alpha,\beta)}$, $n = 0,1,\ldots$, satisfies the nonhomogeneous linear differential equation

$$(2.9)\quad \begin{aligned}(1-x^2)\frac{d^2y}{dx^2} + [(\lambda-3)x+\alpha-\beta]\frac{dy}{dx} + (n+1)(n+\lambda-1)y(x)\\ = 2\lambda\frac{d}{dx}P_n^{(\alpha,\beta)}(x).\end{aligned}$$

Observe that $P_n^{(\alpha,\beta)}$, $Q_n^{(\alpha,\beta)}$ are linearly independent solutions of the homogeneous equation

$$(1-x^2)\frac{d^2z}{dx^2}-[(\lambda+1)x+\alpha-\beta]\frac{dz}{dx}+n(n+\lambda)z=0. \tag{2.10}$$

Substituting the right-hand side of (2.5) for z in (2.10), making use of the above remark and of the formula

$$\frac{d}{dx}Q_0^{(\alpha,\beta)}(x)=-\frac{1}{2}\lambda(x-1)^{-\alpha-1}(x+1)^{-\beta-1}, \tag{2.11}$$

which can be deduced from an identity given in [5, Vol. 2, Section 10.8], we obtain Eq. (2.9) with $y=W_n^{(\alpha,\beta)}$.

2.2. *Differential-Difference Equations.* The following identities hold:

$$\begin{aligned}
(1-x^2)&\frac{d}{dx}W_n^{(\alpha,\beta)}(x)-\lambda P_n^{(\alpha,\beta)}(x)\\
&=(n+\lambda-1)\left(\frac{\beta-\alpha}{2n+\lambda-1}-x\right)W_n^{(\alpha,\beta)}(x)\\
&\quad+\frac{2(n+\alpha)(n+\beta)}{2n+\lambda-1}W_{n-1}^{(\alpha,\beta)}(x)\\
&=(n+1)\left(x+\frac{\beta-\alpha}{2n+\lambda+1}\right)W_n^{(\alpha,\beta)}(x)-\frac{2(n+1)(n+\lambda)}{2n+\lambda+1}W_{n+1}^{(\alpha,\beta)}(x)\\
&=\frac{2(n+1)(n+\alpha)(n+\beta)}{(2n+\lambda-1)_2}W_{n-1}^{(\alpha,\beta)}(x)\\
&\quad+\frac{2(\beta-\alpha)(n+1)(n+\lambda-1)}{(2n+\lambda)^2-1}W_n^{(\alpha,\beta)}(x)\\
&\quad-\frac{2(n+1)(n+\lambda-1)_2}{(2n+\lambda)_2}W_{n+1}^{(\alpha,\beta)}(x),
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
(1+x)&\frac{d}{dx}\left[nW_n^{(\alpha,\beta)}(x)-(n+\alpha)W_{n-1}^{(\alpha,\beta)}(x)\right]\\
&+\lambda\frac{d}{dx}\left[P_n^{(\alpha,\beta)}(x)+\frac{n+\alpha}{n+\lambda-1}P_{n-1}^{(\alpha,\beta)}(x)\right]\\
&=n(n+\lambda-1)W_n^{(\alpha,\beta)}(x)+n(n+\alpha)W_{n-1}^{(\alpha,\beta)}(x),
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
(1-x)&\frac{d}{dx}\left[nW_n^{(\alpha,\beta)}(x)+(n+\beta)W_{n-1}^{(\alpha,\beta)}(x)\right]\\
&+\lambda\frac{d}{dx}\left[\frac{n+\beta}{n+\lambda-1}P_{n-1}^{(\alpha,\beta)}(x)-P_n^{(\alpha,\beta)}(x)\right]\\
&=n(n+\lambda-1)W_n^{(\alpha,\beta)}(x)-n(n+\beta)W_{n-1}^{(\alpha,\beta)}(x).
\end{aligned} \tag{2.14}$$

Clearly, the three equalities in (2.12) are equivalent. The first of them can be proved using (2.5), (2.11), and the identity

$$\begin{aligned}
(2n+\lambda-1)(1-x^2)\frac{dy_n}{dx}&+n[(2n+\lambda-1)x+\alpha-\beta]y_n(x)\\
&+2(n+\alpha)(n+\beta)y_{n-1}(x)=0,
\end{aligned}$$

satisfied by $y_n(x)=P_n^{(\alpha,\beta)}(x)$ and $y_n(x)=Q_n^{(\alpha,\beta)}(x)$ (*ibid.*).

Identity (2.13) is obtained in the following way. In the second equality of (2.12), replace n by $n-1$, subtract the resulting equation from the first equation of (2.12) multiplied by $n/(n+\alpha)$, and use the relation [16, p. 262]

$$(1-x)\frac{d}{dx}\left[(n+\lambda-1)P_n^{(\alpha,\beta)}(x)+(n+\alpha)P_{n-1}^{(\alpha,\beta)}(x)\right] = (n+\lambda-1)\left[(n+\alpha)P_{n-1}^{(\alpha,\beta)}(x)-nP_n^{(\alpha,\beta)}(x)\right].$$

Equation (2.14) readily follows from (2.13) and from the symmetry properties

$$P_n^{(\alpha,\beta)}(-x)=(-1)^nP_n^{(\beta,\alpha)}(x), \qquad W_n^{(\alpha,\beta)}(-x)=(-1)^{n-1}W_n^{(\beta,\alpha)}(x).$$

2.3. *Power Form.* The following expansion can be obtained directly from the definition (2.3):

$$W_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n-1}a_{nk}^{(\alpha,\beta)}\left(\frac{1-x}{2}\right)^k, \qquad n\geqslant 1, \tag{2.15}$$

where

$$a_{nk}^{(\alpha,\beta)} := \frac{(n+\lambda)_{k+1}(1-n)_k(\alpha+2)_{n-1}}{2(n-1)!(k+1)!(\alpha+2)_k} \times {}_4F_3\left(\begin{matrix}k+1-n, n+k+\lambda+1, \alpha+1, 1\\ k+\alpha+2, \lambda+1, k+2\end{matrix}\middle|1\right). \tag{2.16}$$

Note that substituting $1-2x/\beta$ for x in (2.15), multiplied by $2/(1+\lambda)$, and taking $\beta\to\infty$, we obtain the Askey-Wimp formula for the associated Laguerre polynomial $L_{n-1}^{\alpha}(x;1)$ (see [2]).

When $\lambda=0$, the ${}_4F_3(1)$ in (2.16) reduces to a balanced ${}_3F_2(1)$ and so can be summed, which gives

$$a_{nk}^{(\alpha,\beta)}=(-1)^k\frac{(n-k)_{2k+1}(1-\alpha)_{n-1}}{2n!k!(1-\alpha)_k}. \tag{2.17}$$

When $\lambda=1$, the ${}_4F_3(1)$ in (2.16) can be expressed in terms of a balanced ${}_3F_2(1)$ (see Eqs. (15) and (21) of [13, Section 5.2.4]) and also can be summed. The result in this case is

$$\begin{aligned} a_{nk}^{(\alpha,\beta)} &= (-1)^k\frac{(n-k+1)_{2k}}{2n!k!}\left[\frac{(1+\alpha)_n}{(\alpha)_{k+1}}+\frac{(1-\alpha)_n}{(-\alpha)_{k+1}}\right], && \alpha=-\beta\neq 0,\\ &= (-1)^k\frac{(n-k+1)_{2k}}{(k!)^2}\sum_{m=k+1}^{n}\frac{1}{m}, && \alpha=\beta=0.\end{aligned}$$

If $\lambda\neq 0$ and the ${}_4F_3(1)$ in (2.16) is transformed using identities (2.4.1.7) and (2.4.1.2) from [17], we obtain

$$a_{nk}^{(\alpha,\beta)}=\frac{\lambda(n+\lambda)_{k+1}(1-n)_k(\alpha+2)_{n-1}}{(n-1)!k!(\alpha+2)_k(n+k+1)(n-k+\lambda-1)} \times {}_4F_3\left(\begin{matrix}k+1-n, n+k+\lambda+1, k-\alpha+1, 1\\ k+\alpha+2, n+k+2, k-n-\lambda+2\end{matrix}\middle|1\right). \tag{2.18}$$

This result, as well as (2.17), can also be deduced from the recurrence formula

$$(n+k+1)(n-k+\lambda-1)a_{nk}^{(\alpha,\beta)}+(k+1)(k-\alpha+1)a_{n,k+1}^{(\alpha,\beta)}$$
$$=\frac{\lambda(n+\lambda)_{k+1}(1-n)_k(\alpha+2)_{n-1}}{(n-1)!k!(\alpha+2)_k},\qquad 0\leqslant k\leqslant n-2,$$
$$a_{n,n-1}^{(\alpha,\beta)}=\frac{(n+\lambda)_n(-1)^{n-1}}{2n!},$$

which is obtained by substituting the right-hand side of (2.15) for y in the differential equation (2.9).

An easy consequence of (2.15), (2.16) is

$$W_n^{(\alpha,\beta)}(1)=\frac{(n+\lambda)(\alpha+2)_{n-1}}{2(n-1)!}\,{}_4F_3\left(\begin{matrix}1-n,n+\lambda+1,\alpha+1,1\\ \alpha+2,\lambda+1,2\end{matrix}\middle|1\right),$$

which by Eqs. (15), (2), and (21) from [13, Section 5.2.4] can be simplified to

$$\begin{aligned}W_n^{(\alpha,\beta)}(1)&=\frac{\lambda}{2\alpha}\left[\frac{(\alpha+1)_n}{n!}-\frac{(\beta+1)_n}{(\lambda)_n}\right], &&\alpha\neq 0,\ \lambda\neq 0,\\ &=\frac{\beta+1}{2}\sum_{k=1}^{n}\left[\frac{1}{k+\beta}+\frac{1}{k}\right], &&\alpha=0,\\ &=\frac{(1-\alpha)_{n-1}}{2(n-1)!}, &&\lambda=0.\end{aligned}$$

2.4. *Jacobi Polynomial Form*. Inserting the expansion (see [13, Section 11.3.4])

$$\left(\frac{1-x}{2}\right)^k=\frac{(1+\alpha)_k}{(1+\lambda)_k}\sum_{j=0}^{k}\frac{(-k)_j(\lambda+2j)(1+\lambda)_{j-1}}{(1+\alpha)_j(\lambda+k+1)_j}P_j^{(\alpha,\beta)}(x)$$

into (2.15) yields the formula

$$W_n^{(\alpha,\beta)}=\sum_{k=0}^{n-1}b_{nk}^{(\alpha,\beta)}P_k^{(\alpha,\beta)},\qquad n\geqslant 1, \tag{2.19}$$

in which

$$\begin{aligned}b_{nk}^{(\alpha,\beta)}:=&\ \frac{(-1)^{n-1}(1+\alpha)_n(n+\lambda)_n(1-n)_k(1+\lambda)_{k-1}(\lambda+2k)}{2(n-1)!(1+\lambda)_{n-1}(1+\alpha)_k(n+\lambda)_k}\\ &\times\sum_{p=0}^{n-k-1}\frac{(1+k-n)_p(1-n-k-\lambda)_p}{p!(n-p+\alpha)(n-p)(1-\lambda-2n)_p}\\ &\times{}_4F_3\left(\begin{matrix}-p,2n+\lambda-p,\alpha+1,1\\ n-p+\alpha+1,\lambda+1,n-p+1\end{matrix}\middle|1\right).\end{aligned} \tag{2.20}$$

A recurrence relation for the coefficients $b_{nk}^{(\alpha,\beta)}$ can be constructed by a method given in [12]; this result may also be obtained using another approach of Askey and Gasper [1]. Note that both methods start from the differential equation (2.9). We have

$$\frac{(k+\lambda-1)_2(n+k)(n-k+\lambda)(2k+\lambda+1)}{2k+\lambda-2} b_{n,k-1}^{(\alpha,\beta)}$$
$$+(\alpha-\beta)(k+\lambda)[(n+1)(n+\lambda-1)+3k(k+\lambda)] b_{nk}^{(\alpha,\beta)}$$
$$-\frac{(k+\alpha+1)(k+\beta+1)(n+k+2\lambda)(n-k-\lambda)(2k+\lambda-1)}{2k+\lambda+2} b_{n,k+1}^{(\alpha,\beta)} = 0,$$

(2.21) $$1 \leqslant k \leqslant n-1,$$

$$b_{nn}^{(\alpha,\beta)} := 0, \qquad b_{n,n-1}^{(\alpha,\beta)} := \frac{(2n+\lambda-2)_2}{2n(n+\lambda-1)}.$$

2.5. *Beta Integral.* In this subsection we examine the integral

(2.22) $$J_n \equiv J_n(\alpha,\beta;\mu,\nu) := \int_{-1}^{1} (1-t)^{\mu}(1+t)^{\nu} W_n^{(\alpha,\beta)}(t)\,dt,$$
$$\operatorname{Re}\mu > -1,\ \operatorname{Re}\nu > -1,\ n \geqslant 0,$$

which we call a beta integral of $W_n^{(\alpha,\beta)}$. Inserting (2.15), (2.16) into (2.22), one obtains

(2.23)
$$J_n = 2^{\sigma-1}\frac{B(\mu+1,\nu+1)(\alpha+2)_{n-1}}{(n-1)!}$$
$$\times \sum_{k=0}^{n-1} \frac{(n+\lambda)_{k+1}(1-n)_k(\mu+1)_k}{(k+1)!(\alpha+2)_k(\sigma+1)_k}$$
$$\times\ {}_4F_3\left(\begin{matrix} k+1-n, n+k+\lambda+1, \alpha+1, 1 \\ k+\alpha+2, \lambda+1, k+2 \end{matrix}\middle| 1\right),$$

where $\sigma := \mu+\nu+1$.

When $\lambda = 0$ or 1, the ${}_4F_3(1)$ in (2.23) can be summed (see Subsection 2.3). The result is

$$J_n = \frac{C(1-\alpha)_{n-1}}{(n-1)!}\ {}_3F_2\left(\begin{matrix} 1-n, n+1, \mu+1 \\ 1-\alpha, \sigma+1 \end{matrix}\middle| 1\right), \qquad \lambda = 0,$$
$$= \frac{C}{\alpha n!}\left[(1+\alpha)_n\ {}_3F_2\left(\begin{matrix} -n, n+1, \mu+1 \\ 1+\alpha, \sigma+1 \end{matrix}\middle| 1\right)\right.$$
$$\left. -(1-\alpha)_n\ {}_3F_2\left(\begin{matrix} -n, n+1, \mu+1 \\ 1-\alpha, \sigma+1 \end{matrix}\middle| 1\right)\right], \qquad \alpha = -\beta \neq 0,$$
$$= 2C\sum_{k=0}^{n-1} \frac{(-n)_k(n+1)_k(\mu+1)_k}{(k!)^2(\sigma+1)_k} \sum_{m=k+1}^{n} \frac{1}{m}, \qquad \alpha = \beta = 0,$$

where $C := 2^{\sigma-1}B(\mu+1,\nu+1)$.

Using the identity

$$\frac{d}{dt}[(1-t^2)z(t)] + [(\sigma+1)t + \mu - \nu]z(t) = 0,$$

in which $z(t) := (1-t)^{\mu}(1+t)^{\nu}$, and the last equation of (2.12), we obtain the second-order recurrence formula

$$\begin{aligned}(2.24)\qquad &\frac{2(n+1)(n+\lambda)(n+\lambda+\sigma)}{(2n+\lambda)_2}J_{n+1} \\ &+\left[(\alpha-\beta)\frac{2(n+1)(n+\lambda-1)-(\lambda-1)(\sigma+1)}{(2n+\lambda)^2-1}+\mu-\nu\right]J_n \\ &-\frac{2(n+\alpha)(n+\beta)(n-\sigma)}{(2n+\lambda-1)_2}J_{n-1} = \lambda I_n, \qquad n \geqslant 1,\end{aligned}$$

where

$$(2.25)\qquad I_n \equiv I_n(\alpha,\beta;\mu,\nu) := \int_{-1}^{1}(1-t)^{\mu}(1+t)^{\nu}P_n^{(\alpha,\beta)}(t)\,dt, \qquad n \geqslant 0.$$

The initial values are

$$J_0 = 0, \qquad J_1 = C(\lambda+1).$$

Integral (2.25) is studied in [10]. It is known that

$$(2.26)\qquad I_n = 2C\frac{(1+\alpha)_n}{n!}\,{}_3F_2\left(\begin{matrix}-n, n+\lambda, \mu+1\\ \alpha+1, \sigma+1\end{matrix}\middle|\,1\right).$$

See [13, Section 11.3.3]; or [10, p. 152]. In [11], we have shown that (2.25) satisfies

$$\begin{aligned}&\frac{(n+1)(n+\lambda)(n+\sigma+1)}{(2n+\lambda)_2}I_{n+1}+[A(n)-A(n-1)-n-\lambda+\mu+1]I_n \\ (2.27)\qquad &-\frac{(n+\alpha)(n+\beta)(n+\lambda-\sigma-1)}{(2n+\lambda-1)_2}I_{n-1} = 0, \qquad n \geqslant 1,\end{aligned}$$

with the starting values

$$I_0 = 2C, \qquad I_1 = \left[\alpha+1-\frac{(\lambda+1)(\mu+1)}{\sigma+1}\right]I_0.$$

Here, $A(n) := (n+\beta+1)(n+\lambda)(n+\lambda-\sigma)/(2n+\lambda+1)$.

It is difficult to decide which method of computation of J_n using Eq. (2.24) is numerically stable. The asymptotic approximations for a fundamental set $s_1(n)$, $s_2(n)$ for the homogeneous form of (2.24) may be obtained from the Birkhoff-Trjitzinsky theory (see [19, Section B2]). We have

$$s_1(n) \sim n^{-2\mu-\alpha-2}, \qquad s_2(n) \sim (-1)^n n^{-2\nu-\beta-2}, \qquad n \to \infty.$$

However, it is rather difficult to gain asymptotic information about J_n, n large. Satisfactory results can probably be achieved by the use of (2.24) in the forward direction for moderately large n, provided $|\mu-\nu|$ and $|\alpha-\beta|$ are not very large.

The same statement seems to be true for the equation (2.27), which has a fundamental set $t_1(n)$, $t_2(n)$ with the property

$$t_1(n) \sim n^{\alpha-2\mu-1}, \qquad t_2(n) \sim (-1)^n n^{\beta-2\nu-1}, \qquad n \to \infty.$$

If $\mu = 0$, (2.24), (2.27) may be replaced by the first-order recurrence relations

$$\begin{aligned} n(n+\lambda+\nu)J_n &+ (n+\alpha)(n-\nu-1)J_{n-1} \\ &= \frac{\lambda}{2}\left[(n+\lambda)I^*_{n-1} + (n+\alpha)I^*_n\right], \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} (n+\lambda+1)(n+\nu+1)I^*_n &+ (n+\beta+1)(n+\lambda-\nu)I^*_{n-1} \\ &= 2^{\nu+1}(2n+\lambda+1)(\alpha+1)_n/n!, \end{aligned} \tag{2.29}$$

respectively. Here $I^*_n := I_n(\alpha+1, \beta+1; 0, \nu)$. In the derivation of (2.28), Eq. (2.13) was used. Equation (2.29) was obtained in [11].

3. Polynomials Associated with the Gegenbauer Polynomials. The Gegenbauer polynomials C^γ_n are a specialization of the Jacobi polynomials

$$\begin{aligned} C^\gamma_n &:= \frac{(2\gamma)_n}{(\gamma+1/2)_n} P_n^{(\gamma-1/2,\gamma-1/2)}, \qquad \gamma > -1/2,\ \gamma \neq 0, \\ C^0_n &:= \frac{2(n-1)!}{(1/2)_n} P_n^{(-1/2,-1/2)}. \end{aligned} \tag{3.1}$$

We shall assume in the sequel that $\gamma \neq 0$. (It is well known that the nth polynomial associated with C^0_n is a multiple of the Chebyshev polynomial U_{n-1}.)

The polynomials associated with (3.1) are defined by

$$V^\gamma_n(x) := \frac{1}{K_\gamma}\int_{-1}^{1} \frac{C^\gamma_n(x) - C^\gamma_n(t)}{x-t}(1-t^2)^{\gamma-1/2}\,dt, \qquad \gamma > -\frac{1}{2},\ \gamma \neq 0,\ n \geqslant 0,$$

where

$$K_\gamma := 2\gamma\int_{-1}^{1}(1-t^2)^{\gamma-1/2}\,dt = 2\sqrt{\pi}\,\Gamma(\gamma+1/2)/\Gamma(\gamma).$$

The properties of V^γ_n, listed below, are obtained by means of the equation

$$V^\gamma_n = \frac{(2\gamma+1)_{n-1}}{(\gamma+1/2)_n} W_n^{(\gamma-1/2,\gamma-1/2)}$$

from the results on $W_n^{(\alpha,\beta)}$ given in Section 2.

The orthogonality relation of the polynomials V^γ_n reads

$$\int_{-1}^{1} V^\gamma_n(x)V^\gamma_p(x)u^\gamma(x)\,dx = \frac{K_\gamma(2\gamma)_n}{2n!(n+\gamma)}\delta_{np}, \qquad n,p = 1,2,\ldots,$$

where

$$u^\gamma(x) := \frac{(1-x^2)^{1/2-\gamma}}{\left[x\,{}_2F_1(\gamma+1/2,1/2;3/2;x^2)\right]^2 + \pi^2/K^2_\gamma}, \qquad -1 \leqslant x \leqslant 1.$$

The value at $x = 1$ is

$$\begin{aligned} V^\gamma_n(1) &= [(2\gamma)_n/n! - 1]/(2\gamma-1), \quad && \gamma \neq 1/2, \\ &= \sum_{k=1}^{n}\frac{1}{k}, && \gamma = 1/2. \end{aligned}$$

3.1. *Recurrence Relation.*

$$V_{n+1}^{\gamma}(x) - 2\frac{n+\gamma}{n+1}xV_n^{\gamma}(x) + \frac{n+2\gamma-1}{n+1}V_{n-1}^{\gamma}(x) = 0, \qquad n \geqslant 1,$$
$$V_0^{\gamma}(x) = 0, \qquad V_1^{\gamma}(x) = 1.$$

3.2. *Differential Properties.*

$$\left[(1-x^2)\frac{d^2}{dx^2} + (2\gamma-3)x\frac{d}{dx} + (n+1)(n+2\gamma-1)\right]V_n^{\gamma}(x) = 2\frac{d}{dx}C_n^{\gamma}(x),$$

$$\begin{aligned}(1-x^2)\frac{d}{dx}V_n^{\gamma}(x) - C_n^{\gamma}(x) &= (n+1)\left[xV_n^{\gamma}(x) - V_{n+1}^{\gamma}(x)\right]\\ &= (n+2\gamma-1)\left[V_{n-1}^{\gamma}(x) - xV_n^{\gamma}(x)\right]\\ &= \frac{(n+1)(n+2\gamma-1)}{2(n+\gamma)}\left[V_{n-1}^{\gamma}(x) - V_{n+1}^{\gamma}(x)\right].\end{aligned}$$

3.3. *Power Form.*

$$V_n^{\gamma}(x) = \frac{(1+\gamma)_{n-1}}{n!}\sum_{k=0}^{[(n-1)/2]}\frac{(-n)_{2k+1}(2x)^{n-2k-1}}{k!(k-n)(1-\gamma-n)_k} \times {}_3F_2\left(\left.\begin{matrix}-k, n+\gamma-k, 1\\ 1-k-\gamma, n+1-k\end{matrix}\right|1\right), \qquad n \geqslant 1.$$

3.4. *Gegenbauer Polynomial Form.* The following expansion was first obtained by Watson (see [5, Vol. 1, Section 3.15.2]) and then rediscovered by Paszkowski [15], using another approach:

$$V_n^{\gamma} = \sum_{k=0}^{[(n-1)/2]}\frac{(n+\gamma-2k-1)(1-\gamma)_k(2\gamma+n-k)_k}{(n-k)_{k+1}(\gamma)_{k+1}}C_{n-2k-1}^{\gamma}, \qquad n \geqslant 1. \tag{3.2}$$

3.5. *Beta Integral.* Let

$$i_n \equiv i_n(\gamma;\mu,\nu) := \int_{-1}^{1}(1-t)^{\mu}(1+t)^{\nu}C_n^{\gamma}(t)\,dt, \tag{3.3}$$

$$j_n \equiv j_n(\gamma;\mu,\nu) := \int_{-1}^{1}(1-t)^{\mu}(1+t)^{\nu}V_n^{\gamma}(t)\,dt, \tag{3.4}$$

$$\operatorname{Re}\mu > -1, \operatorname{Re}\nu > -1, n \geqslant 0.$$

From (3.1), (2.25), and (2.26),

$$i_n = 2^{\sigma}\frac{(2\gamma)_n B(\mu+1,\nu+1)}{n!}\,{}_3F_2\left(\left.\begin{matrix}-n, n+2\gamma, \mu+1\\ \gamma+1/2, \sigma+1\end{matrix}\right|1\right), \tag{3.5}$$

with $\sigma := \mu + \nu + 1$. Inserting (3.2) into (3.4), and using (3.5) yields

$$\begin{aligned}j_n &= 2^{\sigma}B(\mu+1,\nu+1)\\ &\times \sum_{k=0}^{[(n-1)/2]}\frac{(1-\gamma)_k(2\gamma+n-k)_k(n+\gamma-2k-1)(2\gamma)_{n-2k-1}}{(n-k)_{k+1}(\gamma)_{k+1}(n-2k-1)!}\\ &\times {}_3F_2\left(\left.\begin{matrix}1+2k-n, n+2\gamma-2k-1, \mu+1\\ \gamma+1/2, \sigma+1\end{matrix}\right|1\right).\end{aligned} \tag{3.6}$$

The beta integrals (3.4) satisfy the nonhomogeneous recurrence relation

$$(n+1)(n+2\gamma+\sigma)j_{n+1}+2(\mu-\nu)(n+\gamma)j_n-(n-\sigma)(n+2\gamma-1)j_{n-1}$$
$$=2(n+\gamma)i_n, \qquad n\geqslant 1,$$

with the starting values $j_0=0$, $j_1=2^{\sigma}B(\mu+1,\nu+1)$. The quantities (3.3) obey

$$(n+1)(n+\sigma+1)i_{n+1}+2(\mu-\nu)(n+\gamma)i_n$$
$$-(n+2\gamma-1)(n+2\gamma-\sigma-1)i_{n-1}=0, \qquad n\geqslant 1,$$
$$i_0=2^{\sigma}B(\mu+1,\nu+1), \qquad i_1=2\gamma(\nu-\mu-1)i_0/(\sigma+1).$$

When $\mu=0$, we have the following first-order relationships:

$$n(n+2\gamma+\nu)j_n+(n-\nu-1)(n+2\gamma-1)j_{n-1}=2\gamma\left(i^*_{n-1}+i^*_{n-2}\right),$$
$$(n+\nu+1)i^*_n+(n+2\gamma-\nu)i^*_{n-1}=2^{\nu+1}(2n+2\gamma+1)(2\gamma+2)_{n-1}/n!,$$
$$n\geqslant 1,$$

where $i^*_n:=i_n(\gamma+1;0,\nu)$.

Let us consider the integral

$$h_n\equiv h_n(\gamma;\mu):=j_{2n+1}(\gamma;\mu,\mu)$$
$$=\int_{-1}^{1}(1-t^2)^{\mu}V^{\gamma}_{2n+1}(t)\,dt, \qquad \mathrm{Re}\mu>-1,\ n\geqslant 0.$$

We will show that

$$h_n=\frac{B(1/2,\mu+1)(\gamma-\mu-1/2)_n(1+\gamma)_n}{(n+1)!(\mu+3/2)_n}$$

$$\times\ {}_4F_3\left(\begin{matrix}-n,n+\gamma+\mu+3/2,1-\gamma,1\\ 3/2-\gamma+\mu-n,1+\gamma,n+2\end{matrix}\middle|1\right) \tag{3.7}$$

$$\text{for } \mu-\gamma+1/2\notin\{0,1,\dots,n-1\},$$

$$=(-1)^m\frac{K_{\gamma}(\gamma+1/2)_m(2\gamma)_m(1-\gamma)_{n-m}(n+2\gamma+m+1)_{n-m}}{2(n+1)_{n+1}(\gamma)_{n+1}} \tag{3.8}$$

$$\text{for } \mu-\gamma+1/2=m\in\{0,1,\dots,n\}.$$

Note that for $\mu=\gamma+n-1/2$, the same result is furnished by (3.7) and (3.8). (The function ${}_4F_3(1)$ in (3.7) reduces then to a balanced ${}_3F_2(1)$ and so can be summed.) Formula (3.8) was given by Paszkowski [15]; our proof of this form is based on a different idea. Formula (3.7) seems to be new.

We start with

$$h_n=2^{2\mu+1}B(\mu+1,\mu+1)\sum_{k=0}^{n}\frac{(1-\gamma)_k(-2\gamma-2n)_k(2k-2n-\gamma)(2\gamma)_{2n-2k}}{(\gamma)_{k+1}(-1-2n)_{k+1}(2n-2k)!}f_k, \tag{3.9}$$

where

$$f_k:={}_3F_2\left(\begin{matrix}2k-2n,2n-2k+\gamma,\mu+1\\ \gamma+1/2,2\mu+2\end{matrix}\middle|1\right)$$

(cf. (3.6)).

Let us assume first that $\mu - \gamma + 1/2 \notin \{0, 1, \ldots, n-1\}$. By virtue of Watson's theorem ([5, Vol. 1, Section 4.4]; or [13, Section 5.2.4]; or [17, p. 54]), we have

$$f_k = \frac{(\gamma - \mu - 1/2)_{n-k}(1/2)_{n-k}}{(\mu + 3/2)_{n-k}(\gamma + 1/2)_{n-k}}. \tag{3.10}$$

After a little algebra we obtain

$$h_n = \frac{B(1/2, \mu + 1)(1 + \gamma)_{n-1}(\gamma - \mu - 1/2)_n(2n + \gamma)}{n!(\mu + 3/2)_n(2n + 1)} F, \tag{3.11}$$

where

$$\begin{aligned} F := {} & \frac{1}{\gamma + 2n} \sum_{k=0}^{n} \frac{(2n - 2k + \gamma)(-1/2 - \mu - n)_k(-2\gamma - 2n)_k(1 - \gamma)_k(-n)_k}{(3/2 - \gamma + \mu - n)_k(1 + \gamma)_k(-2n)_k(1 - \gamma - n)_k} \\ = {} & {}_7F_6\left(\left. \begin{matrix} -\gamma - 2n, 1 - \gamma/2 - n, -1/2 - \mu - n, -2\gamma - 2n, 1 - \gamma, 1, -n \\ -\gamma/2 - n, 3/2 - \gamma + \mu - n, 1 + \gamma, -2n, -\gamma - 2n, 1 - \gamma - n \end{matrix} \right| 1 \right). \end{aligned}$$

The above function ${}_7F_6(1)$ is well-poised; according to Whipple's theorem [17, p. 61] it can be expressed in terms of a balanced ${}_4F_3(1)$, namely

$$F = \frac{(2n + 1)(n + \gamma)}{(n + 1)(2n + \gamma)}\, {}_4F_3\left(\left. \begin{matrix} -n, n + \gamma + \mu + 3/2, 1 - \gamma, 1 \\ 3/2 - \gamma + \mu - n, 1 + \gamma, n + 2 \end{matrix} \right| 1 \right).$$

Using this result in (3.11) implies (3.7).

Now let $\mu := \gamma + m - 1/2$ and $m \in \{0, 1, \ldots, n\}$. Equation (3.10) can be written as

$$f_k = \frac{K_\gamma (2n - 2k)!}{B(1/2, \mu + 1)(2\gamma)_{2n-2k}} \cdot \frac{(-m)_{n-k}(\gamma + 1/2)_m}{(m - k)!(\gamma + n - k)_{m+1}}.$$

Inserting this in (3.9) yields, after some manipulations,

$$h_n = \frac{K_\gamma (\gamma + 1/2)_m (1 - \gamma)_n (2\gamma + n + 1)_n}{2(n + 1)_{n+1}(\gamma)_{n+1}(\gamma)_{m+1}} S, \tag{3.12}$$

where

$$\begin{aligned} S := {} & \sum_{k=0}^{n} \frac{(\gamma)_k(2k + \gamma)(n + 1)_k(-\gamma - n)_k(-m)_k}{k!(\gamma + m + 1)_k(\gamma - n)_k(2\gamma + n + 1)_k} \\ = {} & \gamma\, {}_5F_4\left(\left. \begin{matrix} \gamma, \gamma + 1/2, n + 1, -\gamma - n, -m \\ \gamma/2, \gamma - n, 2\gamma + n + 1, \gamma + m + 1 \end{matrix} \right| 1 \right). \end{aligned}$$

The function ${}_5F_4(1)$ is well-poised and so can be summed by Dougall's theorem [17, p. 56]; the result is

$$S = \frac{(\gamma)_{m+1}(2\gamma)_m}{(\gamma - n)_n(2\gamma + n + 1)_m},$$

and (3.8) readily follows.

Further explicit formulae for $j_n(\gamma;\ \mu, \nu)$ may be obtained for some special values of μ or ν. For instance,

$$j_n(\gamma;\ \gamma - 1/2, \nu)$$
$$= 2^{\gamma+\nu+1/2}\frac{B(\gamma + 1/2, \nu + 1)(1 + 2\gamma)_{n-1}(\gamma - \nu - 1/2)_{n-1}(1 - \gamma)_{n-1}}{\gamma n!(\gamma + \nu + 3/2)_n}$$
$$\times\ {}_5F_4\left(\left.\begin{matrix} 1 - n, (1 + 2\gamma - 2\nu - 2n)/4, (3 + 2\gamma - 2\nu - 2n)/4, 1, 1 \\ (7 - 2\gamma + 2\nu - 2n)/4, (5 - 2\gamma + 2\nu - 2n)/4, 2\gamma + 1, 1 - n \end{matrix}\right| 1\right).$$

We have written the $1 - n$ parameters in the ${}_5F_4(1)$ (which is nearly-poised, by the way) only to indicate that the sum terminates after n terms.

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