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Supplement to Stability with Large Step Sizes for Multistep Discretizations of Stiff Ordinary Differential Equations

By George Majda

APPENDIX II

Proof of Lemma 2.2. (All constants which appear in this proof are independent of k, t and ε.)

Write (2.8) as its equivalent one-step method

$$w^n = \alpha(k,t_n,\varepsilon)w^{n-1}, \quad n = r,\ r+1,\ldots, \text{ and all } (k,\varepsilon) \in \Gamma' \tag{A.1}$$

where $w^n = (y^n, y^{n-1}, \ldots, y^{n-r+1})^T$ and

$$\alpha(k,t,\varepsilon) = \begin{pmatrix} Q_1(k,\varepsilon,\tau_0(t),\ldots,\tau_r(t)) & \ldots Q_r(k,\varepsilon,\tau_0(t),\ldots,\tau_r(t)) \\ I & 0\ldots\quad . \\ 0 & I0..\quad . \\ . & \ldots \\ 0 & \ldots I \quad 0 \end{pmatrix}. \tag{A.2}$$

Assumption (2.9) implies that $\alpha(k,t,\varepsilon)$ is uniformly bounded for all $rk \leq t < \infty$ and $(k,\varepsilon) \in \Gamma'$.

Assumptions (2.9) and (2.10) imply that for all $rk \leq t < \infty$ and $(k,\varepsilon) \in \Gamma'$

$$\alpha(k,t,\varepsilon) = \beta(k,t,\varepsilon) + k\gamma(k,t,\varepsilon) \tag{A.3}$$

with

$$\text{(A.4)}\quad \beta(k,t,\varepsilon) = \begin{pmatrix} Q_1(k,\varepsilon,\tau_0(t),\dots,\tau_0(t))\dots & Q_r(k,\varepsilon,\tau_0(t),\dots,\tau_0(t)) & \\ I & 0\dots & . \\ 0 & I\dots & . \\ . & \dots & . \\ 0 & \dots I & 0 \end{pmatrix}$$

and

$$\text{(A.5)}\qquad \gamma(k,t,\varepsilon) = \begin{pmatrix} E_1 & \dots & E_r \\ 0 & 0\dots & . \\ 0 & 0\dots & . \\ . & \dots & . \\ 0 & \dots 0 & 0 \end{pmatrix}.$$

Furthermore, $\beta(k,t,\varepsilon)$ and $\gamma(k,t,\varepsilon)$ are uniformly bounded for all $rk \le t < \infty$ and $(k,\varepsilon) \in \Gamma'$.

Let $\kappa_i(k,t,\varepsilon)$ and $\lambda_i(k,t,\varepsilon)$, $i = 1,\dots,mr$, denote the eigenvalues of $\beta(k,t,\varepsilon)$ and $\alpha(k,t,\varepsilon)$, respectively. By Assumption 2) (lines (2.11) and (2.12)) the eigenvalues of $\beta(k,t,\varepsilon)$ satisfy

$$\text{(A.6)}\quad |\kappa_i(k,t,\varepsilon)| \le \kappa < 1 \quad \text{for } i = 1,\dots,mr, \text{ and all } rk \le t < \infty \text{ and } (k,\varepsilon) \in \Gamma'$$

where κ is a constant. The matrix $\gamma(k,t,\varepsilon)$ is uniformly bounded, so if ε_0 is sufficiently small, there exists a constant $\hat{k}$ satisfying $0 < \hat{k} \le \bar{k}$ and a constant $0 \le \lambda < 1$ such that

$$\text{(A.7)}\quad |\lambda_i(k,t,\varepsilon)| \le \lambda < 1 \quad \text{for } i = 1,\dots,mr, \text{ and all } rk \le t < \infty \text{ and } (k,\varepsilon) \in \Gamma''$$

where $\Gamma'' = \{(k,\varepsilon) \in \mathbb{R}^2 : 0 < \varepsilon \le \varepsilon_0 \text{ and } \sigma^*\varepsilon \le k \le \hat{k}\}$.

We now show that for each point $rk \le t^* < \infty$ and any point $(k,\varepsilon) \in \Gamma''$ there exist constants K and τ, independent of t^*, k and ε, and an invertible matrix $S(k,t^*,\varepsilon)$ satisfying

$$\text{(A.8)}\qquad \sup_{\substack{0<t^*<\infty \\ (k,\varepsilon)\in\Gamma''}} \{ \|S(k,t^*,\varepsilon)\| , \|S^{-1}(k,t^*,\varepsilon)\| \} \le K < \infty$$

such that

$$\text{(A.9)}\quad \|S^{-1}(k,t^*,\varepsilon)\alpha(k,t,\varepsilon)S(k,t^*,\varepsilon)\| \le (\lambda + \tfrac{1-\lambda}{2}) < 1 \text{ for } t \in [t^*,t^*+\tau]$$

and all $(k,\varepsilon) \in \Gamma''$.

By Schur's Theorem, for each $rk \le t < \infty$ and $(k,\varepsilon) \in \Gamma''$ there exists a unitary transformation $U(k,t,\varepsilon)$ such that

$$\text{(A.10)}\quad U^*(k,t,\varepsilon)\alpha(k,t,\varepsilon)U(k,t,\varepsilon) = \hat{\alpha}(k,t,\varepsilon) = D(k,t,\varepsilon) + P(k,t,\varepsilon)$$

where

$$\text{(A.11)}\qquad D(k,t,\varepsilon) = \text{diag}(\lambda_1(k,t,\varepsilon),\dots,\lambda_{mr}(k,t,\varepsilon))$$

and $P(k,t,\varepsilon)$ is an upper triangular matrix with zeros on its diagonal. The matrices $\alpha(k,t,\varepsilon)$ and $D(k,t,\varepsilon)$ are uniformly bounded, so

$$\text{(A.12)}\qquad \sup_{\substack{0<t<\infty \\ (k,\varepsilon)\in\Gamma''}} |(P(k,t,\varepsilon))_{ij}| \le P < \infty \quad \text{for all } 1 \le i,j \le mr$$

where P is some positive constant.

Let

$$\text{(A.13)}\qquad \bar{S} = \text{diag}(1,\xi,\dots,\xi^{mr-1}) \quad \text{with } \xi = \frac{1-\lambda}{4(mr-1)P}.$$

Then straightforward estimates show that

(A.14) $\|(\bar{S})^{-1}\hat{\alpha}(k,t,\varepsilon)\bar{S}\| \leq \frac{1+3\lambda}{4}$ for all $rk \leq t < \infty$ and $(k,\varepsilon) \in \Gamma''$.

By Assumption (2.18), for any $rk \leq t^* < \infty$

(A.15) $\alpha(k,t,\varepsilon) = \alpha(k,t^*,\varepsilon)+\alpha(k,t,\varepsilon)-\alpha(k,t^*,\varepsilon)$

with

(A.16) $\|\alpha(k,t,\varepsilon)-\alpha(k,t^*,\varepsilon)\| \leq \bar{Q}(\dot{t}-t^*)$ for all $(k,\varepsilon) \in \Gamma''$.

Set

(A.17) $S(k,t^*,\varepsilon) = U(k,t^*,\varepsilon)\bar{S}$.

Then $S(k,t^*,\varepsilon)$ satisfies (A.8) with $K = \xi^{1-mr}$. Furthermore, if we set $\tau = \frac{1-\lambda}{4\bar{Q}K}$, then straightforward estimates using (A.15)-(A.17) establish (A.9) and the preliminary result is justified.

For our next preliminary result we set

(A.18) $T^* = \min\{\tau,\hat{k}\}$ and $\omega = \lambda + \frac{1-\lambda}{2} < 1$.

Let q be the smallest integer satisfying $\omega^q K<1$ and set

(A.19) $\varepsilon_0 = \frac{T^*}{q\sigma^*}$, $k_0 = \frac{T^*}{q}$ and $\bar{\delta} = \omega k^{1/q}$.

(Clearly $0 \leq \bar{\delta} < 1$.) We now show that for any integer $\ell \geq r-1$ and any set of $q+1$ consecutive grid points $t_{\ell+i} = (\ell+i)k$, $i = 0,\ldots,q$, the solution of (A.1) satisfies the estimate

(A.20) $\|w^{\ell+i}\| \leq K\omega^i\|w^\ell\|$ for $i = 0,\ldots,q$, and all $(k,\varepsilon) \in \Gamma$

where Γ is defined on line (1.12) with ε_0 and k_0 defined on line (A.19).

Consequently, by the first relation on line (A.19)

(A.21) $\|w^{\ell+q}\| \leq (\bar{\delta})^q\|w^\ell\|$ for all $(k,\varepsilon)\in\Gamma$.

To justify the claim, make the change of variables

(A.22) $w^{\ell+i} = S(k,t_\ell,\varepsilon)v^{\ell+i}$, $i = 0,\ldots,q$,

in equation (A.1) where $S(k,t,\varepsilon)$ is the matrix defined on line (A.17). (Note that $S(k,t_\ell,\varepsilon)$ is constant over the (q+1) grid points.) Then

(A.23) $v^{\ell+i} = (S^{-1}(k,t_\ell,\varepsilon)\alpha(k,t_{\ell+j},\varepsilon)S(k,t_\ell,\varepsilon))v^{\ell+i-1}$, $i = 1,\ldots,q$.

By result (A.9) and the definitions on line (A.18)

(A.24) $\|v^{\ell+i}\| \leq \omega^i\|v^\ell\|$ for $i = 1,\ldots,q$, and all $(k,\varepsilon)\in\Gamma$.

By relations (A.22) and (A.8)

(A.25) $\|w^{\ell+i}\| \leq K\omega^i\|w^\ell\|$, for $i = 1,\ldots,q$, and all $(k,\varepsilon)\in\Gamma$

and estimates (A.20) and (A.21) are justified.

To complete the proof of Lemma 2.2, consider equation (A.1) and set $n-j = \ell q+p$ where ℓ and p are non-negative integers with $0 \leq p \leq q-1$. By results (A.20) and (A.21)

$$\|w^n\| = \|w^{\ell q+p+j}\| \leq K\omega^p\|w^{\ell q+j}\| \leq K\omega^p(\bar{\delta})^{\ell q}\|w^j\|$$

for all $(k,\varepsilon)\in\Gamma$. Now set $\delta = \max\{\omega,\bar{\delta}\}<1$ to obtain

$$\|w^n\| \le K\delta^{n-j}\|w^j\|, \text{ for all } r-1\le j\le n<\infty \text{ and } (k,\varepsilon)\in\Gamma.$$

The proof of Lemma 2.2 is complete. □

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