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The Computation of the Fundamental Unit of Totally Complex Quartic Orders

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Dedicated to Professor Daniel Shanks on the occasion of his 70th birthday

Abstract. We describe an efficient algorithm for the computation of the regulator and a fundamental unit of an arbitrary totally complex quartic order. We analyze its complexity and we present tables with computational results for the orders $\mathbf{Z}[\sqrt[4]{-d}\,]$, $1 \leqslant d \leqslant 500$.

1. Introduction. The computation of fundamental units in orders of algebraic number fields is one of the main problems in computational algebraic number theory.

The simplest fields for this problem are those with only one fundamental unit. There are three types of such fields; the real quadratic, the complex cubic, and the totally complex quartic fields.

It is well known that the fundamental unit of a real quadratic field can be computed by means of the ordinary continued fraction algorithm, cf. [2, II, Section 7]. There are interesting refinements of this algorithm due to Shanks [16] and Lenstra [13].

The fundamental unit of complex cubic fields can be computed using Voronoi's generalized continued fraction algorithm, cf. [5], [18]. This algorithm was discussed and improved in several interesting papers of Williams et al., cf. [20].

For totally complex quartic fields there are only a few results. If the field contains a real quadratic subfield, the computation of a fundamental unit can be reduced to the computation of the fundamental unit of the subfield, cf. [8].

For complex quartic fields containing an imaginary quadratic subfield, Scharlau proved that a fundamental unit is a minimal solution of a certain relative Pell equation, cf. [15], but he did not give a method for solving it.

For complex quartic fields containing quadratic subfields of class number one, there are results due to Amara [1] and Lakein [9], [10].

More generally, the author proved that the generalized Voronoi algorithm (GVA), developed in [3], yields a fundamental unit for any order of a totally complex quartic field. The algorithm of Jeans [7] seems to have some similarities with this method.

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In this paper we describe how to apply the GVA practically. We analyze its complexity and prove that it yields a fundamental unit of any totally complex quartic order in $O(RD^{\varepsilon})$ binary operations (for every $\varepsilon > 0$), where D is the absolute value of the discriminant and R is the regulator of the order.

We establish an analogue of Galois' theorem on the symmetry of the continued fraction expansion of the square root of a rational number. We conclude the paper by presenting computational results for the orders $\mathbf{Z}[\sqrt[4]{-d}\,]$, $1 \leqslant d \leqslant 500$.

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2. Notations. In this paper

$L = \mathbf{Q}(\rho)$ is a totally complex quartic algebraic number field,

σ is a $\mathbf{Q}$-isomorphism of L into $\mathbf{C}$, different from the complex conjugation. For $\xi \in L$ we write

$$\xi^{(1)} = \xi, \quad \xi^{(2)} = \sigma(\xi), \quad \xi^{(3)} = \bar{\xi} \quad \text{and} \quad \xi^{(4)} = \overline{\sigma(\xi)}.$$

$\mathcal{O}$ is an order of L,

D is the absolute value of the discriminant of $\mathcal{O}$,

R is the regulator of $\mathcal{O}$,

$\omega_1, \ldots, \omega_4$ is a $\mathbf{Z}$-basis of $\mathcal{O}$,

$\omega_1^*, \ldots, \omega_4^*$ is the corresponding dual basis,

$W = \max\{|\omega_k^{(i)}| \,|\, 1 \leqslant i, k \leqslant 4\}$,

$W^* = \max\{|\omega_k^{*(i)}| \,|\, 1 \leqslant i, k \leqslant 4\}$.

We assume that

$$W \leqslant D^{1/2}. \tag{2.1}$$

Such a basis can be computed using a basis reduction algorithm, e.g., [11], in the Minkowski lattice corresponding to $\mathcal{O}$. It follows immediately that

$$W^* \leqslant 6D. \tag{2.2}$$

For a (fractional) ideal $\mathfrak{a}$ of $\mathcal{O}$ we fix

$$d(\mathfrak{a}) = \min\{d \in \mathbf{N} \,|\, d\mathfrak{a} \subseteq \mathcal{O}\}, \qquad N(\mathfrak{a}) = \text{norm of } \mathfrak{a}.$$

3. The Method. We recall the main definitions and results of [3]. There, we introduced the map

$$L \to \mathbf{R}^2, \qquad \xi \to \boldsymbol{\xi} = \left(|\xi|^2, |\sigma(\xi)|^2\right)^t$$

which, restricted to the multiplicative group $L^{\times}$, is a homomorphism, if we use the product

$$(y_1, y_2)^t \cdot (y_1', y_2')^t = (y_1 y_1', y_2 y_2')^t.$$

Moreover, $L^{\times}$ acts on $\mathbf{L}$ by

$$\xi * \boldsymbol{\xi}' = \boldsymbol{\xi}\boldsymbol{\xi}' \quad \text{for every } \xi \in L^{\times}, \ \boldsymbol{\xi}' \in \mathbf{L}.$$

For a point $\vec{y} = (y_1, y_2)^t \in \mathbf{R}^2$ its norm is defined by

$$N(\vec{y}) = |y_1 y_2|.$$

Now let $\mathfrak{a}$ be a (fractional) ideal in $\mathcal{O}$. Then the image $\mathfrak{a}$ is a discrete set in $\mathbf{R}^2$. A point $0 \neq \vec{m}$ in $\mathfrak{a}$ is called a *minimal point* of $\mathfrak{a}$, if its norm body

$$Q(\vec{m}) = \left\{\vec{y} \in \mathbf{R}^2 \,|\, 0 \leqslant y_i \leqslant m_i \text{ for } 1 \leqslant i \leqslant 2\right\}$$

does not contain points of $\mathfrak{a}$ aside from $\mathbf{0}$ and $\vec{m}$. Minimal points are of bounded norm

$$N(\vec{m}) \leqslant (4/\pi^2) D^{1/2} N(\mathfrak{a}). \tag{3.1}$$

Moreover, for $\{u, v\} = \{1, 2\}$ the *u-neighbor* of a minimal point $\vec{m}$ is defined to be the (uniquely determined) minimal point $\vec{m}'$ with $m'_u < m_v$ and minimal m'_u. For this neighbor we have by Minkowski's convex body theorem

$$m'_u m_v \leqslant (4/\pi^2) D^{1/2} N(\mathfrak{a}). \tag{3.2}$$

Finally, $\mathbf{1}$ is a minimal point in $\mathcal{O}$ and all the minimal points of $\mathcal{O}$ can be arranged in a two-sided sequence $(\vec{m}_k)_{k \in \mathbf{Z}}$, $\vec{m}_0 = \mathbf{1}$, where $\vec{m}_{k+1}$ is always the 2-neighbor of $\vec{m}_k$ and, in turn, $\vec{m}_k$ is the 1-neighbor of $\vec{m}_{k+1}$ for every $k \in \mathbf{Z}$. This sequence, called the GVA-expansion in $\mathcal{O}$, is of the purely periodic form

$$\begin{gathered} \ldots, \varepsilon^{-1} * \mathbf{1},\ \varepsilon^{-1} * \vec{m}_1, \ldots, \varepsilon^{-1} * \vec{m}_{p-1},\ \mathbf{1},\ \vec{m}_1, \ldots, \vec{m}_{p-1}, \\ \varepsilon * \mathbf{1},\ \varepsilon * \vec{m}_1, \ldots, \varepsilon * \vec{m}_{p-1}, \ldots. \end{gathered} \tag{3.3}$$

Here ε is a unit in $\mathcal{O}$, and if p is chosen minimal, then ε is a fundamental unit of $\mathcal{O}$, and p is called the *period length* of the GVA in $\mathcal{O}$.

4. The Algorithm. Here is our algorithm for computing a fundamental unit ε and the regulator R of $\mathcal{O}$.

ALGORITHM 4.1.

Input: $\omega_1, \ldots, \omega_4$.

Output: R, ε.

1. Initialize: $k \leftarrow 0$, $N \leftarrow 1$, $\eta_0 \leftarrow 1$, $\mathfrak{a} \leftarrow \mathcal{O}$, $R \leftarrow 0$.
2. Repeat until $k \geqslant 1$ and $N = 1$:
   (a) $\mathfrak{a} \leftarrow (1/\eta_k)\mathfrak{a}$.
   (b) Compute η_{k+1} in $\mathfrak{a}$ such that η_{k+1} is the 2-neighbor of $\mathbf{1}$ in $\mathfrak{a}$.
   (c) $k \leftarrow k+1$, $R \leftarrow R - \log|\eta_k|^2$, $N \leftarrow N \cdot |N_{L|\mathbf{Q}}(\eta_k)|$.
3. $\varepsilon \leftarrow \prod_{j=0}^{k} \eta_j$.

Notice that we can compute a maximal system of pairwise nonassociated "minima" in the sense of [21, Section 3], if we calculate in step 2(b) all the η in $\mathfrak{a}$ such that η is the 2-neighbor of $\mathbf{1}$ in $\mathfrak{a}$.

The representation of the principal ideals $\mathfrak{a}$ will be discussed in Section 6 and the computation of η_{k+1} will be described in Section 7.

We conclude this section by giving a justification for our algorithm. We define for $0 \leqslant k \leqslant p$

$$\mu_k = \prod_{j=0}^{k} \eta_j. \tag{4.1}$$

Then we have

$$\mu_k \in \mathcal{O} \quad \text{and} \quad \mu_k = \vec{m}_k \quad \text{for } 0 \leqslant k \leqslant p, \tag{4.2}$$

and in step 2(b)

$$\mathfrak{a} = (1/\mu_k)\mathcal{O}. \tag{4.3}$$

In fact, (4.2) and (4.3) are true for $k = 0$. Now suppose that (4.2) and (4.3) hold for $k \geqslant 0$. Since $\eta_{k+1} \in (1/\mu_k)\mathcal{O}$, we must have $\mu_{k+1} = \mu_k\eta_{k+1} \in \mathcal{O}$. Moreover, $\boldsymbol{\eta}_{k+1}$ is the 2-neighbor of **1** in $(1/\boldsymbol{\mu}_k)\mathcal{O}$, and therefore $\boldsymbol{\mu}_{k+1} = \boldsymbol{\mu}_k\boldsymbol{\eta}_{k+1}$ must be the 2-neighbor of $\boldsymbol{\mu}_k = \vec{m}_k$ in $\mathcal{O}$, i.e., $\boldsymbol{\mu}_{k+1} = \vec{m}_{k+1}$. It follows immediately that in step 2(c),

$$N = N(\vec{m}_k) = \left|N_{L|\mathbf{Q}}(\vec{\mu}_k)\right|. \tag{4.4}$$

If $k = p$, then by (3.3) and (4.4) we must have $N = 1$, and because of the minimality of p, this is the first time that $N = 1$ can happen.

5. The Theorem of Galois. By a theorem of Galois, the period of the continued fraction expansion of the square root of a positive rational number is symmetric, cf. [14, Section 23]. A similar result is proved in this section. We assume that the order under consideration satisfies the condition

$$\sigma(\mathcal{O}) = \mathcal{O}. \tag{5.1}$$

This is true, for example, if $\mathcal{O} = \mathbf{Z}[\sqrt[4]{-d}]$, $d \in \mathbf{N}$. Note that (5.1) implies that L has a quadratic subfield. On the plane $\mathbf{R}^2$ we introduce the reflection

$$\tilde{\sigma}\colon \mathbf{R}^2 \to \mathbf{R}^2, \qquad \vec{y} = (y_1, y_2)^t \to \tilde{\sigma}(\vec{y}) = (y_2, y_1)^t. \tag{5.2}$$

Then we have for every $\xi \in L$

$$\boldsymbol{\sigma}(\boldsymbol{\xi}) = \tilde{\sigma}(\boldsymbol{\xi}).$$

Consequently, the minimal points in $\mathcal{O}$ have the symmetry property:

PROPOSITION 5.1. *For every $k \in \mathbf{Z}$ we have $\vec{m}_k = \tilde{\sigma}(\vec{m}_{-k})$.*

Proof. Let $\vec{m}$, $\vec{m}'$ be minimal points of $\mathcal{O}$. Then $\tilde{\sigma}(\vec{m})$ and $\sigma(\vec{m}')$ are minimal points of $\mathcal{O}$, and $\vec{m}$ is the 1-neighbor of $\vec{m}'$ if and only if $\tilde{\sigma}(\vec{m})$ is the 2-neighbor of $\tilde{\sigma}(\vec{m}')$. □

This yields the following application: If we have computed $\vec{m}_0, \ldots, \vec{m}_k$, then we know $\vec{m}_{-k}, \ldots, \vec{m}_k$. Hence, we can compute the regulator and a fundamental unit of $\mathcal{O}$ by computing only half the period of the GVA-expansion in $\mathcal{O}$. This is done by

ALGORITHM 5.2.
Input: $\omega_1, \ldots, \omega_4$.
Output: R, ε.
1. Initialize: $k \leftarrow 0$, $R \leftarrow 0$, $\eta_0 \leftarrow 1$, $\mathfrak{a} \leftarrow \mathcal{O}$.
2. Repeat:
   (a) $\mathfrak{a} \leftarrow (1/\eta_k)\mathfrak{a}$.
   (b) Compute a complete system of nonassociated η in $\mathfrak{a}$ such that $\boldsymbol{\eta}$ is the 2-neighbor of **1** in $\mathfrak{a}$. Choose one of these η's to be η_{k+1}.
   (c) $R \leftarrow R - \log|\eta_{k+1}|^2$.
   (d) If $\sigma(\mathfrak{a}) = (1/\eta)\mathfrak{a}$ holds for one of the η's of (b), then $\varepsilon \leftarrow \eta\prod_{j=1}^{k}(\eta_j/\sigma(\eta_j))$ and return.
   (e) $R \leftarrow R + \log|\sigma(\eta_{k+1})|^2$.
   (f) If $\sigma((1/\eta_{k+1})\mathfrak{a}) = (1/\eta)\mathfrak{a}$ for one of the η's of (b), then $\varepsilon \leftarrow (\eta/\sigma(\eta_{k+1}))\prod_{j=1}^{k}(\eta_j/\sigma(\eta_j))$, and return.
   (g) $k \leftarrow k + 1$.

For the description of the representation of the ideals $\mathfrak{a}$ and the computation in 2(b), 2(d) and 2(f) we refer to Sections 6 and 7. We conclude this section by giving a justification of Algorithm 5.2.

For

$$\mu_k = \prod_{j=0}^{k} \eta_j, \qquad 0 \leqslant k \leqslant p,$$

one can prove as in Section 4 that

(5.3) $$\mu_k \in \mathcal{O} \quad \text{and} \quad \boldsymbol{\mu}_{\mathbf{k}} = \vec{m}_k,$$

and that in step 2(b), (d) and (f)

(5.4) $$\mathfrak{a} = (1/\mu_k)\mathcal{O},$$

and that

(5.5) $$\vec{m}_{k+1} = \eta * \boldsymbol{\mu}_{\mathbf{k}}$$

for every η computed in step 2(b).

It follows by (5.1) and Proposition 5.1 that

(5.6) $$\sigma(\boldsymbol{\mu}_{\mathbf{k}}) = \vec{m}_{-k},$$

and that in step 2(d)

(5.7) $$\sigma(\mathfrak{a}) = (1/\sigma(\mu_k))\mathcal{O}.$$

Now suppose that in step 2(d), $\sigma(\mathfrak{a}) = (1/\boldsymbol{\eta})\mathfrak{a}$. Since in 2(d)

(5.8) $$\varepsilon = \eta\mu_k/\sigma(\mu_k),$$

it follows from (5.4) and (5.7) that ε is a unit in $\mathcal{O}$. Moreover, by (5.5) and (5.6),

(5.9) $$\varepsilon * \vec{m}_{-k} = \vec{m}_{k+1}.$$

Because of the minimality of the period length p, this can happen only if $p \mid 2k+1$. But if $p = 2k+1$, then (5.9) holds for every fundamental unit ε with $|\varepsilon| < 1$. Let ε be such a fundamental unit. It follows from (5.5), (5.6) and (5.9) that

(5.10) $$\varepsilon \cdot \sigma(\mu_k) = \alpha \cdot \eta_{k+1}\mu_k$$

with $\alpha \in L$, $\boldsymbol{\alpha} = \mathbf{1}$. We set $\eta = \alpha\eta_{k+1}$ and see by (5.4), (5.7) and (5.10) that $\eta \in \mathfrak{a}$, $\boldsymbol{\eta} = \boldsymbol{\eta}_{\mathbf{k+1}}$, and $\sigma(\mathfrak{a}) = (1/\boldsymbol{\eta})\mathfrak{a}$.

Analogously, one can show that in step 2(f) one has $\sigma((1/\eta_{k+1})\mathfrak{a}) = (1/\boldsymbol{\eta})\mathfrak{a}$ only if ε is a unit and $p \mid 2k+2$ and, in turn, that this happens in fact if $p = 2k+2$. Then ε is again a fundamental unit of $\mathcal{O}$.

6. Basis Reduction and Ideal Representation. In this section we describe how we represent the ideals $\mathfrak{a}$ in Algorithm 4.1 and Algorithm 5.2.

First of all, we recall some properties of the basis reduction algorithm of Lenstra, Lenstra and Lovász [11]. Let $\vec{b}_1, \ldots, \vec{b}_n$ be a basis of a lattice Γ in $\mathbf{Z}^n$ and let $B \geqslant 2$, $|\vec{b}_j| \leqslant B$ for $1 \leqslant j \leqslant n$.** This algorithm yields in $O(n^4 \log B)$ binary operations a basis $\vec{a}_1, \ldots, \vec{a}_n$ of Γ which satisfies

(6.1) $$\prod_{j=1}^{n} |\vec{a}_j| \leqslant 2^{n(n-1)/4}\det(\Gamma),$$

where $\det(\Gamma)$ is the determinant of Γ.

**$|\ |$ denotes the Euclidean norm.

Now let $\mathfrak{a}$ be an ideal in $\mathcal{O}$, and let $\alpha_1, \dots, \alpha_4$ be a $\mathbf{Z}$-basis of $\mathfrak{a}$,

$$\alpha_j = \left(\sum_{k=1}^{4} a_{kj} \omega_k \right) / d(\mathfrak{a}), \qquad 1 \leqslant j \leqslant 4. \tag{6.2}$$

Denote by Γ the lattice spanned by the columns of the integral matrix $A = (a_{kj})$. We call A an LLL-*matrix* of $\mathfrak{a}$, if the columns of A form a basis of Γ which is reduced in the sense of [11, p. 516]. The ideals in Algorithm 4.1 and Algorithm 5.2 are represented in terms of their common denominator and an LLL-matrix. It has turned out in our computational experience that these representing integers are always small compared to the discriminant of $\mathfrak{a}$. We are also able to give bounds on the denominators and the elements of the LLL-matrices. These bounds are polynomials in D, and this means that the number of digits of these integers is $O(\log D)$. This statement will be useful in our complexity analysis in Section 8.

PROPOSITION 6.1. *If A is an LLL-matrix of $\mathfrak{a}$, then the column vectors $\vec{a}_1, \dots, \vec{a}_4$ of A satisfy*

$$c_1^{-1} d(\mathfrak{a}) N(\mathfrak{a})^{1/4} \leqslant |\vec{a}_j| \leqslant c_2 d(\mathfrak{a}) N(\mathfrak{a})^{1/4} \; *** \tag{6.3}$$

for $1 \leqslant j \leqslant 4$.

Proof. Let $\alpha_1, \dots, \alpha_4$ be defined as in (6.2). Then we have for $1 \leqslant j \leqslant 4$,

$$\left|\alpha_j^{(1)}\right|^2 \left|\alpha_j^{(2)}\right|^2 = \left|N_{L|\mathbf{Q}}(\alpha_j)\right| \geqslant N(\mathfrak{a}).$$

Hence, we have $|\alpha_j^{(i)}| \geqslant N(\mathfrak{a})^{1/4}$ for $i = 1$ or $i = 2$, and the first inequality of (6.3) follows from (2.1), (6.2), and Cauchy's inequality, whereas the second one follows from the first one and (6.1), since in this case $\det(\Gamma) = N(\mathfrak{a})$. □

COROLLARY 6.2. *If A is an LLL-matrix of $\mathfrak{a}$ and if $\alpha_1, \dots, \alpha_4$ is the corresponding $\mathbf{Z}$-basis of $\mathfrak{a}$, defined in* (6.2), *then we have for* $1 \leqslant i, j \leqslant 4$,

$$\left|\alpha_j^{(i)}\right| \leqslant c_3 N(\mathfrak{a})^{1/4}.$$

Proof. This corollary follows from (2.1), (6.2) and Proposition 6.1. □

COROLLARY 6.3. *If $\mathfrak{a}$ is one of the ideals, used in Algorithm* 4.1 *or Algorithm* 5.2, *if A is an LLL-matrix of $\mathfrak{a}$ with the columns $\vec{a}_1, \dots, \vec{a}_4$, and if $\alpha_1, \dots, \alpha_4$ is the corresponding $\mathbf{Z}$-basis of $\mathfrak{a}$, defined in* (6.2), *then we have for* $1 \leqslant i, j \leqslant 4$,

$$|\vec{a}_j| \leqslant c_4 \quad \text{and} \quad \left|\alpha_j^{(i)}\right| \leqslant c_5.$$

Proof. It follows from (3.1), (4.2), (4.3), (5.3), and (5.4) that $N(\mathfrak{a}) \leqslant 1$ and $d(\mathfrak{a}) \leqslant D^{1/2}$. □

The last result shows that the ideals $\mathfrak{a}$ in Algorithm 4.1 and Algorithm 5.2 can be represented by integral matrices which are—independently of k—of the same "small" size.

***The numbers c_k, $k \in \mathbf{N}$, are of the form uD^v, $u, v > 0$.

We finally remark that the comparison of the ideals in Algorithm 5.2 step 2(d) and 2(f) can be carried out by comparing the denominators and Hermite normal forms of the representation matrices.

7. The Neighbor Computation. In step 2(b) of Algorithm 4.1 we want to know a number η in the ideal $\mathfrak{a}$ such that η is the 2-neighbor of **1** in $\mathfrak{a}$. Following the explanation given in Section 3, this means that we have to find η in $\mathfrak{a}$ with

$$(7.1)\qquad |\eta^{(1)}|^2 < 1 \quad \text{and} \quad |\eta^{(2)}|^2 \leqslant (4/\pi^2) D^{1/2} N(\mathfrak{a})$$

with minimal $|\eta^{(2)}|^2$.

Let $A = (a_{k,j})_{1 \leqslant k,j \leqslant 4}$ be an LLL-matrix of $\mathfrak{a}$ and let $\alpha_1, \ldots, \alpha_4$ be the corresponding **Z**-basis of $\mathfrak{a}$, defined in (6.2). For $\vec{x} = (x_1, \ldots, x_4)^t \in \mathbf{Z}^4$ and $1 \leqslant i \leqslant 2$ we write

$$(7.2)\qquad \eta^{(i)}(\vec{x}) = \begin{cases} \sum_{j=1}^{4} x_j \alpha_j^{(i)} = \left(\sum_{k=1}^{4} \omega_k^{(i)} \sum_{j=1}^{4} a_{kj} x_j \right) / d(\mathfrak{a}) & \text{if } \vec{x} \neq \vec{0}, \\ D^{1/4} N(\mathfrak{a})^{1/2} & \text{if } \vec{x} = \vec{0}. \end{cases}$$

Then we can compute η using

PROCEDURE 7.1.
1. Initialize: $\vec{x}_2 \leftarrow \vec{0}$, $f \leftarrow 2$.
2. Repeat:
 Try to find $\vec{x}_1 \in \mathbf{Z}^4$ satisfying

$$(7.3)\qquad |\eta^{(1)}(\vec{x}_1)| < 1 \quad \text{and} \quad |\eta^{(2)}(\vec{x}_1)| < |\eta^{(2)}(\vec{x}_2)|/f.$$

 If the search is successful, then set $\vec{x}_2 \leftarrow \vec{x}_1$,
else
 if $f = 2$, then set $f \leftarrow 1$,
 else return $\eta = \eta^{(1)}(\vec{x}_2)$. □

Notice that for all $\vec{x} \in \mathbf{Z}^4$ with

$$|\eta^{(1)}(\vec{x})| < 1 \quad \text{and} \quad |\eta^{(2)}(\vec{x})| < (2/\pi) D^{1/4} N(\mathfrak{a})^{1/2},$$

by the well-known dual basis argument [2, p. 403],

$$(7.4)\qquad \left| \sum_{j=1}^{4} a_{kj} x_j \right| \leqslant 4W^* d(\mathfrak{a}) D^{1/4} N(\mathfrak{a})^{1/2}, \qquad 1 \leqslant k \leqslant 4.$$

The comparisons in (7.3) have to be carried out using rational approximations $\hat{\omega}_k^{(i)}$ to $\omega_k^{(i)}$, $1 \leqslant k,\ i \leqslant 4$. We must therefore discuss the question of how this is to be done. Let $\lambda > 0$ have the property

$$(7.5)\qquad \max\left\{ |\omega_k^{(i)} - \hat{\omega}_k^{(i)}| \,|\, 1 \leqslant i,\ k \leqslant 4 \right\} < \lambda.$$

For $\vec{x} \neq 0$, let $\hat{\eta}^{(i)}(\vec{x})$ be the approximation of $\eta^{(i)}(\vec{x})$ obtained by substituting $\omega_k^{(i)}$ by $\hat{\omega}_k^{(i)}$ in (7.2), and let $\hat{\eta}^{(2)}(0)$ be a rational approximation of $D^{1/4}$ such that $|\hat{\eta}^{(2)}(0) - D^{1/4}| < \lambda$. Finally, we set for $\vec{x} \in \mathbf{Z}^4$

$$\delta_1(\vec{x}) = \begin{cases} 4\lambda \left| \sum_{j=1}^{4} a_{kj} x_j \right| / d(\mathfrak{a}) & \text{if } \vec{x} \neq \vec{0}, \\ \lambda & \text{if } \vec{x} = \vec{0}, \end{cases} \tag{7.6}$$

and

$$\delta(\vec{x}) = \delta_1(\vec{x})\big(2D^{1/4} + \delta_1(\vec{x})\big). \tag{7.7}$$

Then it follows for every $\vec{x}$, subject to (7.4), that

$$\delta(\vec{x}) \leqslant \delta = \delta_1\big(2D^{1/4} + \delta_1\big), \tag{7.8}$$

with

$$\delta_1 = 16\lambda W^* D^{1/4} N(\mathfrak{a})^{1/2},$$

and we have for every $\vec{x} \in \mathbf{Z}^4$, subject to (7.4), and $1 \leqslant i \leqslant 2$,

$$\left| \left|\eta^{(i)}(\vec{x})\right|^2 - \left|\hat{\eta}^{(i)}(\vec{x})\right|^2 \right| \leqslant \delta(\vec{x}) \leqslant \delta. \tag{7.9}$$

Hence, (7.3) can be true only if

$$\left|\hat{\eta}^{(1)}(\vec{x}_1)\right|^2 + \left|\hat{\eta}^{(2)}(\vec{x}_1)\right|^2 \leqslant 1 + \left|\hat{\eta}^{(2)}(\vec{x}_2)\right|^2/f^2 + 3\delta. \tag{7.10}$$

The solutions of (7.10) can be computed using an algorithm of Fincke and Pohst [6] which yields all the integral solutions $\vec{x}$ of an inequality $Q(\vec{x}) \leqslant K$, where Q is a positive definite n-dimensional rational quadratic form and $K \geqslant 0$ is a real constant. By (7.9) we are able to decide whether a solution of (7.10) satisfies (7.3) as long as

$$\begin{aligned} &\left| \left|\hat{\eta}^{(1)}(\vec{x}_1)\right|^2 - 1 \right| > \delta(\vec{x}_1) \quad \text{and} \\ &\left| \left|\hat{\eta}^{(2)}(\vec{x}_1)\right|^2 - \left|\hat{\eta}^{(2)}(\vec{x}_2)\right|^2/f^2 \right| > \delta(\vec{x}_1) + \delta(\vec{x}_2). \end{aligned} \tag{7.11}$$

This, in turn, is true if

$$\left|\eta^{(1)}(\vec{x}_1)\right| \neq 1 \quad \text{and} \quad \left|\eta^{(2)}(\vec{x}_1)\right| \neq \left|\eta^{(2)}(\vec{x}_2)\right|/f, \tag{7.12}$$

and λ is small enough. Notice that (7.12) can be tested by using [21, Proposition 2.2].

Concluding these remarks, we can carry out the search for $\vec{x}_1$ in Procedure 7.1 in the following way.

We enumerate the solutions of (7.10) using the algorithm of Fincke and Pohst [6]. If we find a solution $\vec{x}_1$, we check if $\vec{x}_1$ satisfies (7.4). If not, we reject $\vec{x}_1$ as a possible solution of (7.3). Otherwise, we check whether $\vec{x}_1$ is subject to

$$\left|\hat{\eta}^{(1)}(\vec{x}_1)\right|^2 < 1 - \delta(\vec{x}_1) \quad \text{and} \quad \left|\hat{\eta}^{(2)}(\vec{x}_1)\right|^2 < \left|\hat{\eta}^{(2)}(\vec{x}_2)\right|^2/f^2 - \delta(\vec{x}_1) - \delta(\vec{x}_2),$$

or

$$|\hat{\eta}^{(1)}(\vec{x}_1)|^2 > 1 + \delta(\vec{x}_1) \quad \text{or} \quad |\hat{\eta}^{(2)}(\vec{x}_1)|^2 > |\hat{\eta}^{(2)}(\vec{x}_2)|^2/f^2 + \delta(\vec{x}_1) + \delta(\vec{x}_2).$$

In the first case, we have found a solution of (7.3). In the second case, we must reject $\vec{x}_1$ as a possible solution of (7.3).

If neither the first nor the second case holds, a situation which we have never encountered in any of our computations, then we check whether (7.12) is true. If the answer is negative, then we must reject $\vec{x}_1$; otherwise, we have to increase the precision of our approximation to $\omega_k^{(i)}$, i.e., we have to decrease λ. During our computations we found the value $\lambda = 10^{-12}$ always to be sufficient.

If we have enumerated all the solutions of (7.10) without finding a solution of (7.3), then no such solution exists.

In Algorithm 5.2 we need all the $\eta \in \mathfrak{a}$ such that η is the 2-neighbor of **1** in $\mathfrak{a}$. These numbers can be computed by a further application of the algorithm in [6].

For our complexity analysis in the next section, it is necessary to be able to prove that there exists a value of λ such that (7.11) follows from (7.12). This can be done analogously to the proof of [21, Proposition 4.1]. The result is

PROPOSITION 7.12. *We can choose* $\lambda = c_7^{-1}$ *and* $\delta = c_8\lambda$ *such that for every* $\vec{x}_1$, $\vec{x}_2$ *subject to* (7.4) *the following statements hold*:

(i) *If* $\vec{x}_2 = \vec{0}$, *then it follows from*

$$(7.13) \qquad |\eta^{(1)}(\vec{x}_1)| < 1 \quad \text{and} \quad |\eta^{(2)}(\vec{x}_1)| < (2/\pi)D^{1/4}N(\mathfrak{a})^{1/2}/f$$

that

$$(7.14) \qquad |\hat{\eta}^{(1)}(\vec{x}_1)|^2 \leqslant 1 - \delta \quad \text{and} \quad |\hat{\eta}^{(2)}(\vec{x}_1)|^2 \leqslant |\hat{\eta}^{(2)}(\vec{x}_2)|^2/f^2 - \delta.$$

Conversely, (7.14) *implies* (7.3).

(ii) *Let* $\vec{x}_2 \neq 0$; *then* (7.14) *and* (7.3) *are equivalent.*

8. Complexity Analysis. It is well known that the continued fraction algorithm computes a fundamental unit of an order of a real quadratic field in $O(R'D'^{\mu'})$ binary operations (for every $\mu' > 0$), where R' is the regulator and D' is the absolute value of the discriminants of the order, cf. [19]. An analogous result is true for Voronoi's generalized continued fraction algorithm in complex cubic fields, cf. [19]. The purpose of this section is to prove that Algorithm 4.1 and Algorithm 5.2 are of the same complexity.

Since by [4] the period length of the GVA in $\mathcal{O}$ is $O(R)$, also the number of iterations in Algorithm 4.1, step 2, is $O(R)$. Since we use LLL-matrices to represent the ideals $\mathfrak{a}$ in Algorithm 4.1, it follows from Corollary 6.3 and from (7.1) that step 2(a) requires $O(D^{\mu})$ binary operations (for every $\mu > 0$).

We must now analyze step 2(b), i.e., Procedure 7.1. Since **1** is a minimal point in $\mathfrak{a}$, the number of iterations, when $f = 2$, must be $O(\log D)$. But then it follows from the same arguments as used in the proof of [4, Lemma 2] that the number of solutions of (7.3) with $f = 1$ is $O(1)$. The number of iterations of the procedure is therefore $O(\log D)$.

It remains to analyze the complexity of the search for $\vec{x}_1$ in Procedure 7.1. It follows from Proposition 7.12 that, instead of searching the convex body described by (7.3), which has irrational constraints, we can search the convex body described by (7.4) and (7.14), which has rational constraints. Though the algorithm of Fincke and Pohst [6] has turned out to be very efficient in practice, we cannot use the complexity analysis provided for this algorithm in [6]. Therefore, we replace this method by a procedure of Lenstra [12], which, however, is too complicated for practical implementation. This procedure solves our search problem in polynomial time in the size of the input data, because obviously, our convex set is "solvable", cf. [12, Remark (d) in Section 2]. In view of (2.1), (2.2), Corollary 6.3, (7.1), (7.5), and the choice of λ in Proposition 7.12, the input data length is $O(\log D)$. Hence, Procedure 7.1 requires $O(D^{\mu})$ binary operations (for every $\mu > 0$).

Finally, it follows from (7.1) and the fact that by [4] the number of factors in step 3 of Algorithm 4.1 is $O(R)$, that ε can be computed in $O(RD^{\mu})$ steps (for every $\mu > 0$).

Concluding these remarks, we have

THEOREM 8.1. *A fundamental unit of $\mathcal{O}$ can be computed by means of Algorithm* 4.1 *in $O(RD^{\mu})$ binary operations* (*for every* $\mu > 0$).

The same complexity can be proved for Algorithm 5.2.

9. Example. Let $\mathcal{O} = \mathbf{Z}[\rho]$, $\rho = \sqrt[4]{-326}$. Since condition (5.1) is satisfied, we can apply Algorithm 5.2:

1. Initialization: $k \leftarrow 0$, $R \leftarrow 0$, $\eta_0 \leftarrow 1$, $\mathfrak{a} \leftarrow \mathcal{O}$.
2. (a) $\mathfrak{a} \leftarrow \mathcal{O}$.
   (b) $\eta = 18 + 6\rho + \rho^2$ is up to association the only number such that η is the 2-neighbor of $\mathbf{1}$ in $\mathfrak{a}$, $\eta_1 \leftarrow \eta$.
   (c) $R \leftarrow 7.8633$.
   (d) $\alpha_1 \leftarrow 1$, $\alpha_2 \leftarrow \rho$, $\alpha_3 \leftarrow \rho^2/2$, $\alpha_4 \leftarrow \rho^3/2$, $(1/\eta_1)\mathfrak{a} = \oplus_{j=1}^{4} \mathbf{Z}\alpha_j \neq \sigma(\mathfrak{a})$.
   (e) $R \leftarrow 14.3402$.
   (f) $\sigma((1/\eta_1)\mathfrak{a}) = (1/\eta_1)\mathfrak{a}$, $\varepsilon \leftarrow \eta_1/\sigma(\eta_1) = 1 + 108\rho - 36\rho^2 + 6\rho^3$.

10. Numerical Results. We have computed the GVA-expansions in the orders $\mathcal{O} = \mathbf{Z}[\sqrt[4]{-d}]$, $d \in \mathbf{N}$, $d \neq 4k^4$ for $1 \leqslant d \leqslant 500$. In Table 1,

$$E_1 + E_2\rho + E_3\rho^2 + E_4\rho^3$$

is a fundamental unit of $\mathcal{O}$, $\rho = \sqrt[4]{-d}$.

In Table 2 we denote by

PL	the period length,
REG	the regulator,
NR	the relative norm of the fundamental units over $\mathbf{Z}[\sqrt{-d}]$.

TABLE 1

Fundamental units of $\mathbf{Z}[(-d)^{1/4}]$, $1 \leqslant d \leqslant 40$.

D	E_1	E_2	E_3	E_4
1	0	1	-1	1
2	-1	0	1	-1
3	2	-2	1	0
5	-2	2	-1	0
6	1	4	-4	2
7	36	-26	9	2
8	1	2	-2	1
9	-485	198	0	-66
10	-27	12	-1	-3
11	-98	96	-45	6
12	23	-14	4	1
13	-86	28	3	-10
14	-13	2	2	-2
15	16	4	-7	4
16	577	-204	0	51
17	-16	7	-1	-1
18	-13823	58332	-36792	11512
19	14439374	11821890	-11320425	4956000
20	-9	6	-2	0
21	1	36	-24	8
22	-91167	440140	-267972	81146
23	25899588	-8909082	352849	1629810
24	-95	46	-10	-3
25	-4443	1405	0	-281
26	-125	12	17	-13
27	-1414178	220224	135531	-126466
28	57	2	-12	7
29	330206	-411912	189709	-39122
30	-74879	57396	-21012	2218
31	1975104	28184	-371631	217672
32	6913	-360	-1008	663
33	67	-74	32	-6
34	-33	-4	8	-4
35	64926	-77790	34255	-6768
36	1351	-390	0	65
37	-571878	289258	-71847	-6356
38	-267790150327167	-183064217333724	147714465189436	-54441051012990
39	3745	-2544	840	-68
40	-159	-94	78	-29

TABLE 2

Regulators of $\mathbf{Z}[(-d)^{1/4}]$

D	PL	NR	REG	D	PL	NR	REG
1	2	1	1.7627	15	4	1	8.2853
2	3	-1	2.4485	16	10	1	14.1020
3	5	1	3.3258	17	2	1	6.9942
5	1	-1	3.5796	18	20	1	25.3105
6	2	1	5.9660	19	38	1	36.8971
7	8	1	8.9161	20	2	1	6.4677
8	2	1	4.8969	21	6	1	10.7870
9	12	1	13.7546	22	26	1	29.4799
10	13	-1	7.9923	23	38	1	35.5300
11	16	1	11.7560	24	6	1	10.7298
12	6	1	7.9666	25	13	-1	18.1845
13	7	-1	10.3107	26	9	-1	11.4356
14	2	1	6.8013	27	36	1	29.9319

TABLE 2 (*continued*)

D	PL	NR	REG	D	PL	NR	REG
28	8	1	10.2793	88	40	1	35.4889
29	23	-1	29.1595	89	44	1	54.5712
30	36	1	25.0462	90	22	1	33.7471
31	28	1	31.1193	91	34	1	50.8547
32	10	1	19.5876	92	28	1	31.3328
33	4	1	11.9390	93	56	1	60.0945
34	4	1	9.4774	94	16	1	19.6221
35	28	1	25.9225	95	32	1	37.4436
36	14	1	15.8035	96	18	1	23.8641
37	27	-1	28.3596	97	30	1	34.3796
38	70	1	70.3594	98	2	1	11.0465
39	16	1	18.9285	99	26	1	49.2976
40	10	1	13.8874	100	18	1	28.8727
41	9	-1	11.5768	101	13	-1	23.6552
42	14	1	20.9260	102	76	1	64.0178
43	72	1	65.4578	103	186	1	174.6268
44	58	1	44.6300	104	60	1	54.3461
45	28	1	30.5438	105	8	1	16.5349
46	6	1	10.6957	106	53	-1	64.0716
47	62	1	65.8973	107	106	1	121.6505
48	8	1	13.3031	108	68	1	71.6995
49	40	1	47.6012	109	27	-1	44.3041
50	35	-1	38.7416	110	64	1	77.2438
51	26	1	29.8215	111	10	1	22.1403
52	16	1	19.2904	112	26	1	35.6645
53	21	-1	24.6862	113	12	1	20.0114
54	104	1	96.5684	114	78	1	69.3097
55	10	1	19.9494	115	116	1	103.6233
56	4	1	8.1859	116	38	1	34.0075
57	12	1	21.0419	117	28	1	31.8778
58	11	-1	14.3255	118	136	1	143.8799
59	120	1	120.9619	119	126	1	131.9000
60	4	1	9.6399	120	20	1	19.8271
61	19	-1	25.0618	121	22	1	23.9054
62	4	1	11.0587	122	97	-1	98.6025
63	2	1	6.2305	123	10	1	26.3885
65	1	-1	6.2461	124	16	1	34.7878
66	2	1	11.1212	125	9	-1	17.8982
67	104	1	114.9532	126	14	1	23.6952
68	2	1	9.7650	127	166	1	187.6628
69	40	1	42.9765	128	34	1	39.1752
70	36	1	43.1795	129	72	1	79.2322
71	64	1	75.4121	130	43	-1	48.4233
72	2	1	8.4368	131	96	1	107.5991
73	14	1	26.4517	132	20	1	21.0849
74	9	-1	13.7289	133	76	1	88.0299
75	48	1	54.9405	134	344	1	309.1082
76	22	1	34.8990	135	16	1	31.3204
77	10	1	14.0040	136	2	1	8.7844
78	44	1	53.3431	137	3	-1	10.6102
79	116	1	143.0598	138	16	1	21.9224
80	4	1	14.3186	139	272	1	282.8793
81	52	1	63.4589	140	54	1	61.9243
82	15	-1	17.2367	141	2	1	9.1079
83	112	1	100.7354	142	88	1	112.8690
84	20	1	20.6834	143	68	1	101.5713
85	33	-1	33.3199	144	32	1	55.0184
86	126	1	103.5648	145	3	-1	11.6309
87	8	1	13.9061	146	22	1	35.1488

TABLE 2 (*continued*)

D	PL	NR	REG	D	PL	NR	REG
147	38	1	54.8322	206	44	1	50.1061
148	42	1	55.1209	207	8	1	12.2720
149	189	-1	194.0644	208	14	1	20.6215
150	34	1	52.7798	209	120	1	116.8674
151	40	1	53.3334	210	74	1	74.1316
152	42	1	51.3585	211	184	1	204.1566
153	34	1	56.7027	212	86	1	85.6995
154	36	1	51.5545	213	40	1	55.5989
155	4	1	13.8075	214	250	1	252.4171
156	2	1	8.5164	215	158	1	179.6845
157	105	-1	98.3062	216	48	1	53.6943
158	18	1	29.2540	217	8	1	15.4263
159	140	1	140.2070	218	53	-1	61.8690
160	56	1	63.9380	219	16	1	23.9217
161	10	1	17.0583	220	20	1	25.9591
162	168	1	176.2886	221	20	1	31.5731
163	248	1	255.1354	222	12	1	29.2041
164	36	1	42.4639	223	256	1	287.3029
165	46	1	45.3833	224	12	1	27.2051
166	224	1	226.4442	225	68	1	92.6691
167	258	1	262.0549	226	2	1	12.2875
168	24	1	28.3465	227	292	1	300.2000
169	39	-1	60.1234	228	30	1	37.5111
170	57	-1	57.0557	229	7	-1	18.6889
171	30	1	45.0703	230	114	1	137.2790
172	12	1	29.3731	231	8	1	20.7683
173	165	-1	151.4328	232	70	1	82.9100
174	188	1	195.8745	233	92	1	94.4689
175	12	1	23.1631	234	14	1	25.1475
176	14	1	23.5120	235	154	1	162.1329
177	6	1	14.8157	236	106	1	120.1679
178	40	1	56.5423	237	188	1	161.0306
179	430	1	436.3769	238	48	1	45.0959
180	22	1	31.3516	239	88	1	95.7492
181	71	-1	93.3738	240	8	1	16.5705
182	48	1	42.8558	241	50	1	67.9555
183	46	1	59.6027	242	322	1	355.5986
184	52	1	51.7444	243	270	1	269.3875
185	4	1	11.7772	244	134	1	145.6830
186	108	1	114.2256	245	206	1	194.9075
187	134	1	147.8083	246	114	1	107.7667
188	8	1	20.1458	247	174	1	177.9984
189	50	1	60.6175	248	22	1	32.0201
190	26	1	43.3695	249	116	1	142.7931
191	164	1	190.6173	250	159	-1	173.5928
192	44	1	63.7329	251	318	1	334.1780
193	50	1	71.7403	252	32	1	41.5299
194	108	1	118.2099	253	34	1	49.4254
195	40	1	49.7549	254	4	1	13.2797
196	62	1	77.5225	255	70	1	78.7884
197	155	-1	128.6433	256	90	1	112.8158
198	12	1	15.4730	257	13	-1	19.5379
199	170	1	162.9320	258	74	1	83.6537
200	76	1	77.4831	259	96	1	89.4168
201	38	1	47.7578	260	12	1	23.0330
202	147	-1	151.4960	261	28	1	33.4639
203	38	1	38.4180	262	360	1	388.5790
204	86	1	89.9429	263	94	1	106.1759
205	56	1	46.7214	264	10	1	18.1441

TABLE 2 (*continued*)

D	PL	NR	REG	D	PL	NR	REG
265	46	1	53.4454	325	1	-1	7.8617
266	42	1	52.5164	326	2	1	14.3402
267	174	1	171.0466	327	2	1	13.5324
268	50	1	65.1967	328	2	1	12.9601
269	273	-1	273.0119	329	222	1	222.7142
270	62	1	68.0688	330	2	1	12.1552
271	128	1	108.5475	331	166	1	200.8754
272	40	1	55.9533	332	4	1	23.1719
273	28	1	45.2243	333	10	1	34.0602
274	4	1	13.9617	334	126	1	154.5054
275	90	1	100.3821	335	240	1	272.3563
276	8	1	15.6401	336	2	1	10.7870
277	217	-1	226.3740	337	16	1	36.3692
278	314	1	355.4098	338	9	-1	17.7100
279	38	1	46.5751	339	270	1	307.0294
280	40	1	46.5511	340	10	1	20.4472
281	38	1	53.5975	341	190	1	194.7969
282	64	1	78.7796	342	6	1	29.9819
283	498	1	534.2976	343	454	1	436.8904
284	128	1	160.7741	344	76	1	78.8263
285	4	1	13.3667	345	56	1	59.2368
286	116	1	136.5902	346	49	-1	61.1623
287	150	1	165.4356	347	122	1	160.9133
288	14	1	25.3105	348	4	1	18.8724
289	118	1	144.4419	349	257	-1	304.3413
290	147	-1	151.2632	350	144	1	152.6403
291	18	1	26.7751	351	46	1	82.8841
292	46	1	52.1155	352	80	1	82.0567
293	271	-1	217.7830	353	78	1	80.3032
294	288	1	314.4971	354	148	1	179.1556
295	16	1	33.6647	355	56	1	70.2960
296	86	1	95.7995	356	52	1	53.3035
297	60	1	81.3816	357	144	1	123.0712
298	83	-1	71.4501	358	616	1	646.7992
299	86	1	92.5525	359	28	1	35.6185
300	8	1	18.5757	360	6	1	25.9792
301	26	1	36.0290	361	120	1	163.5475
302	216	1	218.2099	362	215	-1	222.2816
303	232	1	235.1681	363	66	1	75.3363
304	134	1	147.5885	364	182	1	191.6409
305	51	-1	67.4837	365	73	-1	100.3562
306	18	1	29.6483	366	134	1	148.8015
307	482	1	490.9257	367	470	1	420.3721
308	12	1	20.2496	368	114	1	142.1201
309	370	1	368.9610	369	66	1	103.1073
310	62	1	64.4781	370	189	-1	181.5445
311	94	1	120.4224	371	36	1	38.7761
312	4	1	10.7129	372	10	1	16.2358
313	61	-1	87.0739	373	168	1	166.4816
314	11	-1	18.2544	374	168	1	161.2338
315	20	1	33.8935	375	232	1	261.0032
316	8	1	23.0731	376	12	1	26.1178
317	79	-1	80.5413	377	57	-1	59.7611
318	4	1	12.1182	378	38	1	71.0890
319	58	1	72.4672	379	820	1	865.4943
320	4	1	12.9354	380	24	1	30.6399
321	4	1	13.5139	381	218	1	223.3955
322	4	1	14.3279	382	215	-1	256.2240
323	2	1	7.8586	383	502	1	524.9965

TABLE 2 (*continued*)

D	PL	NR	REG	D	PL	NR	REG
384	74	1	85.8386	443	158	1	185.2916
385	72	1	76.2725	444	50	1	54.7087
386	14	1	19.7568	445	16	1	34.7171
387	36	1	42.9218	446	22	1	38.3673
388	80	1	68.9839	447	52	1	62.8728
389	91	-1	88.9151	448	68	1	82.2346
390	32	1	33.6724	449	50	1	88.9369
391	162	1	171.7357	450	34	1	37.2955
392	64	1	77.3254	451	144	1	158.3658
393	64	1	89.0531	452	8	1	22.1945
394	493	-1	541.8796	453	308	1	266.8515
395	46	1	43.7689	454	1062	1	1211.1332
396	24	1	44.2289	455	188	1	187.0916
397	260	1	248.2283	456	12	1	21.6822
398	2	1	13.9520	457	41	-1	50.0053
399	28	1	41.2984	458	206	1	229.7405
400	108	1	145.4757	459	448	1	491.2462
401	80	1	99.6167	460	156	1	159.9725
402	58	1	66.0425	461	71	-1	73.7591
403	208	1	265.5586	462	126	1	114.3615
404	160	1	215.3186	463	778	1	766.1231
405	236	1	257.7341	464	42	1	58.3189
406	44	1	43.1812	465	6	1	16.5710
407	60	1	59.3021	466	24	1	31.8975
408	42	1	58.9399	467	428	1	471.6362
409	223	-1	275.1346	468	30	1	36.9668
410	68	1	81.7698	469	226	1	274.8130
411	64	1	76.6110	470	352	1	371.6888
412	142	1	152.9192	471	82	1	83.2163
413	334	1	288.8929	472	164	1	190.5559
414	206	1	223.2892	473	148	1	176.9863
415	324	1	306.4599	474	72	1	94.1183
416	82	1	91.4846	475	66	1	100.7383
417	32	1	46.7187	476	10	1	22.9074
418	124	1	151.9582	477	284	1	294.5090
419	104	1	92.2576	478	14	1	25.8042
420	38	1	41.1450	479	364	1	361.1309
421	18	1	23.1981	480	86	1	100.1847
422	562	1	566.4572	481	51	-1	65.8861
423	34	1	38.9541	482	88	1	116.3825
424	176	1	193.3631	483	78	1	76.4443
425	50	1	66.1080	484	108	1	119.7289
426	116	1	133.3298	485	161	-1	183.4538
427	232	1	256.8214	486	294	1	311.9160
428	422	1	401.2819	487	184	1	185.0248
429	14	1	30.5790	488	174	1	156.7782
430	28	1	35.7034	489	264	1	277.8649
431	468	1	436.4574	490	220	1	274.5838
432	38	1	59.8639	491	566	1	559.1352
433	168	1	175.7602	492	188	1	190.6584
434	8	1	25.3625	493	203	-1	197.4414
435	128	1	131.6412	494	244	1	286.6531
436	70	1	103.3975	495	46	1	66.3733
437	376	1	329.6845	496	50	1	62.2386
438	64	1	69.6490	497	6	1	19.6465
439	434	1	388.0275	498	154	1	204.6873
440	40	1	68.4719	499	46	1	79.0547
441	40	1	52.1058	500	140	1	161.6921
442	58	1	69.9187				

Notice that NR $= -1$ if and only if PL is odd for $5 \leqslant d \leqslant 500$.

The computations were carried out on the CDC-Cyber 76 of the Universität zu Köln and the VAX 11/785 of the Department of Electrical Engineering of The Ohio State University.

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