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A Table of Fundamental Pairs of Units in Totally Real Cubic Fields

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In honor of Daniel Shanks' 70th birthday

Abstract. We apply a method of Cusick [5] to tabulate data on the first 250 totally real cubic fields F having discriminant $D \leqslant 6{,}885$. Apart from D, we list the class number H and the regulator R of F. Also given are the integer coefficients A, B, C of a defining polynomial $g(x) = x^3 - Ax^2 + Bx - C$, its index I, and its largest zero R_0. For $j = 1,2$, we also tabulate both the integer coefficients X_j, Y_j, Z_j for the two units $E_j = (X_j + R_0 Y_j + R_0^2 Z_j)/I$ with norm $+1$, forming a fundamental pair, as well as the E_j and the integers $F_j = \operatorname{trace}(E_j^2)$.

1. Introduction. The first author [4], [5] recently gave a new method for finding a fundamental pair of units in totally real cubic fields. This method is an improvement of one due to Godwin [8]. The earlier version [4] of the main theorem contained some possible exceptional cases (presumed not to exist) for which the determination of the fundamental pair was less simple. Godwin [10] gave a clever and very short argument which proved that the exceptional cases do not exist. The later paper [5] provided a much simpler proof of the main theorem in which there are no longer any exceptional cases.

The purpose of the present paper is to apply the algorithm of [5] to the actual computation of units and to tabulate the fundamental pairs of units E_1 and E_2 and other information for all the 250 totally real fields F with field discriminant $D \leqslant 6{,}885$. The resulting table has been photographically reproduced from the computer output.

The fundamental pairs of units are standardized so as to have a norm of $+1$; in common with all integers in F, they have a representation

$$(1) \qquad E_j = \left(X_j + R_0 Y_j + R_0^2 Z_j\right)/I \qquad (j = 1,2).$$

Here X_j, Y_j, Z_j are rational integers, R_0 is the largest zero of a certain irreducible, monic defining polynomial $g(x)$ with rational integer coefficients and polynomial discriminant D_g, and $I = I_g$, the index of $g(x)$, is the positive integer $(D_g/D)^{1/2}$.

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The table lists the coefficients A, B, C in

$$(2)\qquad \begin{aligned} g(x) &= x^3 - Ax^2 + Bx - C \\ &= (x - R_0)(x - R_1)(x - R_2), \qquad R_0 > R_1 > R_2 > 0, \end{aligned}$$

as well as the index I of $g(x)$ and an approximation to the largest zero R_0 of $g(x)$. In addition, the coefficients X_j, Y_j, Z_j in (1) and approximations to the units E_j are given; also tabulated are the quantities

$$(3)\qquad F_j = \operatorname{trace}\left(E_j^2\right) \qquad (j = 1, 2),$$

which play an important role in our determination of the E_j. Of course, $g(x)$ is not unique but, as we explain in Section 3, we have tried to pick it so that I is minimal.

The table also lists the discriminant D and class number H of the field F as well as an approximation to its regulator R defined by

$$(4)\qquad R = \left| \det \begin{pmatrix} \log|E_1| & \log|E_1'| \\ \log|E_2| & \log|E_2'| \end{pmatrix} \right|,$$

where E_1', E_2' are the conjugates of E_1, E_2 obtained by replacing R_0 in (1) by R_1. The values for D and H have been copied from the table of Angell [1], as have most of the triples A, B, C. For $D < 20{,}000$, a list of discriminants and class numbers has been computed independently by Godwin [9] and others.

In Section 3 we give some of the details of the computer program which determined the X_j, Y_j, Z_j of (1). Here we give a brief outline of Cusick's algorithm on which the program is based. If F is a totally real cubic field with a defining integer r, then any integer λ in F is of the form

$$(5)\qquad \lambda = x + yr + zQ(r),$$

where $Q(r)$ is a fixed quadratic in r with rational coefficients and x, y, z are rational integers determined by λ. For any α in F, we define

$$(6)\qquad T(\alpha) = \operatorname{trace}(\alpha^2) = \alpha^2 + \alpha'^2 + \alpha''^2, \qquad \operatorname{norm}(\alpha) = \alpha\alpha'\alpha'',$$

where $\alpha, \alpha', \alpha''$ denote the conjugates of $\alpha = \alpha^{(0)}$. As the function $T(\lambda)$ is a positive-definite ternary quadratic form in x, y, z whose values are rational integers, we can systematically change x, y, z and seek a minimal value of $T(\lambda)$ for λ's which are units. The minimizing unit λ is denoted by ε_1. We then continue computing $T(\lambda)$, if necessary, until another minimizing unit $\varepsilon_2 \neq \pm\varepsilon_1^m$, m a rational integer, is obtained. Cusick's result [5, Theorem 1] is:

THEOREM. *The units ε_1, ε_2 are a fundamental pair of units for the totally real cubic field.*

In the table, the defining integer r of F is called R_0 and the units ε_1, ε_2 are called E_1, E_2.

It is a consequence of (3) and $|E_j E_j' E_j''| = 1$ that, for example, $F_j \geqslant E_j''^2 = 1/(E_j^2 E_j'^2) \geqslant 1/(F_j F_j)$ for $j = 1, 2$; more generally, for $i = 0, 1, 2$, we have $1/F_j \leqslant |E_j^{(i)}| \leqslant F_j^{1/2}$. As F_j is minimal, $|E_j^{(i)}|$ is not too small, and the relative error in computing $|E_j^{(i)}|$ should be good; the same is true of the absolute error in calculating $\log|E_j^{(i)}|$. Hence, the absolute error in finding the regulator R in (4) should be satisfactory.

Case	Field F			$g(x) = x^3 - Ax^2 + Bx - C$					Fundamental Pair of Units $E_j = (X_j + R_0 Y_j + R_0^2 Z_j)/I$									
No.	D	H	R	A	B	C	I	R_0	E_1	X_1	Y_1	Z_1	F_1	E_2	X_2	Y_2	Z_2	F_2
1	49	1	.53	7	14	7	1	3.801938	1.247E+0	2	-4	1	5	-4.450E-1	-5	5	-1	5
2	81	1	.85	6	9	1	1	3.532089	1.879E+0	0	-3	1	6	-3.473E-1	-2	4	-1	6
3	148	1	1.66	8	18	10	1	4.481194	1.481E+0	-3	1	0	7	-1.939E-1	-7	6	-1	11
4	169	1	1.37	8	17	5	1	4.377203	2.651E+0	1	-4	1	9	-2.739E-1	-3	5	-1	9
5	229	1	2.36	9	23	14	1	4.860806	-1.861E+0	3	-1	0	8	1.392E-1	5	-1	0	20
6	257	1	1.97	7	12	1	1	4.198691	1.199E+0	-3	1	0	10	1.987E-1	-4	1	0	17
7	316	1	3.91	7	12	2	1	4.342923	-1.464E-1	-3	5	-1	15	-1.854E+0	1	-5	1	31
8	321	1	2.57	8	17	7	1	4.699628	1.700E+0	-3	1	0	9	-1.113E-1	6	-6	1	18
9	361	1	1.95	10	27	11	1	5.285142	3.507E+0	2	-5	1	14	-2.219E-1	-4	6	-1	14
10	404	1	3.76	8	16	2	1	4.655442	-1.655E+0	3	-1	0	11	5.137E-2	-3	-4	1	43
11	469	1	3.85	10	28	17	1	5.391382	1.391E+0	-4	1	0	12	-6.406E-2	14	-8	1	68
12	473	1	2.84	9	22	11	1	5.128419	-2.128E+0	3	-1	0	10	-3.659E+0	-3	5	-1	21
13	564	1	5.40	10	28	18	1	5.514137	-4.514E+0	1	-1	0	27	-7.835E+0	-5	5	-1	63
14	568	1	6.09	11	34	22	1	5.761557	-1.354E-1	7	-7	1	23	-6.252E+0	-9	12	-2	67
15	621	1	5.40	9	21	6	1	5.145103	1.451E-1	-5	1	0	24	-3.290E+0	7	-2	0	51
16	697	1	2.71	9	20	1	1	5.166013	1.166E+0	-4	1	0	17	1.660E-1	-5	1	0	26
17	733	1	5.31	10	26	7	1	5.518199	-1.518E+0	4	-1	0	16	3.640E-2	-11	2	0	115
18	756	1	5.69	9	21	7	1	5.261802	-1.157E-1	-4	6	-1	27	1.493E+0	4	-11	2	111
19	761	1	3.53	11	34	23	1	5.891954	-1.892E+0	4	-1	0	13	1.080E-1	6	-1	0	29
20	785	1	4.10	10	27	13	1	5.575773	1.576E+0	-4	1	0	14	-5.884E-2	-8	7	-1	34
21	788	1	5.99	11	33	17	1	5.877404	1.226E-1	6	-1	0	31	2.755E+0	-9	2	0	67
22	837	1	6.80	9	21	8	1	5.361469	-2.361E+0	3	-1	0	12	-6.667E-3	-21	20	-3	300
23	892	1	8.32	10	25	2	1	5.597735	6.346E+0	3	-5	1	47	-3.504E+0	1	-12	2	195
24	940	1	8.91	9	20	2	1	5.292402	-6.548E+0	-5	5	-1	63	-3.730E-2	9	-7	1	79
25	961	1	12.20	13	46	32	2	7.297071	-2.741E+0	-26	32	-4	323	1.776E+1	2	-10	2	323
26	985	1	3.72	8	15	1	1	5.093763	2.094E+0	-3	1	0	13	9.376E-2	-5	1	0	29
27	993	1	5.55	10	27	15	1	5.773387	-2.692E+0	-4	6	-1	30	-8.171E-2	7	-7	1	33
28	1016	1	10.13	8	15	2	1	5.177410	8.148E-2	1	5	-1	55	-8.436E+0	1	-7	1	183
29	1076	1	6.93	12	40	26	1	6.328872	1.329E+0	-5	1	0	19	1.338E-2	19	-3	0	291
30	1101	1	9.18	13	47	32	1	6.687400	7.597E+0	3	-6	1	60	-1.819E+1	-9	12	-2	339
31	1129	1	6.73	9	20	3	1	5.397662	-2.251E+0	1	-6	1	66	9.544E+0	2	-4	1	93
32	1229	1	8.23	10	26	9	1	5.751532	1.752E+0	-4	1	0	16	6.128E-3	-23	4	0	515
33	1257	1	6.20	10	25	3	1	5.723956	-1.724E+0	4	-1	0	18	-2.797E-2	13	-8	1	153
34	1300	1	6.55	12	38	14	1	6.423622	-1.424E+0	5	-1	0	23	1.875E-2	1	-13	2	183
35	1304	1	11.93	11	10	2	1	10.022119	-2.800E+0	7	-11	1	23	2.207E-3	-49	55	-5	1303
36	1345	1	4.92	9	20	5	1	5.571201	-2.571E+0	3	-1	0	14	-3.988E-2	-8	7	-1	54
37	1369	1	3.13	14	53	29	1	7.187101	5.345E+0	4	-7	1	30	-1.576E-1	-6	8	-1	30
38	1373	1	9.42	12	40	27	1	6.439312	-4.976E-2	10	-8	1	44	-3.779E+0	-11	14	-2	131
39	1384	1	10.38	13	46	26	1	6.785667	-2.571E+0	11	-2	0	99	-2.573E-2	15	-9	1	103
40	1396	1	8.15	10	26	10	1	5.843404	8.495E-2	1	-6	1	59	3.758E+0	-3	7	-1	99
41	1425	1	6.68	11	32	13	1	6.113538	1.135E-1	-6	1	0	33	-7.694E+0	-7	6	-1	78
42	1436	1	12.70	12	37	8	1	6.483612	3.278E-2	13	-2	0	163	1.624E+1	-3	-10	2	291
43	1489	1	3.36	11	30	1	1	6.142495	1.142E+0	-5	1	0	26	1.425E-1	-6	1	0	37
44	1492	1	7.65	11	31	7	1	6.126552	1.266E-1	-6	1	0	35	5.649E+0	11	-7	1	119
45	1509	1	11.30	10	26	11	1	5.925423	7.458E-2	6	-1	0	36	-5.994E+0	30	-12	1	612
46	1524	1	10.45	11	33	21	1	6.273073	2.273E+0	-4	1	0	15	-1.161E-3	-65	48	-6	1731
47	1556	1	8.38	13	47	33	1	6.809786	-1.051E-1	8	-8	1	43	3.210E+0	-13	16	-2	259
48	1573	1	8.44	11	33	22	1	6.346462	-5.346E+0	1	-1	0	36	-1.177E-2	-21	16	-2	179
49	1593	1	6.33	12	39	21	1	6.487051	1.487E+0	-5	1	0	21	2.590E-2	13	-2	0	147
50	1620	1	10.17	12	36	2	1	6.552475	-6.760E-2	-3	7	-1	39	1.186E+1	1	-18	3	219

Case No.	Field F			$g(x)=x^3-Ax^2+Bx-C$					Fundamental Pair of Units $E_j=(X_j+R_0Y_j+R_0^2Z_j)/I$									
	D	H	R	A	B	C	I	R_0	E_1	X_1	Y_1	Z_1	F_1	E_2	X_2	Y_2	Z_2	F_2
51	1708	1	12.87	11	32	14	1	6.210184	-9.510E-2	-5	7	-1	47	1.532E+1	15	-31	5	927
52	1765	1	9.44	13	45	20	1	6.890547	1.095E-1	7	-1	0	44	-1.816E+1	1	11	-2	368
53	1772	1	15.37	43	32	6	1	42.245892	2.202E+0	77	-213	5	31	-6.028E-4	-1399	3793	-89	4687
54	1825	1	4.49	10	25	5	1	5.919089	1.919E+0	-4	1	0	18	8.091E-2	6	-1	0	38
55	1849	1	18.92	14	51	22	2	7.888236	6.615E-3	-14	-14	2	323	-1.023E+1	-38	18	-2	323
56	1901	1	10.66	11	31	8	1	6.230725	3.461E+0	-9	2	0	83	-2.386E-2	11	-8	1	100
57	1929	1	8.22	13	46	27	1	6.902223	9.778E-2	7	-1	0	42	1.113E+1	-2	-5	1	150
58	1937	1	6.54	11	32	15	1	6.295416	-2.295E+0	4	-1	0	17	-2.648E-2	17	-9	1	157
59	1940	1	11.09	9	19	1	1	5.693995	5.694E+0	0	1	0	43	-9.776E+0	13	-4	0	259
60	1944	1	15.60	12	39	22	1	6.584225	-1.343E+1	-3	5	-1	183	-1.236E+1	-31	16	-2	579
61	1957	2	4.55	13	47	34	1	6.911503	-1.912E+0	5	-1	0	20	8.850E-2	7	-1	0	40
62	2021	1	11.52	9	19	2	1	5.763724	-2.764E+0	3	-1	0	16	-5.087E+1	1	-9	0	3324
63	2024	1	15.77	11	30	2	1	6.254820	-1.870E+1	-3	10	-2	355	2.455E+1	1	-15	3	615
64	2057	1	6.78	12	37	9	1	6.602581	-1.603E+0	5	-1	0	25	2.141E-2	4	6	-1	102
65	2089	1	20.76	12	35	8	2	7.440552	2.116E+1	6	-10	2	451	-2.876E+1	2	-8	0	1091
66	2101	1	8.54	14	54	37	1	7.283186	1.283E+0	-6	1	0	28	-7.751E-3	25	-18	2	259
67	2177	1	7.52	10	25	7	1	6.073562	7.356E-2	-6	1	0	38	-1.152E+1	-5	5	-1	145
68	2213	1	12.68	14	52	23	1	7.355968	-1.356E+0	6	-1	0	32	1.013E-3	1	44	-6	2515
69	2228	1	11.09	15	61	37	1	7.714645	-8.513E+0	-3	7	-1	75	5.480E+0	22	-33	4	535
70	2233	1	5.52	11	32	17	1	6.442683	2.443E+0	-4	1	0	17	-3.330E-2	10	-8	1	61
71	2241	1	8.26	12	39	23	1	6.669664	-8.466E+0	-4	6	-1	78	-1.314E+1	-2	5	-1	174
72	2292	1	14.36	13	13	3	1	11.931522	3.183E+0	4	-12	1	27	6.447E+1	-47	69	-5	4707
73	2296	1	14.27	14	51	16	1	7.401976	2.458E-2	3	7	-1	87	-8.829E+0	3	-9	1	279
74	2300	1	18.12	11	32	18	1	6.507903	1.581E-2	-13	2	0	163	-5.447E+1	1	11	-3	3023
75	2349	1	11.92	12	36	3	1	6.670622	8.473E+0	4	-6	1	84	9.947E+0	1	-12	2	147
76	2429	1	13.28	14	50	9	1	7.455204	-6.157E-2	4	-8	1	48	3.521E+0	-10	54	-7	2560
77	2505	1	10.68	11	30	3	1	6.349967	-1.057E+1	-2	5	-1	114	1.814E+1	1	-10	2	339
78	2557	1	10.72	11	31	10	1	6.401222	1.437E+1	-1	-4	1	212	-3.658E-2	23	-10	1	380
79	2589	1	16.29	26	180	141	1	12.739724	-5.557E-2	16	-14	1	36	2.832E+1	10	-75	6	3540
80	2597	3	4.80	10	24	1	1	6.079119	2.079E+0	-4	1	0	20	7.912E-2	-6	1	0	40
81	2636	1	18.38	14	49	2	1	7.515853	3.171E-2	-15	2	0	227	-2.669E+1	15	17	-3	959
82	2673	1	7.76	12	39	25	1	6.816914	-6.500E-2	8	-8	1	66	2.065E+0	-6	8	-1	102
83	2677	1	11.16	12	38	17	1	6.726235	1.088E+1	6	-6	1	128	-4.568E+0	4	-8	1	152
84	2700	1	20.37	15	60	30	1	7.805603	-4.066E+0	-29	11	-1	543	-3.744E-3	31	-43	5	1263
85	2708	1	12.95	14	54	38	1	7.383185	-1.666E+1	-11	14	-2	291	-2.204E+1	-9	13	-2	487
86	2713	1	12.34	15	62	45	1	7.745141	2.608E-2	2	-8	1	93	-9.052E+0	-13	16	-2	147
87	2777	2	3.95	13	42	1	1	7.124784	1.125E+0	-6	1	0	37	1.248E-1	-7	1	0	50
88	2804	1	15.24	11	31	11	1	6.474194	-2.474E+0	4	-1	0	19	-2.124E-4	-170	91	-10	19219
89	2808	1	20.31	12	39	26	1	6.882021	-2.714E+1	-15	12	-2	771	-4.986E+1	-11	15	-3	2487
90	2836	1	9.69	10	24	2	1	6.151148	7.026E-2	1	6	-1	91	2.859E+0	1	-12	2	259
91	2857	1	4.87	13	46	29	1	7.085532	-2.086E+0	5	-1	0	22	8.553E-2	-7	1	0	42
92	2917	1	11.93	13	43	8	1	7.112228	1.122E-1	-7	1	0	48	1.536E+1	-21	-2	1	692
93	2920	1	17.94	12	11	2	1	11.018119	-1.284E+1	9	-13	1	199	1.809E-3	1	55	-5	1703
94	2941	1	13.72	17	79	58	1	8.583687	-2.167E+0	15	-2	0	179	1.018E-2	-5	-8	1	204
95	2981	1	14.63	13	45	22	1	7.093000	9.300E-2	-7	1	0	44	2.550E+1	-39	2	1	2056
96	2993	1	7.51	13	44	15	1	7.101785	1.018E-1	-7	1	0	46	-3.204E+0	11	-2	0	115
97	3021	1	17.40	10	24	3	1	6.217581	2.858E+1	1	-8	2	819	-3.253E+0	50	-21	2	2244
98	3028	1	20.35	12	38	18	1	6.803542	9.663E+0	11	-7	1	147	-1.556E+1	17	-32	4	2307
99	3124	1	19.56	15	59	33	2	8.552799	-4.864E+0	-5	8	-1	39	-1.530E+2	-5	76	-13	23799
100	3132	1	22.49	15	57	15	2	8.506642	1.328E-2	-34	4	0	291	-5.250E+0	134	-34	2	3903

Case	Field F			$g(x)=x^3-Hx^2+Bx-C$					Fundamental Pair of Units $E_j=(X_j+R_0Y_j+R_0^2Z_j)/I$									
No.	D	H	R	A	B	C	I	R_0	E_1	X_1	Y_1	Z_1	F_1	E_2	X_2	Y_2	Z_2	F_2
101	3137	1	7.81	12	37	11	1	6.787759	1.788E+0	-5	1	0	25	-1.616E-2	15	-9	1	173
102	3144	1	24.86	14	49	12	2	8.209050	-6.277E-3	-118	39	-3	2919	-2.412E+0	146	-43	3	4551
103	3173	1	15.47	15	61	38	1	7.825476	-1.669E-2	17	-10	1	136	5.983E+0	23	-10	1	292
104	3221	1	16.39	11	31	14	1	6.662465	7.414E+0	3	-6	1	84	2.849E+1	13	-11	2	868
105	3229	1	14.78	10	24	5	1	6.337517	1.681E+1	2	-4	1	284	-4.456E+0	6	-27	4	1584
106	3252	1	17.26	11	31	15	1	6.717741	-4.937E-2	22	-10	1	267	4.351E+0	-104	43	-4	6315
107	3261	1	18.76	14	54	39	1	7.469990	4.083E-2	4	-8	1	96	8.907E+1	50	-62	9	8160
108	3281	1	7.45	14	51	17	1	7.512680	-1.513E+0	6	-1	0	34	2.536E-2	-15	2	0	211
109	3305	1	9.53	11	30	5	1	6.509040	4.313E+0	1	-6	1	58	1.282E+1	3	-5	1	169
110	3316	1	9.43	15	59	23	1	7.901190	9.881E-2	8	-1	0	59	3.704E+0	-20	3	0	363
111	3325	1	13.26	12	38	19	1	6.874076	-4.748E+0	9	-2	0	83	8.461E-3	-6	-6	1	248
112	3356	1	21.93	14	50	24	2	8.366910	-3.367E+0	10	-2	0	31	-3.992E-5	-734	322	-28	86575
113	3368	1	23.05	17	78	50	1	8.658042	3.932E-2	3	-9	1	71	6.238E+1	-39	55	-5	12103
114	3496	1	17.20	15	62	46	1	7.841222	-8.624E-2	9	-9	1	71	5.564E+1	59	-71	9	4295
115	3508	1	10.04	13	45	23	1	7.174972	-8.045E-2	-6	8	-1	75	2.416E+0	13	-23	3	651
116	3540	1	20.31	14	50	10	1	7.567030	1.443E-2	-1	-15	2	303	-3.602E+1	-7	34	-5	1419
117	3569	1	7.79	13	46	31	1	7.232331	2.232E+0	-5	1	0	22	-1.670E-2	20	-10	1	169
118	3576	1	27.75	14	50	28	2	8.512976	9.367E+0	10	-16	2	195	-1.271E+2	2	38	-8	16311
119	3580	1	21.09	13	41	13	2	8.184732	1.988E+1	4	-12	2	419	-8.781E+0	94	-30	2	1823
120	3592	1	18.70	16	67	26	1	8.341806	-1.894E-2	-11	18	-2	227	-1.624E+1	45	1	-1	2311
121	3596	1	23.05	12	37	12	1	6.864885	7.245E-2	1	-7	1	95	-1.745E+2	-43	125	-21	36175
122	3604	1	9.32	16	68	34	1	8.305810	-1.306E+0	7	-1	0	43	8.386E-3	9	-26	3	603
123	3624	1	28.01	11	30	6	1	6.578021	1.138E+1	1	-5	1	135	1.762E+2	-91	-12	8	39747
124	3721	1	13.71	30	239	207	1	14.674990	4.005E+1	127	-138	9	1605	-2.436E-2	-131	141	-9	1605
125	3732	1	19.08	13	43	9	1	7.206330	4.952E+0	-5	23	-3	927	2.175E-2	7	-37	5	1935
126	3736	1	17.86	13	42	2	1	7.225800	5.895E+0	1	-21	3	471	-3.280E+1	1	17	-3	1079
127	3753	1	8.04	15	60	31	1	7.910752	8.925E-2	8	-1	0	57	2.822E+0	-13	2	0	147
128	3873	1	12.01	14	49	3	1	7.627161	1.041E+1	-2	-6	1	114	-4.687E+0	1	-16	2	435
129	3877	1	13.44	14	52	25	1	7.555673	1.556E+0	-6	1	0	32	1.055E-3	-68	9	0	4188
130	3889	1	7.03	11	30	7	1	6.641892	-2.642E+0	4	-1	0	21	-2.041E-2	9	-8	1	98
131	3892	1	13.04	12	38	22	1	7.057087	5.709E-2	-7	1	0	47	-1.429E+1	21	-5	0	503
132	3941	1	17.77	17	77	42	1	8.737949	-2.774E-2	11	-10	1	76	-2.600E+1	-31	53	-6	1252
133	3957	1	22.29	17	78	60	2	9.326577	-1.185E+1	-2	7	-1	144	2.980E+1	154	-178	18	14067
134	3969	3	4.20	18	87	55	1	9.137763	7.259E+0	6	-9	1	54	-1.211E-1	-8	10	-1	54
135	3969	3	12.59	18	87	62	2	9.651205	-1.328E+1	-14	18	-2	195	-1.750E-2	26	-22	2	195
136	3973	1	13.15	12	38	23	1	7.111039	-3.111E+0	4	-1	0	20	6.485E-4	64	-9	0	3972
137	3981	2	10.63	13	45	24	1	7.248898	-2.249E+0	5	-1	0	24	4.408E-3	29	-4	0	771
138	3988	1	11.49	11	19	7	1	8.968502	-1.494E+1	3	-2	0	227	1.694E+1	-1	2	0	291
139	4001	1	10.59	12	37	13	1	6.935152	6.485E-2	7	-1	0	49	6.805E+0	-14	3	0	210
140	4065	1	14.12	13	46	33	1	7.357463	-6.357E+0	1	-1	0	54	-2.962E+1	-17	13	-2	918
141	4104	1	23.76	15	57	19	2	8.697700	2.698E+0	-12	2	0	39	-2.497E-5	606	-348	32	85731
142	4193	1	11.10	14	53	33	1	7.589844	1.407E+1	2	-6	1	202	-1.607E+1	-4	6	-1	262
143	4212	3	5.46	12	36	6	1	6.930454	1.930E+0	-5	1	0	27	6.955E-2	7	-1	0	51
144	4281	1	15.30	13	44	17	1	7.267567	-1.848E+1	-2	5	-1	354	-2.445E-2	14	-31	4	1449
145	4312	3	7.23	13	40	8	2	8.292951	-6.824E-2	-6	9	-1	55	-1.932E+0	2	-9	1	87
146	4344	1	30.53	19	98	74	1	9.545803	1.287E-2	59	-73	7	2871	7.486E+0	207	-250	24	31203
147	4345	1	9.16	10	23	1	1	6.467859	6.468E+0	0	1	0	54	5.936E+0	-7	2	0	83
148	4360	1	23.67	11	30	10	1	6.810821	-2.695E+1	-1	3	-1	743	4.662E-4	37	15	-3	4359
149	4364	1	25.75	16	66	28	2	9.046483	4.648E-2	-18	2	0	79	-3.265E+1	-890	254	-18	150335
150	4409	1	9.62	11	30	11	1	6.861319	4.847E-2	1	-7	1	129	-1.477E+1	-2	5	-1	226

Case No.	Field F: D	H	R	$g(x)=x^3-Ax^2+Bx-C$: A	B	C	I	R_0	Fundamental Pair of Units $E_j=(X_j+R_0Y_j+R_0^2Z_j)/I$: E_1	X_1	Y_1	Z_1	F_1	E_2	X_2	Y_2	Z_2	F_2
151	4481	1	40.12	15	58	32	2	8.863876	-8.587E+0	-22	36	-4	835	1.689E-4	-62	202	-22	37891
152	4489	1	19.70	17	74	43	3	10.085427	-6.375E-3	-47	45	-4	714	2.603E+1	15	-24	3	714
153	4493	1	20.11	13	43	10	1	7.288665	1.239E+1	3	-6	1	156	9.753E+1	13	-54	9	9696
154	4596	1	21.92	11	29	1	1	6.683060	6.683E+0	0	1	0	63	-1.233E-4	81	-99	13	16947
155	4597	1	15.31	14	50	11	1	7.660286	-1.660E+0	6	-1	0	36	5.549E-4	-74	2	1	6000
156	4628	1	16.72	14	52	26	1	7.636249	-2.435E+1	-7	13	-2	595	-5.334E+1	-11	25	-4	2851
157	4641	1	13.33	13	42	3	1	7.312560	-2.695E-2	5	-8	1	90	5.571E+0	1	-14	2	243
158	4649	1	12.92	12	37	15	1	7.060432	6.043E-2	-7	1	0	49	2.473E+1	-11	-2	1	749
159	4684	1	18.88	17	76	34	1	8.822294	-1.951E+1	-5	16	-2	387	1.664E+1	-1	2	0	483
160	4692	1	21.19	14	48	12	2	8.550309	3.353E+0	2	-8	1	51	-2.777E-4	94	-182	20	24891
161	4729	1	4.49	15	56	1	1	8.110974	1.111E+0	-7	1	0	50	1.110E-1	-8	1	0	65
162	4749	1	23.60	16	70	51	1	8.339652	1.415E-2	-11	18	-2	387	-4.090E+0	76	-93	10	9216
163	4764	1	32.10	14	53	34	1	7.661600	-1.007E+1	-5	7	-1	111	-4.359E+2	33	138	-26	206115
164	4765	1	17.65	13	45	26	1	7.379390	2.379E+0	-5	1	0	24	-1.179E-4	-257	116	-11	34388
165	4825	1	36.40	14	47	2	2	8.495135	9.730E-3	34	-4	0	323	4.299E+1	-526	140	-8	69443
166	4841	1	9.11	16	69	43	1	8.370366	1.370E+0	-7	1	0	41	-1.100E-2	22	-11	1	246
167	4844	1	28.52	13	44	18	1	7.338368	1.616E+1	-1	-5	1	271	-2.320E+2	-43	99	-17	53983
168	4852	1	15.26	11	17	5	1	9.213847	-4.757E+0	-12	10	-1	91	-3.609E-3	27	-49	5	1083
169	4853	1	19.96	14	14	3	1	12.935648	-3.768E+0	10	-14	1	48	-2.168E-4	-151	348	-26	11827
170	4857	1	14.15	17	78	51	1	8.763635	1.069E+1	4	-8	1	117	6.577E+0	-28	39	-4	1677
171	4860	1	30.54	30	237	178	1	14.765898	-1.147E-2	-33	17	-1	399	1.995E+1	889	-1122	72	116643
172	4892	1	30.19	12	37	16	1	7.117176	1.365E+0	29	-11	1	559	-1.076E-4	129	-132	16	24515
173	4933	1	15.44	11	29	2	1	6.743614	1.465E-2	-5	-6	1	188	-2.458E+0	1	-14	2	515
174	5073	1	14.29	16	67	27	1	8.444316	-1.444E+0	7	-1	0	45	1.154E-3	76	-9	0	5322
175	5081	1	9.90	12	37	17	1	7.170782	-5.386E-2	-6	8	-1	102	1.140E+1	3	-6	1	149
176	5089	1	11.65	14	51	19	1	7.686845	1.687E+0	-6	1	0	34	-2.398E-3	25	-34	4	1459
177	5172	1	23.69	16	64	12	2	9.145480	-2.907E+0	2	-10	1	111	-3.957E-4	-70	218	-23	32943
178	5204	1	22.71	14	48	20	2	8.808464	-1.537E+1	-6	6	-1	239	-4.754E+1	14	-30	2	3979
179	5261	1	18.48	16	66	19	1	8.488250	2.350E-2	17	-2	0	275	-6.441E+0	36	-5	0	1228
180	5281	1	9.94	14	53	35	1	7.727511	-3.894E+0	-6	8	-1	78	1.635E+1	3	-6	1	269
181	5297	1	11.61	12	37	19	1	7.270200	-3.270E+0	4	-1	0	22	-1.825E-3	-45	28	-3	1177
182	5300	1	17.89	28	208	166	1	13.775451	5.907E+0	9	-14	1	43	2.991E+1	297	-350	24	12323
183	5325	1	27.47	14	52	27	1	7.708850	4.936E-3	97	-28	2	6483	1.311E+2	28	-56	9	17208
184	5329	1	19.25	16	61	3	3	9.818119	-2.571E+0	3	-60	6	2163	-4.644E+1	-9	26	-4	2163
185	5333	1	22.00	13	45	28	1	7.493311	1.338E-2	15	-2	0	211	1.945E+1	-33	7	0	1132
186	5353	1	8.19	13	44	19	1	7.403519	-2.404E+0	5	-1	0	26	1.254E-2	3	7	-1	201
187	5356	1	24.63	13	12	2	1	12.015112	-1.893E+0	-13	61	-5	959	1.258E-3	-71	78	-6	3171
188	5368	1	24.46	18	83	20	1	9.367485	1.232E+0	-89	19	-1	7095	-5.059E-3	-143	34	-2	18083
189	5369	1	11.04	15	58	17	1	8.092261	9.226E-2	-8	1	0	61	-2.159E+1	-12	15	-2	521
190	5373	1	21.31	18	84	29	1	9.329348	1.196E-2	28	-3	0	732	-6.550E+1	-16	60	-7	4320
191	5468	1	30.66	18	85	38	1	9.296159	-1.759E+1	1	-2	0	547	-4.417E+1	-49	47	-5	2671
192	5477	1	22.72	12	22	9	1	9.861687	2.836E+1	10	-8	1	836	5.981E+1	2	-4	1	3580
193	5497	1	39.31	16	67	44	2	9.313195	-2.654E+1	-10	14	-2	707	-7.514E+1	-3030	868	-60	1411907
194	5521	1	5.71	12	35	1	1	7.068398	2.068E+0	-5	1	0	29	6.840E-2	-7	1	0	53
195	5529	1	13.78	16	65	11	1	8.537114	-4.829E-2	-4	9	-1	78	-4.895E+1	-5	29	-4	2433
196	5556	1	21.74	17	77	43	1	8.838905	-2.678E+0	15	-2	0	195	-7.806E-4	63	-69	7	4083
197	5613	1	28.27	17	79	60	1	8.783959	4.363E+0	-17	20	-2	531	9.614E-4	61	-86	9	7920
198	5620	1	15.78	11	29	5	1	6.905030	2.905E+0	-4	1	0	23	9.524E+1	14	-78	13	17579
199	5621	1	22.79	17	75	26	1	8.909973	9.003E-2	9	-1	0	76	-1.126E+2	175	39	-8	48256
200	5624	1	29.64	17	72	8	2	9.543212	6.184E+0	2	-18	2	163	-2.137E-4	82	-209	21	26519

Case	Field F			$g(x)=x^3-Ax^2+Bx-C$					Fundamental Pair of Units $E_j=(X_j+R_0Y_j+R_0^2Z_j)/I$									
No.	D	H	R	A	B	C	I	R_0	E_1	X_1	Y_1	Z_1	F_1	E_2	X_2	Y_2	Z_2	F_2
201	5629	1	13.75	18	86	47	1	9.267435	-1.267E+0	8	-1	0	56	-1.404E-3	50	-61	6	2720
202	5637	1	27.21	17	17	4	1	15.949885	3.610E+1	5	-14	1	1320	2.102E-4	-63	-12	1	9528
203	5684	3	5.73	15	61	41	1	8.073185	-2.073E+0	6	-1	0	31	7.318E-2	-8	1	0	55
204	5685	1	29.72	13	45	30	1	7.595348	7.306E+1	17	-23	4	5364	-8.425E+1	-13	21	-4	7104
205	5697	1	12.18	15	60	33	1	8.078653	7.865E-2	-8	1	0	57	-3.806E+1	-20	22	-3	1497
206	5724	1	29.55	12	21	8	1	9.975172	1.695E+1	-3	2	0	291	1.338E+2	57	-112	12	24771
207	5741	1	19.78	11	29	6	1	6.953570	4.643E-2	7	-1	0	56	-1.536E+1	125	-48	4	13315
208	5780	1	22.07	13	45	31	1	7.642840	-1.540E-2	18	-10	1	139	-2.879E+1	40	-9	0	1839
209	5821	1	18.36	13	45	32	1	7.688338	-6.688E+0	1	-1	0	56	7.723E+1	33	-25	4	6128
210	5853	1	27.69	13	43	12	1	7.430084	-6.236E+1	-5	22	-4	3891	-1.151E+2	-11	38	-7	13272
211	5901	1	27.58	16	64	3	1	8.590936	-2.074E+1	-2	15	-2	432	3.383E+1	10	-212	25	39492
212	5912	1	31.96	14	51	20	1	7.759849	1.290E+1	7	-7	1	183	1.558E+2	63	-151	21	35671
213	5925	1	29.50	22	28	9	1	20.666204	-1.407E+2	4	-7	0	19788	-2.183E+0	694	-943	44	28968
214	5940	1	21.00	12	36	10	1	7.180140	-5.360E+0	9	-2	0	99	-9.574E+1	-9	31	-6	9243
215	5980	1	27.52	16	69	44	1	8.453029	-2.756E+1	-3	14	-2	803	-5.460E+1	-27	39	-5	3039
216	6053	1	23.69	18	88	65	1	9.221237	1.221E+0	-8	1	0	52	-1.035E-5	-696	389	-34	207192
217	6088	1	26.75	12	35	2	1	7.131567	6.172E-2	1	7	-1	135	3.205E+2	9	-156	28	111491
218	6092	1	32.63	18	85	48	2	9.916692	2.038E+0	-86	19	-1	1375	3.017E-4	-238	24	0	13251
219	6108	1	40.49	34	49	18	1	32.509793	5.274E+0	535	-699	21	2703	-4.006E+2	583	-843	25	164367
220	6133	1	20.23	21	119	92	1	10.514049	2.810E-2	-21	2	0	371	-6.598E+0	67	-7	0	3716
221	6153	1	16.34	14	53	37	1	7.846027	-5.410E-2	9	-9	1	117	5.540E+1	21	-27	4	3177
222	6184	1	32.17	15	56	20	2	9.067092	-1.413E+1	-24	63	-7	3239	-5.018E-3	32	-67	7	3879
223	6185	1	12.42	14	49	5	1	7.800609	-1.801E+0	6	-1	0	38	3.047E-3	-39	5	0	1553
224	6209	1	13.38	13	44	21	1	7.520893	2.521E+0	-5	1	0	26	1.148E-3	-34	-3	1	1858
225	6237	1	25.53	12	36	11	1	7.233194	-1.285E-2	20	-10	1	300	-1.396E+1	-74	30	-3	4296
226	6241	1	4.70	20	107	71	1	10.121509	8.230E+0	7	-10	1	69	-1.083E-1	-9	11	-1	69
227	6268	1	25.82	15	56	2	1	8.202724	4.046E+1	1	-28	4	1667	1.062E+2	1	-61	9	11279
228	6289	1	11.30	13	42	5	1	7.459324	-3.308E-2	4	-8	1	109	-3.033E+0	1	-8	1	178
229	6396	1	40.03	16	65	26	2	9.336008	1.958E+1	-20	-3	1	495	-1.352E+2	22	-50	2	29475
230	6401	1	16.56	19	98	75	1	9.649566	-3.110E-2	13	-11	1	89	-3.762E+0	-53	63	-6	2694
231	6420	1	28.31	19	95	47	1	9.757997	-3.077E-3	-34	23	-2	699	4.592E+0	43	-82	8	4323
232	6452	1	21.94	13	43	13	1	7.492505	1.499E-2	15	-2	0	227	-7.742E+0	56	-16	1	2635
233	6453	1	24.53	15	57	10	1	8.186658	1.795E+0	9	-50	6	3459	-2.974E-3	-87	27	-2	6756
234	6508	1	27.05	14	53	38	1	7.900053	2.379E+0	-15	18	-2	611	5.255E-4	-79	10	0	5603
235	6549	1	27.17	14	50	13	1	7.814589	1.815E+0	-6	1	0	36	-1.743E-6	1102	-774	81	1207848
236	6556	1	32.25	13	44	22	1	7.574473	2.970E+1	21	-14	2	1059	5.546E-4	-39	-10	2	3619
237	6557	1	25.11	14	52	29	1	7.836919	4.674E+0	-11	2	0	115	-1.590E-4	-134	166	-19	42856
238	6584	1	31.18	15	58	18	1	8.172656	3.345E+0	-13	2	0	163	7.788E-5	505	-119	7	217367
239	6588	1	33.60	15	60	34	1	8.149602	-6.960E-2	-7	9	-1	111	-4.750E+2	533	-849	89	1957503
240	6601	1	20.24	12	35	3	1	7.190470	-1.137E-2	13	-9	1	201	4.343E+1	2	-23	4	2190
241	6616	1	48.49	15	56	24	2	9.191483	-5.922E-3	70	-26	2	867	-8.182E+1	1642	-463	29	511959
242	6637	1	16.63	15	59	26	1	8.160394	-1.191E-2	15	-10	1	172	-5.481E+0	19	-3	0	336
243	6669	1	21.97	12	36	13	1	7.331596	-1.233E+1	17	-4	0	387	5.213E-3	22	-3	0	516
244	6681	1	15.56	14	51	21	1	7.826545	-1.064E-2	17	-10	1	213	5.184E+0	-4	9	-1	246
245	6685	1	22.20	16	70	53	1	8.491479	1.704E-2	17	-2	0	243	3.763E+1	68	-80	9	6264
246	6728	1	35.02	12	19	6	1	10.193875	5.795E+1	5	-5	1	3367	1.842E+2	5	-13	3	33991
247	6809	2	6.33	14	53	39	1	7.951255	-2.951E+0	5	-1	0	25	4.874E-2	8	-1	0	58
248	6856	1	34.05	23	14	2	1	22.378390	4.911E+0	-9	23	-1	39	-1.452E+3	-3795	16978	-754	9988579
249	6868	1	12.63	17	79	61	1	8.863517	-7.324E-2	10	-10	1	107	4.220E+0	-26	30	-3	1227
250	6885	3	8.95	12	36	15	1	7.421658	-3.422E+0	4	-1	0	24	-1.384E+1	1	-2	0	243

2. Other Tables and Methods. Most other tables of fundamental units use a method of Voronoĭ [15]. An account of this is given in the book of Delone and Faddeev [6, Chapter IV, Part A]. However, the Voronoĭ algorithm is very complicated, since it depends on a detailed consideration of sequences of points in various lattices.

We know of only two published tables of more than a few cases of fundamental pairs of units in general totally real cubic fields. The first of these, due to Billevich [2], lists the 33 fields (plus 7 duplications) with discriminant $D < 1{,}300$. He gives the field discriminant, an integral basis, a defining polynomial for the field, and the coefficients of the fundamental pair with respect to this basis. Billevich uses his own method for the calculation of the units. With this method, computational difficulties arise for discriminants larger than the small ones considered by Billevich (see Steiner and Rudman [14]). The second table, by Williams and Zarnke [16], gives fundamental pairs for various fields determined by irreducible cubic equations; for example, they list the coefficients of a fundamental pair with respect to the natural integral basis for all the totally real cubic fields defined by $x^3 - px - q = 0$ with $|p| \leqslant 15$, $|q| \leqslant 15$. The discriminants of the fields are not given, and no attempt is made to indicate different pairs p, q that give the same field. The Voronoĭ algorithm was used to compute the units, and an account of how this algorithm was implemented is given.

Among the unpublished tables is a large one of Angell who describes it in [1]. This was obtained by using the Voronoĭ procedure on the 4,804 fields with $D < 100{,}000$. The actual table gives the field discriminant, the class number of the field, a defining polynomial with its index, and the coefficients in a representation (1) of the fundamental units. It is known that about ten fields with $D > 30{,}000$ are missing from this table, and that some of the fundamental units are wrong (see Ennola and Turunen [7], and Llorente and Oneto [12]). There are other errors in [1], and apparently also in [12] (see Ennola and Turunen [7]). Some of these errors presumably result from mistakes in programming the Voronoĭ algorithm for the computer, but neither [1] nor [12] gives details about how this was done.

There is a still larger unpublished table of Ennola and Turunen [7] who applied Voronoĭ's method to the 26,440 fields with $D < 500{,}000$. No errors are known in this table, but 300-digit precision was used to handle the worst cases. The actual table gives the ordinal number for each field in the list, the field discriminant and class number, coefficients for an integral basis, a defining polynomial, and coefficients in a representation of two fundamental units with respect to the given integral basis.

The Voronoĭ procedure is certainly much faster than the method of the present paper for the worst cases (essentially those in which the fundamental units have at least one large conjugate). On the other hand, the necessary programming for the Voronoĭ algorithm is quite intricate, as is shown by the errors in previous tables produced by that method.

One advantage of the Cusick procedure [5] is its extreme simplicity. Another advantage of the procedure used here is that it makes easier the computation of the regulator R of the cubic fields. As noted in Section 1, for our units, the $|E_j^{(i)}|$ are not small, so that R can be computed with good accuracy. However, other algorithms, including Voronoĭ's, can perform badly in this respect. There is a good example of this in the small table of units and regulators given by Pohst, Weiler and Zassenhaus [13, p. 301]. Even for as small a discriminant as 961, their method (which is based on the geometry of numbers, but is different from Voronoï's) gives a value for the regulator which is in error by 0.16% in spite of a machine precision of 14 digits. This happened, in part, because one of the units in their fundamental pair has a small conjugate.

Regulators are not given in the first four tables mentioned. However, the units in the table of Ennola and Turunen [7] can be used for the accurate computation of regulators. This is because the unit pairs given in their table are not the pairs produced by the Voronoï algorithm, but rather are normalized pairs derived from the Voronoï units. A comparison of Ennola and Turunen's normalized pairs with our pairs shows that in each of the 250 fields, our first unit E_1 agrees with one of their units, but this is not always the case for the unit E_2.

We remark that in the special case of cyclic cubic fields, the discriminant is a square, and there always exists a fundamental pair made up of a unit and one of its conjugates. Cohn and Gorn [3] and Gras [11] use procedures simpler than Voronoï's to construct tables of units in the cyclic case, although Gras does not list the units. The method of the present paper becomes much simpler in the cyclic case and always gives a fundamental pair made up of two conjugate units [4, Theorem 2], but we have chosen not to make a separate table for the cyclic case.

3. The Computer Program. This was initially written for and tested on the HP-85 microcomputer and subsequently modified and run on the CDC 6400 mainframe which produced the table above. With the exception of R_0, E_1, E_2, R and certain bounds, all computations were done in integer arithmetic.

For each of the totally real cubic fields F with discriminant $D \leqslant 6{,}885$, we used the table of Angell [1] to copy the value of D, H and the coefficients a, b, c of a polynomial

$$h(x) = x^3 - ax^2 + bx - c,$$

one of whose roots, say r, generates F. Let I_h be the index of $h(x)$ and let D_h be its discriminant, so that $D_h = -4a^3c + a^2b^2 + 18abc - (4b^3 + 27c^2)$.

If $D_h = D$, then $I_h = 1$ and $1, r, r^2$ form an integral basis for F, called a "power basis". In this case, the program takes $g(x)$ in (2) to be $h(x)$. Then, by Angell's choice of a, b, c, the smallest zero R_2 of $g(x)$ is in the interval $(0, 1)$ and $A, B, C > 0$.

If $D_h \neq D$, we attempted to reduce I_h so as to obtain a power basis. To this end, the program forms the irreducible monic polynomial $H(z) = (\gamma z + \delta)^3 h(\zeta)/\phi$, where $\zeta = (\alpha z + \beta)/(\gamma z + \delta)$ and $\phi = \gamma^3 h(\alpha/\gamma)$, for various rational integers $\alpha, \beta, \gamma, \delta$ in $[-10, 10]$ such that $\alpha\delta \neq \beta\gamma$ and $H(z)$ has integral coefficients; of course, $H(z)$ also generates F. If any $H(z)$ has $I_H = 1$, then we let $G(z) = H(z)$; otherwise, $G(z)$ was taken to be an $H(z)$ with the least index I_H that arose from this procedure. Letting ρ be the smallest zero of $G(z)$, we defined $g(x) = G(x + [\rho])$, guaranteeing that the smallest zero R_2 of $g(x)$ is in $(0, 1)$ and $A, B, C > 0$; also, $I_g = I_G$.

In the 250 cases we calculated, this algorithm resulted in 25 cases of index 2 compared with 45 cases in Angell's table [1] and 54 cases in the table of Ennola and Turunen [7]; we also had 2 cases of index 3, compared with 4 cases in the tables of

Angell and Ennola-Turunen. It is possible, of course, that further calculation would reduce the index in some, but not all, of these cases; e.g., in the case of $D = 961$, no polynomial has index 1.

Given the polynomial $g(x)$, the program computes its zeros $R_0 > R_1 > R_2$ in (2). Next the program finds an integral basis 1, R_0, S for the field F. If $I_g = 1$, the program takes $S = R_0^2 = Q(R_0)$. Otherwise, a standard algorithm of Voronoĭ (see Delone and Faddeev [6, pp. 108–112]) was used to determine

$$S = \left(u + vR_0 + R_0^2\right)/I = Q(R_0)$$

for suitable rational integers u, v, which are obtained from a solution of a simultaneous pair of congruences involving $g(x)$ and $g'(x)$.

For any integer λ in F, (6) and (5) with $r = R_0$ give $T(\lambda) = F(x, y, z)$, where

$$\begin{aligned} F(x, y, z) &= \sum_{k=0}^{2} \{x + yR_k + zQ(R_k)\}^2 \\ &= a_0x^2 + b_0y^2 + c_0z^2 + 2(a_1yz + b_1xz + c_1xy) \end{aligned}$$

for suitable rational integers a_0, b_0, c_0, a_1, b_1, c_1; of course, $a_0 = 3$ and all these coefficients are expressible in terms of the power symmetric functions of the zeros R_k of $g(x)$ or, alternatively, in terms of A, B, C. The program calculates these six integers and also the ten integers occurring in $N(x, y, z) = \operatorname{norm}(\lambda)$, where

$$\begin{aligned} N(x, y, z) &= \prod_{k=0}^{2} \{x + yR_k + zQ(R_k)\} \\ &= a_2x^3 + b_2y^3 + c_2z^3 + a_3x^2y + b_3x^2z \\ &\quad + c_3xy^2 + d_3y^2z + e_3xz^2 + f_3yz^2 + g_3xyz. \end{aligned} \tag{7}$$

The last function is needed to test whether $N(x, y, z) = \pm 1$, i.e., whether λ is a unit of F.

To facilitate the search for the minima of the positive-definite quadratic form $F(x, y, z)$, the program next determines the real coefficients k_j appearing in the identity

$$F(x, y, z) = (k_1x + k_2y + k_3z)^2 + (k_4y + k_5z)^2 + (k_6z)^2, \tag{8}$$

where

$$k_1 = a_0^{1/2}, \quad k_2 = c_1a_0^{-1/2}, \quad k_3 = b_1a_0^{-1/2}, \quad k_4 = \left(b_0 - k_2^2\right)^{1/2},$$
$$k_5 = (a_1 - k_2k_3)/k_4, \qquad k_6 = \left(c_0 - k_3^2 - k_5^2\right)^{1/2} = (D/a_0)^{1/2}/k_4.$$

The final step is to search the values of $F(x, y, z)$ until the fundamental units ε_1, ε_2 of the Theorem above are found. This is done by picking a number L (say $D^{1/2}$) and only considering x, y, z such that $F(x, y, z) \leqslant L$, a condition which imposes bounds on z, y, x, as we now explain. If the last inequality holds, then (8) shows that $|z| \leqslant L^{1/2}/k_6 \equiv Z_2$; as $F(x, y, -z) = F(-x, -y, z)$, it suffices to consider $z \geqslant 0 \equiv Z_1$. Thus, z must lie in the interval $[Z_1, Z_2]$. For a given z of this kind, (8) also gives $|k_4y + k_5z| \leqslant (L - k_6^2z^2)^{1/2}$, so that y is confined to some interval $[Y_1, Y_2]$ depending on z. Similarly, for given z, y, (8) implies that x must lie in some $[X_1', X_2']$. Finally, if we let $V_k = yR_k + zQ(R_k)$, then (7) shows that $N(x, y, z) < -1$ if

$x < X_1''$, and $N(x, y, z) > 1$ if $x > X_2''$, where

$$X_1'' = \min(-V_0, -V_1, -V_2) - 1, \qquad X_2'' = \max(-V_0, -V_1, -V_2) + 1.$$

Hence, x can be restricted to $[X_1, X_2]$, where the bounds $X_1 = \max(X_1', X_1'')$ and $X_2 = \min(X_2', X_2'')$ depend on z, y. (Actually, x can be confined to the union of three subintervals of $[X_1, X_2]$.)

For given z, y, x in their respective ranges, the program checks the condition $N(x, y, z) = \pm 1$; if this is not satisfied, then the corresponding λ is not a unit and the next largest x is tried. Otherwise, the triple (x, y, z) is stored in a table for units after standardizing so that $N(x, y, z) = +1$. The value $F(x, y, z)$ is computed and compared with the previous minimum value for F, thereby keeping this minimum current. Once a unit λ_0 satisfying $T(\lambda_0) \leqslant L$ has been found, the program checks the condition $z > T(\lambda_0)^{1/2}/k_6$. If this holds, then $F(x, y, z) > T(\lambda_0)$, so that no additional z's need be tried; i.e., the current minimum value for F is F_1, and the z, y, x that produces F_1 defines $\varepsilon_1 = E_1$.

The program lets z, y, x run over their respective ranges. If no unit λ with $T(\lambda) \leqslant L$ is found, then L is doubled and the process is repeated.

Once ε_1 has been found, the program continues its search with the current value of L. Whenever it finds a unit λ, it checks that $\lambda \neq \pm\varepsilon_1^m$ for each integral m. If this is satisfied, then λ becomes a candidate for ε_2; the value $T(\lambda)$ is now used to replace L, and appropriate bounds on z, y, x are recomputed. If no independent unit λ is found after z, y, x have run through all their values, then L is doubled and the computation is started from the beginning. Ultimately, a unit $\lambda \neq \pm\varepsilon_1^m$ is found, and the minimum of $T(\lambda)$ for such λ is defined as F_2 with the associated λ defined as $\varepsilon_2 = E_2$.

In passing, we remark that our original HP-85 program reduced the positive-definite ternary form $F(x, y, z)$ to Gauss form. However, the running time for this search was no better than for the reduction in (8). We also tried changing the roles of x, y, z in (8), but here, also, we achieved no improvement.

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