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Defect Corrections for Multigrid Solutions of the Dirichlet Problem in General Domains

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Abstract. Recently, the technique of defect correction for the refinement of discrete solutions to elliptic boundary value problems has gained new acceptance in connection with the multigrid approach. In the present paper we give an analysis of a specific application, namely to finite-difference analogues of the Dirichlet problem for Helmholtz's equation, emphasizing the case of nonrectangular domains. A quantitative convergence proof is presented for a class of convex polygonal domains.

1. Introduction. The purpose of this paper is to study the behavior of a defect correction method for the linear elliptic boundary value problem

$$(1.1)\qquad \begin{aligned} -\Delta u(x, y) + c(x, y)u(x, y) &= f(x, y), \qquad (x, y) \in \Omega, \\ u(x, y) &= g(x, y), \qquad (x, y) \in \partial\Omega, \end{aligned}$$

in a general domain $\Omega \subset \mathbf{R}^2$. The method and, in particular, its combination with the multigrid approach, has been discussed by Auzinger and Stetter [4] and Hackbusch [10]. (We also refer to the work of Frank, Hertling, and Monnet [7].)

Defect correction is a way to obtain, in an iterative or semi-iterative fashion, solutions to complex problems by means of solving related, simpler problems. In the present application, the "complex problem" is a high-order discretization, whereas the "simpler problem" is a low-order scheme which is solved by standard multigrid. It is our aim to show, in a concrete nontrivial situation, that this is a reasonable way to get a high-order solution. More precisely, we prove that contraction rates can be obtained which yield the usual multigrid efficiency. Our approach has the advantage that standard multigrid software can be used for the solution phase in a black box manner. The high-order scheme is only involved in an outer iteration. Thus, high accuracy is introduced in such a way as to be (theoretically and computationally) clearly separated from the inversion process.

After some general remarks in Section 2, we present in Section 3 an account of the model problem analysis given in Auzinger [1]. In Section 4, which is the heart of the paper, we derive explicit bounds for the contraction number of the defect correction iteration for a class of convex polygonal domains. That section includes a quantitative H^2-regularity estimate for the discrete Poisson equation. Our analysis does not

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depend on asymptotic error expansions. Some remarks on general domains can be found in Section 5.

Throughout we have adopted the convention of "generic constants" C, $C_1, \ldots$. These are always independent of the discretization level.

2. Basic Properties. Let

$$Lu = f \tag{2.1}$$

denote a linear elliptic boundary value problem of second order (e.g., (1.1)) and

$$L_h u_h = f_h \tag{2.2}$$

its discretization on a grid with mesh size h. (2.2) is assumed to be stable in some norm $\|\cdot\|$ and will be referred to as the "*basic discretization*" of (2.1). In our application, (2.2) is a 2nd-order method.

Let, in addition,

$$L'_h u_h = f'_h \tag{2.3}$$

be a discretization of higher order. Within our defect correction approach, (2.3) will not be inverted but evaluated: Given a discrete approximation $u_h^{(i)}$, its defect with respect to (2.3) defines a corrected version of (2.2) in the following way:

$$L_h u_h^{(i+1)} = L_h u_h^{(i)} - \left(L'_h u_h^{(i)} - f'_h\right). \tag{2.4}$$

This yields an iteration procedure possessing any solution of (2.3) as a fixed point. Within this context, (2.3) will be referred to as the "*target discretization*" of (2.1). (See Stetter [12] for the general principle of defect correction.)

With the notation

$$\Delta L_h := L_h - L'_h, \tag{2.5}$$

the iteration (2.4) reads

$$u_h^{(i+1)} := L_h^{-1} \Delta L_h u_h^{(i)} + L_h^{-1} f'_h. \tag{2.4'}$$

The general structure of our analysis of the defect correction iteration (2.4) can be described as follows: Let u_h^* be the restriction of the true solution u^* of (2.1) to the grid with mesh size h. For the error function

$$e_h^{(i)} := u_h^{(i)} - u_h^*$$

we obtain the "error iteration" equivalent to (2.4):

$$e_h^{(i+1)} := L_h^{-1} \Delta L_h e_h^{(i)} - L_h^{-1}\left(L'_h u_h^* - f'_h\right). \tag{2.6}$$

Thus, we will have to investigate

(i) the contraction behavior of $L_h^{-1} \Delta L_h$,

(ii) the role of the inhomogeneous term $L_h^{-1}(L'_h u_h^* - f'_h)$.

We have not presupposed stability of (2.3). In fact, for the purpose of defect correction, unstable target discretizations are usually admitted. It must, however, be pointed out that the "global" contractivity of the defect correction operator $L_h^{-1} \Delta L_h$ and the stability of (2.3) are directly interrelated: $L_h^{-1} \Delta L_h$ cannot be contractive, independent of h, if L'_h is unstable or even fails to be invertible. (See, however, our remarks at the end of this section. See also Section 5 for further comments on unstable target discretizations.)

PROPOSITION 2.1. *Let L_h be stable and $L_h^{-1}\Delta L_h$ have a contraction number $\|L_h^{-1}\Delta L_h\| \leqslant k < 1$ independent of h. Then L'_h is invertible and stable*:

$$(2.7) \qquad \left\| L_h'^{-1} \right\| \leqslant \frac{1}{1-k} \left\| L_h^{-1} \right\|.$$

For the error $e'_h := u'_h - u^*_h$ (i.e., the fixed point of (2.6)) *we obtain*

$$(2.8) \qquad \left\| e'_h \right\| \leqslant \frac{1}{1-k} \left\| L_h^{-1}\left(L'_h u^*_h - f'_h \right) \right\|.$$

Proof. Since $\|I_h - L_h^{-1}L'_h\| < 1$, $L_h^{-1}L'_h$ is invertible, and so is L'_h. Moreover,

$$\left\| L_h'^{-1} \right\| \leqslant \left\| L_h^{-1} \right\| + \left\| \left(I_h - L_h^{-1}L'_h \right) L_h'^{-1} \right\| \leqslant \left\| L_h^{-1} \right\| + k \left\| L_h'^{-1} \right\|,$$

from which we infer (2.7). A similar argument establishes (2.8) □

For an extension of this simple result, see Proposition 2.2 below.

The convergence behavior of the defect correction iteration (2.4) is essentially retained if L_h is only approximately inverted: Assume

$$(2.9) \qquad \left\| I_h - K_h L_h \right\| \leqslant l < 1,$$

independent of h. (This can be expected if K_h represents a suitable multigrid cycle.) Then the iteration

$$(2.10) \qquad u_h^{(i+1)} := \left(I_h - K_h L'_h \right) u_h^{(i)} + K_h f'_h$$

has a contraction number $\leqslant \tilde{k} < 1$ if

$$(2.11) \qquad \tilde{k} := k + l + kl < 1$$

(k from Proposition 2.1; cf. Auzinger and Stetter [4]).

Since, in our situation, the actual convergence rate depends crucially on the smoothness of the "algebraic error" (i.e., the error with respect to the fixed point), we shall also consider the following modification of (2.4) (or (2.10)):

$$(2.12) \qquad \bar{u}_h^{(i)} := \left(I_h - T_h L_h \right) u_h^{(i)} + T_h f_h, \qquad u_h^{(i+1)} := L_h^{-1}\Delta L_h \bar{u}_h^{(i)} + L_h^{-1} f'_h$$

(or K_h instead of L_h^{-1}, respectively). Here, the defect correction is "preconditioned" by a smoothing sweep relative to the *basic* discretization. We expect that a suitable multigrid smoother T_h will improve the convergence considerably.

On the other hand, a fixed point $\hat{u}_h$ of (2.12) does no longer satisfy (2.3). In the following we give a representation of $\hat{u}_h$. (See also Hackbusch [10] for a related result.)

PROPOSITION 2.2. Fixed-point shift. *Let $M_h := L_h^{-1}\Delta L_h(I_h - T_h L_h)$, and assume $\|M_h\| \leqslant m < 1$, m independent of h. Then there is a unique fixed point $\hat{u}_h$ of* (2.12) *satisfying*

$$(2.13) \qquad \hat{L}_h \hat{u}_h = \hat{f}_h,$$

where $\hat{L}_h$ and $\hat{f}_h$ are defined by

$$(2.14) \qquad \hat{L}_h := L'_h + \Delta L_h T_h L_h, \qquad \hat{f}_h := f'_h + \Delta L_h T_h f_h.$$

$\hat{L}_h$ is invertible and stable with

$$(2.15) \qquad \left\| \hat{L}_h^{-1} \right\| \leqslant \frac{1}{1-m} \left\| L_h^{-1} \right\|.$$

The error $\hat{e}_h := \hat{u}_h - u_h^*$ *is*

$$\hat{e}_h = -(I_h - M_h)^{-1} L_h^{-1}\big[(L_h' u_h^* - f_h') + \Delta L_h T_h (L_h u_h^* - f_h)\big]. \tag{2.16}$$

Proof. Combining the steps in (2.12), we obtain the fixed-point equation

$$\hat{u}_h = M_h \hat{u}_h + L_h^{-1}\big[f_h' + \Delta L_h T_h f_h\big],$$

hence, $L_h(I_h - M_h)\hat{u}_h = \hat{f}_h$. Moreover, $L_h(I_h - M_h) = \hat{L}_h$ as defined in (2.14), since

$$\begin{aligned} M_h &= (I_h - L_h^{-1} L_h')(I_h - T_h L_h) \\ &= I_h - L_h^{-1}(L_h' + \Delta L_h T_h L_h) = I_h - L_h^{-1}\hat{L}_h. \end{aligned}$$

Together with $\|M_h\| \leqslant m < 1$, this implies (2.15). The error $\hat{e}_h$ is given by

$$\hat{e}_h = -\hat{L}_h^{-1}(\hat{L}_h u_h^* - \hat{f}_h),$$

which is easily seen to be equivalent to (2.16). □

Proposition 2.2 shows that $\hat{L}_h$ is stable under weaker assumptions than $\|L_h^{-1}\Delta L_h\| < 1$. Hence it is possible that $\hat{L}_h$ is stable even if L_h' is not; $\hat{L}_h$ may be considered a "stabilization" of L_h'. Although this observation has no immediate consequence in our application (cf. Sections 3 and 4), it may be useful in other cases.

Naturally, the modification (2.12) will only make sense if, within (2.16), the "perturbation" involving the low-order truncation error $L_h u_h^* - f_h$ does not destroy the accuracy controlled by $L_h^{-1}(L_h' u_h^* - f_h')$. We shall return to this question in the sections which follow.

3. Model Problem Analysis. In this section we consider Helmholtz's equation in the unit square. This example has already been discussed in Auzinger and Stetter [4] and, in more detail, in Auzinger [1]. Let $\Omega = (0,1) \times (0,1)$, and let Helmholtz's equation (1.1) be given. The quoted results apply to the case of $c \equiv \text{const} > 0$. (See [1] for the handling of variable $c(x, y)$ by partial summation.) On a uniform grid with mesh spacing $h = 2^{-m}$, $m \in \mathbf{N}$, the basic discretization $L_h u_h = f_h$ is defined by the usual five-point stencil

$$\frac{1}{h^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix} \tag{3.1}$$

and by straightforward point evaluation for cu and f. $L_h u_h = f_h$ is a stable, 2nd-order discretization.

Let the target discretization $L_h' u_h = f_h'$ be given by the well-known stable, 4th-order "Mehrschrittverfahren". Then, $\Delta L_h = L_h - L_h'$ is given by

$$\Delta L_h u_h \sim \frac{1}{6h^2}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} u_h + \frac{1}{12}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix} c u_h. \tag{3.2}$$

Now let $\|\cdot\|_2$ denote the (properly scaled) Euclidean norm as well as its associated operator norm. With respect to $\|\cdot\|_2$, $L_h^{-1}\Delta L_h$ has a contraction number independent of h:

PROPOSITION 3.1.

$$\big\|L_h^{-1}\Delta L_h\big\|_2 \leqslant \tfrac{1}{3} + O(ch^2) \leqslant \tfrac{2}{3}. \tag{3.3}$$

Proof. Given in [1]. □

Here and in the sequel, $O(ch^2)$ means $\psi(c) \cdot O(h^2)$, where $\psi(c) = O(c)$ for small c, but uniformly bounded for arbitrary c. In (3.3), the bound $\frac{2}{3}$ is valid independent of h and c. The analysis of the modified iteration (2.12) is based on the following estimate.

LEMMA 3.2.

$$\|L_h^{-1}\Delta L_h L_h^{-1}\|_2 \leqslant \frac{h^2}{24}(1 + O(ch^2)) \leqslant \frac{h^2}{6}. \tag{3.4}$$

Proof. Given in [1]. □

Assume that, in (2.12), the smoothing step consists in $\nu \geqslant 1$ applications of an appropriate relaxation procedure, say

$$I_h - T_h L_h = S_h^\nu,$$

and assume further that S_h has the *smoothing property* (defined in Hackbusch [9]) for $\|\cdot\|_2$ and $\alpha = 2$:

$$\|L_h S_h^\nu\|_2 \leqslant C(\nu)h^{-2}, \tag{3.5}$$

with $C(\nu) \to 0$ as $\nu \to \infty$.

PROPOSITION 3.3. *Let S_h satisfy* (3.5). *Then,*

$$\|L_h^{-1}\Delta L_h S_h^\nu\|_2 \leqslant \frac{1}{24}C(\nu)(1 + O(ch^2)) \leqslant \frac{1}{6}C(\nu). \tag{3.6}$$

Proof. Use the splitting $L_h^{-1}\Delta L_h S_h^\nu = L_h^{-1}\Delta L_h L_h^{-1} L_h S_h^\nu$ and apply Lemma 3.2. □

The smoothing property (3.5) has been proved in [9] for damped Jacobi and "red-black" Gauss/Seidel smoothers with

$$C(\nu) \leqslant \frac{6}{2\nu + 1}(1 + O(ch^2)), \tag{3.7a}$$

$$C(\nu) \leqslant \frac{8}{3\sqrt{3}\,\nu}(1 + O(ch^2)), \tag{3.7b}$$

respectively.

The following table shows the resulting bounds for $\|L_h^{-1}\Delta L_h S_h^\nu\|_2$, $\nu = 1, 2, \ldots,$ in the case of Poisson's equation.

ν	Jacobi	Gauss/Seidel
1	.083	.064
2	.050	.032
3	.036	.021
4	.028	.016

These numbers are comparable in size to typical multigrid convergence factors.

We conclude this section by showing that the "fixed-point shift" (cf. Proposition 2.2) is $O(h^4)$. Hence, the smoother does not affect the order of accuracy.

LEMMA 3.4. *For $T_h = (I_h - S_h^\nu)L_h^{-1}$, there exists $C'(\nu)$ independent of h such that*

$$\|T_h\|_2 \leqslant C'(\nu)h^2. \tag{3.8}$$

This holds for both of the smoothing procedures considered. For Jacobi relaxation, $C'(\nu) \leqslant C \cdot \nu$; for Gauss/Seidel, $C'(\nu) \leqslant C \cdot (2^\nu - 1)$.

Proof. See [2]. □

PROPOSITION 3.5. *Let $\|L_h^{-1}\Delta L_h S_h^\nu\|_2 \leqslant m < 1$. There exist constants C_1, C_2 independent of h such that the error $\hat{e}_h$ of Proposition 2.2 satisfies*

$$(3.9)\qquad \|\hat{e}_h\|_2 \leqslant \frac{1}{1-m}\left[C_1\|L_h' u_h^* - f_h'\|_2 + C_2 C'(\nu) h^2 \|L_h u_h^* - f_h\|_2\right].$$

Thus, $\|\hat{e}_h\|_2 = O(h^4)$.

Proof. Use the representation (2.16), (3.3), (3.8) and the stability of L_h. □

4. Convergence Analysis for a Class of Convex Polygonal Domains. For simplicity, we shall from now on restrict our considerations to Poisson's equation; the results can be transformed to the general case (1.1).

Let $\Omega \subset \mathbf{R}^2$ be a bounded polygonal domain such that, for some sequence of uniform grids $\Omega_h \subset \Omega$ (with mesh size h), its boundary consists of horizontal, vertical or diagonal grid lines (see Figure 4.1). For this type of domains, explicit bounds are known for multigrid convergence rates; see Braess [5]. We are going to establish bounds for the contraction rate of the defect correction method.

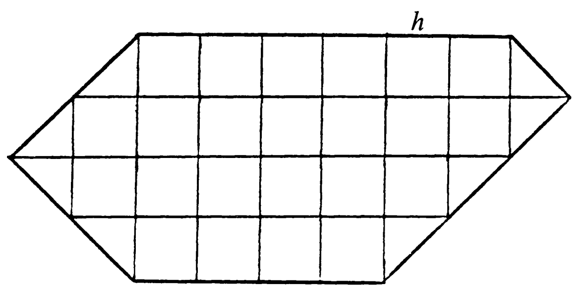

FIGURE 4.1
"Polygonal" domain

By $\partial\Omega_h$ we denote the intersection of $\partial\Omega$ with the grid lines. Let $\overline{\Omega}_h := \Omega_h + \partial\Omega_h$. $P = (x, y) \in \overline{\Omega}_h$ is called (interior or boundary) grid point, respectively. $u_h: \overline{\Omega}_h \to \mathbf{R}$ is called *grid function*; often we shall tacitly extend a grid function to the infinite grid by $u_h(P) := 0$, $P \notin \overline{\Omega}_h$. The linear space of grid functions is denoted by $\mathcal{U}_h$. $\mathcal{U}_h^0$ is the subspace of functions vanishing on $\partial\Omega_h$. Since the Dirichlet boundary condition will always be trivially satisfied, any "error function" henceforth considered is contained in $\mathcal{U}_h^0$.

Let $\mathring{\Omega}_h \subset \Omega_h$ be the set of grid points P far enough away from the boundary such that the nine-point stencil (3.2) is well defined within $\overline{\Omega}_h$; let $\Gamma_h := \Omega_h \setminus \mathring{\Omega}_h$. Define projection operators R_h, B_h: $\mathcal{U}_h^0 \to \mathcal{U}_h^0$ by

$$(4.1a)\qquad R_h u_h(P) := \begin{cases} u_h(P), & P \in \mathring{\Omega}_h, \\ 0, & P \in \Gamma_h \end{cases}$$

and

$$(4.1b)\qquad B_h := I_h - R_h.$$

With respect to the discrete L_∞-norm $\|\cdot\|_\infty$, the following estimate holds for the basic discretization L_h (defined via (3.1)).

PROPOSITION 4.1.

$$\|L_h^{-1}B_h\|_\infty \leqslant h^2. \tag{4.2}$$

Proof. (4.2) is the reformulation of a "discrete Green's function estimate" in Bramble and Hubbard [6] (see also [1]). □

Proposition 4.1 shows that the full order of consistency is not required near the boundary.

The target discretization $L'_h u_h = f'_h$ can be defined by the "Mehrstellenoperator" for $P \in \mathring{\Omega}_h$; for $P \in \Gamma_h$ we use the basic five-point scheme (3.1). Thus, the order of consistency is

$$|(L'_h u_h^* - f'_h)(P)| = \begin{cases} O(h^4), & P \in \Omega_h, \\ O(h^2), & P \in \Gamma_h. \end{cases} \tag{4.3}$$

$\Delta L_h = L_h - L'_h$ is given by (3.2) for $P \in \mathring{\Omega}_h$; it is 0 for $P \in \Gamma_h$. Let $\mathscr{U}_h$ be equipped with the scalar product

$$\langle u_h, v_h \rangle := h^2 \sum_{P \in \bar{\Omega}_h} u_h(P) v_h(P). \tag{4.4}$$

Introducing discrete Sobolev norms, we shall write

$$|u_h|_{H^0} := \|u_h\|_2 = \langle u_h, u_h \rangle^{1/2}. \tag{4.5}$$

On $\mathscr{U}_h^0$, a H^1-norm can be defined by

$$|u_h|_{H^1} := \left[|\partial_x u_h|_{H^0}^2 + |\partial_y u_h|_{H^0}^2 \right]^{1/2}, \tag{4.6}$$

where ∂x, ∂y are the first (forward-) difference quotients. (The corresponding backward-difference quotients will be denoted by $\bar{\partial} x$, $\bar{\partial} y$.) The dual of $|\cdot|_{H^1}$ is

$$|u_h|_{H^{-1}} := \sup_{0 \neq v_h \in \mathscr{U}_h^0} \frac{|\langle u_h, v_h \rangle|}{|v_h|_{H^1}}. \tag{4.7}$$

For $u_h \in \mathscr{U}_h^0$ and L_h as defined above, partial summation yields

$$|u_h|_{H^1} = \langle L_h u_h, u_h \rangle^{1/2} = |L_h^{1/2} u_h|_{H^0}. \tag{4.8}$$

(Note that L_h is a symmetric, positive definite operator.) A discrete H^2-norm will be introduced later in this section. For the norm of an operator A_h: $\mathscr{U}_h^0 \to \mathscr{U}_h$ we adopt the notation

$$|A_h|_{s,t} := \sup_{0 \neq u_h \in \mathscr{U}_h^0} \frac{|A_h u_h|_{H^s}}{|u_h|_{H^t}}. \tag{4.9}$$

By virtue of the following lemma, the estimate (4.2) carries over to an estimate relative to $|\cdot|_{1,1}$:

LEMMA 4.2.

$$|L_h^{-1}B_h|_{1,1} \leqslant h^2. \tag{4.10}$$

Proof. (4.8) is equivalent to $|L_h^{-1}|_{1,-1} = 1$. Hence it follows from (4.2) that

$$\begin{aligned} |L_h^{-1}B_h|_{1,1}^2 &\leqslant |L_h^{-1}|_{1,-1}^2 |B_h|_{-1,1}^2 \\ &= |L_h^{-1/2}B_hL_h^{-1/2}|_{0,0}^2 = \rho\left(L_h^{-1}B_hL_h^{-1}B_h\right) \\ &\leqslant \|L_h^{-1}B_hL_h^{-1}B_h\|_\infty \leqslant \|L_h^{-1}B_h\|_\infty^2 \leqslant (h^2)^2. \quad \square \end{aligned}$$

We shall need the following estimates for the projection operator R_h: $\mathscr{U}_h^0 \to \mathscr{U}_h^0$.

LEMMA 4.3. *We have*

$$|R_h|_{1,1} \leqslant \sqrt{7}. \tag{4.11a}$$

If Ω is convex,

$$|R_h|_{1,1} \leqslant \sqrt{2}. \tag{4.11b}$$

Proof. (a) is established by the estimate

$$\begin{aligned} |R_h|_{1,1}^2 &= |L_h^{1/2}R_hL_h^{-1/2}|_{0,0}^2 = \rho\left(R_hL_h^{-1}R_hL_h\right) \\ &\leqslant \|R_h\|_\infty \|L_h^{-1}R_hL_h\|_\infty \leqslant 1 + \|L_h^{-1}B_hL_h\|_\infty. \end{aligned}$$

It is easy to see that $\|B_hL_h\|_\infty \leqslant 6h^{-2}$. Hence,

$$|R_h|_{1,1} \leqslant (1 + h^2 6h^{-2})^{1/2} = \sqrt{7}$$

follows from Proposition 4.1.

For convex Ω, the sharper bound (b) can be derived via direct estimates of the scalar products $\langle \partial_x R_h u_h, \partial_x R_h u_h \rangle$ and $\langle \partial_y R_h u_h, \partial_y R_h u_h \rangle$. See [2] for details. $\square$

Let $\bar{\Delta} L_h$ refer to the application of the Mehrstellenoperator (3.2) in every point of the infinite grid, irrespective of the boundary condition. In other words,

$$\bar{\Delta} L_h := \frac{h^2}{6} \bar{\partial}_x \partial_x \bar{\partial}_y \partial_y. \tag{4.12}$$

Clearly, $\Delta L_h = R_h \bar{\Delta} L_h$.

LEMMA 4.4. *We have*

$$|\bar{\Delta} L_h|_{-1,1} \leqslant \tfrac{1}{3} \quad \textit{independent of } h. \tag{4.13}$$

Proof. We consider the scalar product $\langle \bar{\Delta} L_h u_h, v_h \rangle$ and apply partial summation:

$$\begin{aligned} |\langle \bar{\Delta} L_h u_h, v_h \rangle| &= \frac{h^2}{12}\left[|\langle \bar{\partial}_x \partial_x \bar{\partial}_y \partial_y u_h, v_h \rangle| + |\langle \bar{\partial}_x \partial_x \bar{\partial}_y \partial_y u_h, v_h \rangle| \right] \\ &= \frac{h^2}{12}\left[|\langle \bar{\partial}_y \partial_x \partial_y u_h, \partial_x v_h \rangle| + |\langle \bar{\partial}_x \partial_x \partial_y u_h, \partial_y v_h \rangle| \right] \\ &\leqslant \frac{h^2}{12} |\partial_x \partial_y u_h|_{H^1} |v_h|_{H^1} \leqslant \frac{1}{3} |u_h|_{H^1} |v_h|_{H^1}, \end{aligned}$$

where we have used the Cauchy-Schwarz inequality and crude estimation of $|\partial_x \partial_y u_h|_{H^1}$. Hence,

$$|\bar{\Delta} L_h u_h|_{H^{-1}} = \sup_{0 \neq v_h \in \mathscr{U}_h^0} \frac{|\langle \bar{\Delta} L_h u_h, v_h \rangle|}{|v_h|_{H^1}} \leqslant \frac{1}{3} |u_h|_{H^1}. \quad \square$$

THEOREM 4.5. *With respect to* $|\cdot|_{1,1}$, $L_h^{-1}\Delta L_h$ *is a contraction*:

$$|L_h^{-1}\Delta L_h|_{1,1} \leqslant .882. \tag{4.14a}$$

If Ω *is convex*,

$$|L_h^{-1}\Delta L_h|_{1,1} \leqslant .472. \tag{4.14b}$$

Proof. By (4.8),

$$\begin{aligned} |L_h^{-1}\Delta L_h u_h|_{H^1} &\leqslant |\Delta L u_h|_{H^{-1}} = |R_h \bar{\Delta} L_h u_h|_{H^{-1}} \\ &= \sup_{0 \neq v_h \in \mathscr{U}_h^0} \frac{|\langle \bar{\Delta} L_h u_h, R_h v_h \rangle|}{|v_h|_{H^1}} \leqslant \sup_{0 \neq v_h \in \mathscr{U}_h^0} \frac{|\bar{\Delta} L_h u_h|_{H^{-1}} |R_h v_h|_{H^1}}{|v_h|_{H^1}} \\ &\leqslant |R_h|_{1,1} |\bar{\Delta} L_h|_{-1,1} |u_h|_{H^1}. \end{aligned}$$

The result follows from Lemmas 4.3 and 4.4. □

The analysis of the modified iteration (2.12) will require stronger properties of L_h than used so far. We define $\partial_{xx}, \partial_{yy}: \mathscr{U}_h^0 \to \mathscr{U}_h^0$ by

$$\partial_{xx} u_h(P) := \begin{cases} \bar{\partial}_x \partial_x u_h(P), & P \in \Omega_h, \\ 0, & \text{else}, \end{cases} \tag{4.15}$$

and similarly for ∂_{yy}. We introduce a *discrete* H^2*-seminorm on* $\mathscr{U}_h^0$:

$$|u_h|_{H^2} := \left[|\partial_{xx} u_h|_{H^0}^2 + |\partial_{yy} u_h|_{H^0}^2 \right]^{1/2}. \tag{4.16}$$

In the following theorem we present a quantitative discrete H^2-regularity estimate.

THEOREM 4.6. Discrete H^2-regularity. *If* Ω *is convex*,

$$|u_h|_{H^2} \leqslant |L_h u_h|_{H^0} \tag{4.17}$$

holds independently of h for all $u_h \in \mathscr{U}_h^0$.

Proof. Let $u_h \in \mathscr{U}_h^0$. Since $L_h = -(\partial_{xx} + \partial_{yy})$,

$$\begin{aligned} |u_h|_{H^2}^2 &= \langle \partial_{xx} u_h, \partial_{xx} u_h \rangle + \langle \partial_{yy} u_h, \partial_{yy} u_h \rangle \\ &= \langle L_h u_h, L_h u_h \rangle - 2\langle \partial_{xx} u_h, \partial_{yy} u_h \rangle \\ &= |L_h u_h|_{H^0}^2 - 2\langle \bar{\partial}_x \partial_x u_h, \partial_{yy} u_h \rangle. \end{aligned}$$

Now,

$$\begin{aligned} \langle \bar{\partial}_x \partial_x u_h, \partial_{yy} u_h \rangle &= \langle \bar{\partial}_x \partial_x u_h, \bar{\partial}_y \partial_y u_h \rangle + \langle \bar{\partial}_x \partial_x u_h, (\partial_{yy} - \bar{\partial}_y \partial_y) u_h \rangle \\ &= \langle \partial_x \partial_y u_h, \partial_x \partial_y u_h \rangle - \langle \bar{\partial}_x \partial_x u_h, C_h \bar{\partial}_y \partial_y u_h \rangle. \end{aligned} \tag{4.18}$$

The first part is $\geqslant 0$, as required. In the second part, C_h denotes restriction to the boundary points (because these are the only points where, in general, $\partial_{yy} u_h(P) \neq \bar{\partial}_y \partial_y u_h(P)$ for $u_h \in \mathscr{U}_h^0$). We shall now investigate

$$\gamma_h(P) := \bar{\partial}_x \partial_x u_h(P) \bar{\partial}_y \partial_y u_h(P)$$

for all $P \in \partial\Omega_h$.

(a) P lies on a horizontal or vertical grid line, but is not a corner point. Then $\gamma_h(P) = 0$, since $\bar{\partial}_x \partial_x u_h(P) = 0$ or $\bar{\partial}_y \partial_y u_h(P) = 0$.

(b) P is a corner point, the corner being *not reentrant*. Then again $\gamma_h(P) = 0$, since either $\bar{\partial}_x \partial_x u_h(P) = 0$ or $\bar{\partial}_y \partial_y u_h(P) = 0$.

(a) and (b) imply H^2-regularity for rectangles.

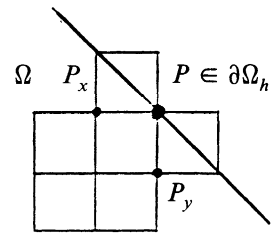

FIGURE 4.2
Boundary point on diagonal grid line

(c) P lies on a diagonal grid line and is not a corner point. One of the possible situations is shown in Figure 4.2. Let $P_x := P - he_x$, $P_y := P - he_y$. Clearly, $\bar{\partial}_x\partial_x u_h(P) = h^{-2}u_h(P_x)$ and $\bar{\partial}_y\partial_y u_h(P) = h^{-2}u_h(P_y)$. The inequality $ab \leqslant \frac{1}{2}(a^2 + b^2)$ yields

$$|\gamma_h(P)| \leqslant \frac{1}{2}\left[\left(h^{-2}u_h(P_x)\right)^2 + \left(h^{-2}u_h(P_y)\right)^2\right].$$

Since $u_h \in \mathscr{U}_h^0$, we have

$$|h^{-2}u_h(P_x)| = |\partial_x\partial_y u_h(P_x)|, \qquad |h^{-2}u_h(P_y)| = |\partial_x\partial_y u_h(P_y)|.$$

Similar conclusions hold for all other cases in question.

Since (a), (b) and (c) characterize arbitrary convex domains of the type under consideration, we have shown that, for convex Ω,

$$\begin{aligned}\left|\left\langle \bar{\partial}_x\partial_x u_h, C_h\bar{\partial}_y\partial_y u_h\right\rangle\right| &\leqslant \sum_{P\in\partial\Omega_h} |\gamma_h(P)| \\ &\leqslant \frac{1}{2}\sum_{P\in\partial\Omega_h}\left[\left(\partial_x\partial_y u_h(P_x)\right)^2 + \left(\partial_x\partial_y u_h(P_y)\right)^2\right],\end{aligned}$$

where it is sufficient to cover the points of type (c). It is obvious that, for h not too large, the latter sum is bounded by $\langle \partial_x\partial_y u_h, \partial_x\partial_y u_h\rangle$, since no point P_x, P_y will appear more than twice. Hence it follows from (4.18) that

$$\left\langle \bar{\partial}_x\partial_x u_h, \partial_{yy}u_h\right\rangle \geqslant 0.$$

This establishes H^2-regularity in the convex case. □

We shall from now on assume that Ω is convex. Theorem 4.6 enables us to establish a strengthened estimate for $L_h^{-1}\Delta L_h$.

LEMMA 4.7.

$$|L_h^{-1}\Delta L_h L_h^{-1}|_{1,0} \leqslant .236h. \tag{4.19}$$

Proof. By (4.8) and Theorem 4.6 it suffices to show that

$$|\Delta L_h u_h|_{H^{-1}} \leqslant .236h|u_h|_{H^2}, \qquad u_h \in \mathscr{U}_h^0. \tag{4.20}$$

Let $v_h \in \mathscr{U}_h^0$. Then,

$$\begin{aligned}|\langle \Delta L_h u_h, v_h\rangle| &= |\langle \Delta L_h u_h, R_h v_h\rangle| \\ &= \frac{h^2}{12}\left[\left|\left\langle \bar{\partial}_x\partial_x u_h, \bar{\partial}_y\partial_y R_h v_h\right\rangle\right| + \left|\left\langle \bar{\partial}_y\partial_y u_h, \bar{\partial}_x\partial_x R_h v_h\right\rangle\right|\right] \\ &= \frac{h^2}{12}\left[\left|\left\langle \partial_{xx}u_h, \bar{\partial}_y\partial_y R_h v_h\right\rangle\right| + \left|\left\langle \partial_{yy}u_h, \bar{\partial}_x\partial_x R_h v_h\right\rangle\right|\right],\end{aligned}$$

because the definition of R_h implies that for any $P \in \partial\Omega_h$ either $\bar{\partial}_x\partial_x u_h(P)$ or $\bar{\partial}_y\partial_y R_h v_h(P)$ vanishes (similarly for the pair $\bar{\partial}_y\partial_y, \bar{\partial}_x\partial_x R_h$). Further partial summation and application of the Cauchy-Schwarz inequality yield

$$|\langle \Delta L_h u_h, v_h\rangle| = \frac{h^2}{12}\left[|\langle \partial_y \partial_{xx} u_h, \partial_y R_h v_h\rangle| + |\langle \partial_x \partial_{yy} u_h, \partial_x R_h v_h\rangle|\right]$$

$$\leqslant \frac{h^2}{12}\left[|\partial_y \partial_{xx} u_h|^2_{H^0} + |\partial_x \partial_{yy} u_h|^2_{H^0}\right]^{1/2} |R_h|_{1,1} |v_h|_{H^1}.$$

Hence, $|\Delta L_h u_h|_{H^{-1}}$ is bounded by

$$|\Delta L_h u_h|_{H^{-1}} \leqslant \frac{h^2}{12}\left[\left(\frac{2}{h}\right)^2 |u_h|^2_{H^2}\right]^{1/2}\sqrt{2} = \frac{2\sqrt{2}}{12} h |u_h|_{H^2} < .236 h |u_h|_{H^2}$$

by (4.11b) and crude estimates for ∂_x, ∂_y. □

Consider now the modified defect correction iteration (2.12) involving a smoothing operator S_h as described in Section 3. For S_h, the following smoothing property is required:

$$|L_h S_h^\nu|_{0,1} \leqslant C(\nu) h^{-1}, \tag{4.21}$$

with $C(\nu) \to 0$ as $\nu \to \infty$.

PROPOSITION 4.8. *Let S_h satisfy* (4.21). *Then,*

$$|L_h^{-1}\Delta L_h S_h^\nu|_{1,1} \leqslant 0.24 C(\nu). \tag{4.22}$$

Proof. Apply Lemma 4.7. □

It follows from a result in Hackbusch [9, Section 3.3] that (4.21) is satisfied for Jacobi relaxation (with damping factor $\frac{1}{2}$) with

$$C(\nu) \leqslant \frac{2\sqrt{2}}{\sqrt{4\nu - 1}}. \tag{4.23a}$$

We also have

PROPOSITION 4.9. *For Gauss/Seidel relaxation with red-black ordering,* (4.21) *is satisfied with*

$$C(\nu) \leqslant \frac{2\sqrt{2\sqrt{2}}}{3\sqrt{3\sqrt{3}\,\nu}}. \tag{4.23b}$$

Proof. See [2]. □

We summarize as follows.

COROLLARY 4.10. *For convex* Ω,

$$|L_h^{-1}\Delta L_h|_{1,1} \leqslant .472$$

by Theorem 4.5. $|L_h^{-1}\Delta L_h S_h^\nu|_{1,1}$ *is bounded by the numbers given in the following table*:

ν	Jacobi	Gauss/Seidel
1	.386	.349
2	.253	.247
3	.202	.202
4	.173	.175

Our bounds are rigorous but certainly not optimal (cf. the model problem analysis in Section 3). The reason for this is that (4.19) as well as (4.21) are somewhat weaker than the corresponding estimates from Section 3. One of the open questions is whether something like $|L_h^{-1}\Delta L_h L_h^{-1}|_{1,1} \leqslant Ch^2$ is satisfied.

Since our bounds are valid only in H^1, the investigation of the "fixed point shift" becomes more difficult than in Section 3. In particular, we get the following nonoptimal result. We note, in this connection, that Lemma 3.4 (applied to $|\cdot|_{0,0}$) carries over to the present case without modification.

Let $D_h\colon \mathscr{U}_h^0 \to \mathscr{U}_h^0$ be defined by

$$D_h := R_h + h^2 B_h. \tag{4.24}$$

PROPOSITION 4.11. *Let* $|L_h^{-1}\Delta L_h S_h^\nu|_{1,1} \leqslant m < 1$ (*cf. Corollary* 4.10). *There exist constants* C_1, C_2 *independent of h such that the error* $\hat{e}_h$ *of Proposition* 2.2 *satisfies*

$$|\hat{e}_h|_{H^1} \leqslant \frac{1}{h}\frac{1}{1-m}\left[C_1\left|D_h\left(L_h' u_h^* - f_h'\right)\right|_{H^0} + C_2 C'(\nu) h^2 \left|L_h u_h^* - f_h\right|_{H^0}\right] \tag{4.25}$$

(*with* $C'(\nu)$ *from Lemma* 3.4). *Thus,* $|\hat{e}_h|_{H^1} = O(h^3)$.

Proof. By construction, the truncation error of the target discretization is $O(h^4)$ in the interior points of $P \in \Omega_h$ and $O(h^2)$ for $P \in \Gamma_h$. Therefore,

$$\left|D_h\left(L_h' u_h^* - f_h'\right)\right|_{H^0} \leqslant Ch^4.$$

By (4.10), $|L_h^{-1} D_h^{-1}|_{1,1} \leqslant C$. Hence it follows from Lemmas 3.4 and 4.7 that

$$\begin{aligned} |\hat{e}_h|_{H^1} &\leqslant \frac{1}{1-m}\Big[|L_h^{-1}D_h^{-1}|_{1,1}\left|D_h\left(L_h' u_h^* - f_h'\right)\right|_{H^1} \\ &\qquad + |L_h^{-1}\Delta L_h L_h^{-1}|_{1,0}|L_h|_{0,0}|T_h|_{0,0}|L_h u_h^* - f_h|_{H^0}\Big] \\ &\leqslant \frac{1}{1-m}\left[C_1 h^{-1}\left|D_h\left(L_h' u_h^* - f_h'\right)\right|_{H^0} + .236hCh^{-2}C'(\nu)h^2h^2\right] \\ &= O(h^3). \quad \square \end{aligned}$$

We have shown $O(h^3)$ for $|\hat{e}_h|_{H^0}$ and for $|\partial_x \hat{e}_h|_{H^0}$, $|\partial_y \hat{e}_h|_{H^0}$. It is easy to see that $O(h^4)$ follows if the truncation errors can be measured in H^1 rather than H^0 without loss of order. If u^* is sufficiently smooth, this can be expected in an "interior sense", but not up to the boundary.

On the other hand, estimates for $|L_h^{-1}\Delta L_h|_{0,0}$ seem to be very hard to obtain. In particular, numerical experience tells us that $|L_h^{-1}\Delta L_h|_{0,0} < 1$ cannot be expected in general. We have found an example where $|L_h^{-1}\Delta L_h|_{0,0} > 1$, even though Ω is convex (cf. [1]). (In contrast to this, $|\Delta L_h L_h^{-1}|_{0,0} < 1$ can easily be derived from Theorem 4.6.)

5. General Domains; Concluding Remarks. On the basis of the work of Hackbusch [8], [11], much of the reasoning from Section 4 can (at least qualitatively) be extended to the case of (1.1) in a domain Ω with curved boundary. The remarks below contain a summary of our analysis for general domains, which can be found in [2].

Suitable (basic and target) discretizations of (1.1) involve special difference formulae for "irregular" points near the boundary (see [2] for details). For the usual "Shortley-Weller"-operator L_h, Proposition 4.1 remains valid without modification (cf. [6]). The *discrete regularity* properties of the Shortley-Weller scheme have been studied by Hackbusch [11] (see also [1]). For discrete H^1- and H^2-norms appropriately defined, the following estimates hold independently of the mesh size h:

$$|L_h^{-1}|_{1,-1} \leqslant 1.02 \tag{5.1}$$

is valid under very weak assumptions on Ω, whereas

$$|L_h^{-1}|_{2,0} \leqslant C \tag{5.2}$$

is only true if the boundary $\partial\Omega$ is sufficiently smooth. At present, the constant in (5.2) is not explicitly known.

Using (5.1), (5.2) and analogues of Lemmas 4.2–4.4, it is shown in [2] that $|L_h^{-1}\Delta L_h|_{1,1}$ is $O(1)$. A further result is

$$|L_h^{-1}\Delta L_h L_h^{-1}|_{1,\theta-1} = O(h^\theta), \qquad 0 \leqslant \theta < \tfrac{1}{2}, \tag{5.3}$$

where $|\cdot|_{H^\theta}$ is a discrete Sobolev norm of noninteger order. Combining (5.3) with a generalized smoothing property, which is proved in Hackbusch [8] for Jacobi and Gauss/Seidel smoothers S_h, we obtain

$$|L_h^{-1}\Delta L_h S_h^\nu|_{1,1} \leqslant C(\nu), \tag{5.4}$$

where $C(\nu) \to 0$ for $\nu \to \infty$. Thus, the defect correction is contractive if a sufficient (h-independent) number of smoothing sweeps are performed.

The defect correction algorithm described in this paper has been implemented on the basis of the standard multigrid solver MG01 for Helmholtz's equation (cf. Stüben and Trottenberg [13]). A detailed description is given in [3]. Numerical experiences are reported in [1] and [3].

Our final remark deals with the question of stability of the target discretization. On the one hand, stability of L_h' is a direct consequence of $\rho(L_h^{-1}\Delta L_h) < 1$, independently of h (cf. Sections 3 and 4). However, Proposition 2.2 shows how, in principle, error smoothing may help when using an unstable L_h'. This could be of particular interest for such types of problems where high-order schemes inevitably are unstable (nonelliptic or singularly perturbed problems as, e.g., the convection diffusion equation). Future work will be concerned with this subject.

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