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Local Error Estimates for Some Petrov-Galerkin Methods Applied to Strongly Elliptic Equations on Curves

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Abstract. In this article we derive local error estimates for some Petrov-Galerkin methods applied to strongly elliptic equations on smooth curves of the plane. The results, e.g., cover the basic first-kind and second-kind integral equations appearing in the boundary element solution of the potential problem. The discretization model includes the Galerkin method and the collocation method using smoothest splines as trial functions. Asymptotic error estimates are given for a large scale of the Sobolev norms.

1. Introduction. We study local convergence properties of some Petrov-Galerkin methods applied to the general class of strongly elliptic pseudodifferential operators given on smooth closed Jordan curves of the plane. This framework covers various types of problems such as periodic differential equations on the real line, boundary integral equations of the second kind or Symm's first-kind integral equations with logarithmic single-layer potential. These integral equations are of fundamental importance in solving interior or exterior boundary value problems of potential theory. Other applications are singular integral equations with Cauchy type singularity in the kernel. These occur, e.g., in elasticity.

We are mainly interested in Galerkin and collocation methods. These methods seem to be those best analyzed from among the Petrov-Galerkin methods, and various global error estimates are given in [1], [2], [9], [13], and [14]. In particular, the collocation method is widely used in engineering applications.

As far as we know, the literature concerning local or interior error estimates is not very extensive. In [11] some local estimates were proved for the L_2-orthogonal projection on spline spaces. In [12] and [6] interior error estimates were derived for the Galerkin approximation of the solution of differential equations.

In the case of integral equations no local error estimates seem to exist. The validity of such results is not obvious, since the local properties of the integral operators differ considerably from the local properties of the differential operators, the former being only pseudolocal, in contrast to the local nature of the differential operators.

In the present article we derive local error estimates for general Petrov-Galerkin methods when smoothest splines are used as trial and test functions. In deriving the error estimates we adapt some techniques of [11] and [12]. The error estimates are

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given for a full range of the Sobolev norms corresponding to the known global estimates. When the trial and test space coincide, we have the Galerkin method. We briefly describe our results.

Let $A\colon H^s \to H^{s-2\alpha}$ be an isomorphism between the Sobolev spaces $H^s = H^s(\Gamma)$ given by the strongly elliptic pseudodifferential operator A of real order 2α. Furthermore, let

$$S^d = S^d(\Delta) \quad \text{and} \quad T^{d'} = T^{d'}(\tilde{\Delta})$$

be spaces of smoothest splines of degree d and d', respectively, and the meshes Δ and $\tilde{\Delta}$ be quasiuniform. Assuming essentially only that the Petrov-Galerkin method: Find $u_h \in S^d$ such that

$$(1.1) \qquad (Au_h \mid v) = (Au \mid v), \qquad v \in T^{d'},$$

is stable with respect to the norm $\|\cdot\|_x$, there holds the optimal-order global error estimate

$$(1.2) \qquad \|u - u_h\|_t \leqslant ch^{s-t}\|u\|_s$$

for $2\alpha - (d' + 1) \leqslant t \leqslant s \leqslant d + 1$, $t < d + \frac{1}{2}$, $x \leqslant s$. In Section 4 we prove corresponding local estimates of the following type. Let Γ_0 and Γ_1 be open subarcs of Γ such that $\overline{\Gamma}_0 \subset \Gamma_1$. We assume that the solution u satisfies local regularity $u \in H^s(\Gamma_1)$ together with global regularity $u \in H^r$ such that $x \leqslant r \leqslant s \leqslant d + 1$. By Theorem 4.3 there holds

$$(1.3) \qquad \|u - u_h\|_t(\Gamma_0) \leqslant c\big(h^{s-t}\|u\|_s(\Gamma_1) + h^{d'+1+r-2\alpha}\|u\|_r\big)$$

for values $2\alpha - d' - 1 \leqslant t < x$ and

$$(1.4) \qquad \|u - u_h\|_t(\Gamma_0) \leqslant c\big(h^{s-t}\|u\|_s(\Gamma_1) + h^{d'+1+r+x-2\alpha-t}\|u\|_r\big)$$

for $x \leqslant t < d + \frac{1}{2}$. Let us take as a special case the Galerkin method where $d = d'$ and $x = \alpha$. Then one concludes from (1.4) that the optimal-order convergence for values $\alpha \leqslant t < d + \frac{1}{2}$ is achieved already by the requirement $u \in H^\alpha$, i.e., that the solution belongs to the energy space. On the other hand, if $t < \alpha$, we need by (1.3) for the values $s > t + d + 1 - \alpha$ an additional global regularity to get optimal-order convergence. Results of a similar kind were obtained for differential operators in [12]. Also, other conclusions can be drawn from the above results. For example, it turns out (see Subsection 4.3) that for fixed trial functions the locality of the approximation increases by raising the degree of the test functions.

The other application which is discussed in Section 5 is the collocation method. More precisely, we consider two different variants of the collocation method. First we analyze the local properties of the collocation in the case of smoothest splines of odd degree, when the nodes of the mesh are used as the collocation points. The other example uses smoothest splines of even degree collocating at the midpoints of the mesh. Our framework for Petrov-Galerkin methods is applicable to obtain results for these collocation methods, since these methods can be considered as perturbations of a Galerkin method (odd-degree splines) or of a Petrov-Galerkin method (even-degree splines). For example, with splines of odd degree $d = 2j - 1$ the result is formally included already in estimates (1.3) and (1.4) with the choice $x = j + \alpha$ and $d' = -1$.

2. Preliminaries.

2.1. *Strongly Elliptic Equations.* We consider equations of the form

$$Au = f, \tag{2.1.1}$$

where A is a strongly elliptic pseudodifferential operator of real order 2α on the smooth closed Jordan curve $\Gamma \subset R^2$. For example, the logarithmic potential

$$Au(x) = \int_\Gamma u(y) \ln|x - y| \, dy$$

defines a strongly elliptic pseudodifferential operator of order -1. Since A is of order 2α, it defines for all s a Fredholm operator A: $H^s \to H^{s-2\alpha}$ with vanishing index [15], [16]. Here, H^s is the Sobolev space of functions defined on Γ. We assume that Eq. (2.1.1) is uniquely solvable. This implies that the operator A is an isomorphism for all s.

The extension of the results to cover the cases where the initial problem is not uniquely solvable, but becomes such by introducing auxiliary parameters and side conditions as in [1], [14], is straightforward and is omitted. The same remark is valid when considering strongly elliptic systems instead of single equations.

2.2. *Identification with Periodic Spaces.* By means of the parametric representation $t \to x(t)$ of the curve Γ we identify the functions on the curve with 1-periodic functions $u(t)$ defined on the whole real line. The global scalar products and norms can therefore be given by means of the Fourier representation as

$$(u \mid v)_s = Ju \cdot J\bar{v} + \sum_{0 \neq n \in \mathbf{Z}} |2\pi n|^{2s} \hat{u}(n) \overline{\hat{v}(n)},$$
$$\|u\|_s = (u \mid u)_s^{1/2}.$$

For $s = 0$ we shall often omit the subindex. We have used here the Fourier coefficients

$$\hat{u}(n) = \int_0^1 u(t) e^{-in2\pi t} \, dt$$

and the mean value $Ju = \hat{u}(0)$. Let $\Delta = \{x_k\}$, $x_k < x_{k+1}$, $k \in \mathbf{Z}$, be a set of points on the real axis such that $x_{k+N} = x_k + 1$ for some $N \in \mathbf{N}$ and all $k \in \mathbf{Z}$. We consider the space $S^d = S^d(\Delta)$ of 1-periodic splines $\varphi \in S^d(\Delta)$ such that φ is a polynomial of degree at most d in every subinterval (x_k, x_{k+1}) and the function φ has continuous derivatives up to order $d - 1$ for $d \geqslant 1$. The space $S^0(\Delta)$ means piecewise constants.

2.3. *Approximation and Inverse Properties.* In deriving the asymptotic local error estimates we need the approximation and inverse properties for splines. We assume the sequence of meshes to be quasiuniform, which means that the ratio

$$\bar{h}_\Delta / \underline{h}_\Delta$$

remains bounded when N approaches infinity. Here, $\bar{h}_\Delta = \max(x_{k+1} - x_k)$ and $\underline{h}_\Delta = \min(x_{k+1} - x_k)$.

For periodic spaces the following approximation and inverse properties can be found in [8].

LEMMA 2.1 (Approximation property). *If $t_0 < d + \frac{1}{2}$ and if $t_0 \leqslant s \leqslant d + 1$, then for any $u \in H^s$ there exists a function $\zeta \in S^d$ such that*

$$\|u - \zeta\|_t \leqslant c(t) h^{s-t} \|u\|_s \quad \text{for all } t \leqslant t_0. \tag{2.3.1}$$

LEMMA 2.2 (Inverse property). *We have*

$$\|\varphi\|_s \leqslant ch^{t-s} \|\varphi\|_t \quad \text{for all } \varphi \in S^d, \tag{2.3.2}$$

$t \leqslant s < d + \frac{1}{2}$.

The proof in [8] extends earlier results to cover the range of indices up to $t_0 < d + \frac{1}{2}$ in Lemma 2.1, or to $s < d + \frac{1}{2}$ in Lemma 2.2, from the previously known $t_0 \leqslant d$ in Lemma 2.1 or $s \leqslant d$ in Lemma 2.2.

A standard application of the inverse estimate combined with the approximation property yields the estimate (2.3.1) also for the higher-order norms. Thus we have

$$\|u - \zeta\|_t \leqslant c(t) h^{s-t} \|u\|_s \tag{2.3.3}$$

if $t \leqslant s \leqslant d + 1$, $t < d + \frac{1}{2}$.

In addition to periodic splines, we shall need nonperiodic smoothest splines defined on a given interval I. The corresponding spaces are denoted by $S^d(I)$. Furthermore, we shall need the Sobolev spaces $H^s(I)$ for all $s \in \mathbf{R}$. In particular, the spaces with negative indices are given by the duality $H^{-s}(I) = (H_0^s(I))'$, $s \geqslant 0$. Here, $H_0^s(I)$ is the $H^s(I)$-closure of the space $C_0^\infty(I)$ of the infinitely differentiable functions which are compactly supported in I. Thus the norm of $H^{-s}(I)$ is given by

$$\|u\|_{-s}(I) = \sup_{0 \neq \varphi \in H_0^s(I)} \frac{(u \mid \varphi)(I)}{\|\varphi\|_s(I)}. \tag{2.3.4}$$

It was proven in [3] that for the splines $S^d(I)$ there exist inverse estimates covering also the negative Sobolev norms. However, the negative norms used in [3] are those which arise when one defines the negative spaces $H^{-s}(I)$ by means of the duality $H^{-s}(I) = (H^s(I))'$. This yields the negative norms

$$|u|_{-s}(I) = \sup_{0 \neq \varphi \in H^s(I)} \frac{(u \mid \varphi)(I)}{\|\varphi\|_s(I)}, \tag{2.3.5}$$

which, in general, are stronger than those defined by (2.3.4).

With the weaker norm (2.3.4) we can prove the following

LEMMA 2.3. *We have the inverse estimate*

$$\|\varphi\|_s(I) \leqslant ch^{t-s} \|\varphi\|_t(I) \tag{2.3.6}$$

for all $\varphi \in S^d(I)$ and for all $t \leqslant s$, $s < d + \frac{1}{2}$, $s \neq p + \frac{1}{2}$, $p \in \mathbf{Z}$, $p \leqslant -1$.

Proof. For the fractional-order norms $\|u\|_s$, $s = d + \delta$, $0 < \delta < 1$, we have the Sobolev-Slobodetskiĭ representation

$$\|u\|_s^2 = \|u\|_d^2 + [D^d u]_\delta^2,$$

where

$$[v]_\delta^2 = \int_I \int_I \frac{|v(x) - v(y)|^2}{|x - y|^{1+2\delta}} \, dx \, dy.$$

An explicit calculation carried out in [8] shows that

$$[\psi]_\delta \leqslant ch^{\tau - \delta} [\psi]_\tau \tag{2.3.7}$$

for all piecewise constant splines $\psi \in S^0(I)$, if $0 < \tau \leqslant \delta < \frac{1}{2}$. If $\varphi \in S^d(I)$, we have $D^d\varphi \in S^0(I)$ and (2.3.7) implies

$$\|\varphi\|_s^2 = \|\varphi\|_d^2 + [D^d\varphi]_\delta^2 \leqslant \|\varphi\|_d^2 + ch^{2(\tau-\delta)}[D^d\varphi]_\tau^2$$
$$\leqslant ch^{2(t-s)}\|\varphi\|_t^2$$

for $d < t \leqslant s < d + \frac{1}{2}$, $t = d + \tau$, $s = d + \delta$. Since the inverse estimate holds for the values $d < t \leqslant s < d + \frac{1}{2}$, it holds also for the values $-\infty < t \leqslant s < d + \frac{1}{2}$, if the negative norms are defined by means of (2.3.5) [3].

In particular, we have

$$(2.3.8) \qquad \|\varphi\|_s(I) \leqslant ch^{t-s}\|\varphi\|_t(I), \qquad 0 \leqslant t \leqslant s < d + \tfrac{1}{2},$$

and

$$(2.3.9) \qquad \|\varphi\|_0(I) \leqslant ch^t|\varphi|_t(I), \qquad t \leqslant 0.$$

For any $s \leqslant 0$ we define X_s to be the intermediate space (independent of m) $X_s = [H^0(I), H^m(I)]_\theta$, where $m \in \mathbf{Z}$, $m < s$, and $s = \theta m$. Apart from the exceptional values $s = p + \frac{1}{2}$, where p is a negative integer, the space X_s coincides with the Sobolev space $H^s(I)$, and the interpolation norm $\|\cdot\|(X_s)$ is equivalent to the Sobolev norm $\|\cdot\|_s(I)$. On the other hand, the norms defined by (2.3.4) and (2.3.5) are the same for $-\frac{1}{2} \leqslant s \leqslant 0$, since for these values, $H^{-s}(I) = H_0^{-s}(I)$ [10, p. 55, Theorem 11.1]. Starting from the estimate (2.3.9), that is,

$$\|\varphi\|(X_0) \leqslant ch^t\|\varphi\|(X_t), \qquad -\tfrac{1}{2} \leqslant t \leqslant 0,$$

we obtain, again by means of the procedure in [3],

$$(2.3.10) \qquad \|\varphi\|(X_s) \leqslant ch^{t-s}\|\varphi\|(X_t), \qquad t \leqslant s \leqslant 0.$$

For the exceptional values $t = p + \frac{1}{2}$, $p \in \mathbf{Z}$, $p \leqslant -1$, there holds

$$(2.3.11) \qquad \|\varphi\|(X_t) \leqslant c\|\varphi\|_t,$$

since we have $X_{p+1/2} = (H_{00}^{-p-1/2}(I))'$, where the space $H_{00}^{-p-1/2}(I)$ is continuously embedded in $H_0^{-p-1/2}(I)$ [10, p. 66, Theorem 11.7]. Combining (2.3.8), (2.3.10), and (2.3.11), we have the assertion of the lemma. □

We shall need the following form of inverse estimates without exceptional values. Let $\omega \in C_0^\infty(I)$.

LEMMA 2.4. *For all $t \leqslant s < d + \frac{1}{2}$ and $\psi \in S^d$ there holds*

$$(2.3.12) \qquad \|\omega\psi\|_s \leqslant ch^{t-s}\|\psi\|_t(I).$$

Proof. Since we have $\|\omega\psi\|_s \leqslant c\|\psi\|_s(I)$, we may, by Lemma 2.3, assume that $s \leqslant 0$. For fixed $t < s$ we can choose by (2.3.3) an approximation ζ of $\omega\psi$ such that

$$(2.3.13) \qquad \|\omega\psi - \zeta\|_t \leqslant c\|\psi\|_t(I),$$

$$(2.3.14) \qquad \|\omega\psi - \zeta\|_s \leqslant ch^{-s}\|\psi\|_0(I).$$

But then we have by (2.3.14), Lemma 2.2, and Lemma 2.3,

$$(2.3.15) \qquad \|\omega\psi\|_s \leqslant \|\omega\psi - \zeta\|_s + \|\zeta\|_s \leqslant c\big(h^{t-s}\|\psi\|_t(I) + h^{t-s}\|\zeta\|_t\big).$$

From (2.3.13) there follows $\|\zeta\|_t \leqslant c\|\psi\|_t(I)$, and therefore (2.3.15) yields the assertion. □

We now choose a sequence of intervals $I_0 \subset\subset I_0' \subset\subset I_1 \subset\subset I$ such that the length $l(I) \leqslant 1$. Moreover, let $\omega \in C_0^\infty(I_0')$.

LEMMA 2.5. *Assume that* $t_0 < d + \frac{1}{2}$ *and* $q \in \mathbf{N}$. *Then*

(a) *For any* $\psi \in S^d$ *there exists a function* $\zeta \in S^d$ *such that*

$$\|\omega\psi - \zeta\|_t \leqslant ch^{s+1-t}\|\psi\|_s(I_0'), \tag{2.3.16}$$

$$\|\omega\psi - \zeta\|_t \leqslant ch^{s+1-t}\|\psi\|_s, \tag{2.3.17}$$

when $-q \leqslant t \leqslant t_0$, $s \leqslant d$.

(b) *For any* $u \in H^s$, $t_0 \leqslant s \leqslant d + 1$, *there exists a function* $\zeta \in S^d$ *such that*

$$\|\omega u - \zeta\|_t \leqslant ch^{s-t}\|u\|_s(I_0') \tag{2.3.18}$$

for all $-q \leqslant t \leqslant t_0$. *In both cases* (a) *and* (b) *one can choose the function* ζ *such that* $\operatorname{supp}\zeta \subset I_1$ *for* $0 < h \leqslant h_0$.

Proof. There exists a function $\eta \in S^d(I)$ such that $v = \omega\psi + \eta$ belongs to $H^{d+1}(I)$. Moreover, in the set $I \setminus I_0'$, η is a polynomial of degree $\leqslant d$. We introduce the integration operator D^{-1},

$$D^{-1}f(x) = \int_a^x f(\tau)\,d\tau, \qquad a < x < b,$$

assuming $I = (a, b)$.

The function $w = D^{-q}v$ belongs to $H^{d+1+q}(I)$ and, in $I \setminus I_0'$, is a polynomial of degree $\leqslant d + q$. We fix an interval I_2 such that $I_1 \subset\subset I_2 \subset\subset I$. There exists an approximation $\xi \in S^{d+q}(I)$ of w such that for $0 < h \leqslant h_0$, where h_0 is independent of w, the function ξ coincides with w in the set $I_2 \setminus I_1$ and such that

$$\|D^k(w - \xi)\|(I) \leqslant ch^{d+q+1-k}\|D^{d+q+1}w\|(I) \tag{2.3.19}$$

for $0 \leqslant k \leqslant d + q$.

The above approximation ξ can be defined as in [4], [5]. Then the asserted property is a consequence of the locality of the approximation and the fact that the approximation reproduces polynomials up to degree $d + q$. Define $\varphi = D^q\xi$ and $\zeta \in S^d$ such that

$$\zeta(x) = \begin{cases} -(\eta(x) - \varphi(x)), & x \in I_2, \\ 0, & x \in I \setminus I_2. \end{cases}$$

In the interval I_2 one has $\omega\psi - \zeta = D^q(w - \xi)$, which upon integration by parts yields

$$\begin{aligned} \|\omega\psi - \zeta\|_{-q}(I_2) &= \sup_{0 \neq f \in H_0^q(I_2)} \frac{(D^q(w - \xi) \mid f)(I_2)}{\|f\|_q(I_2)} \\ &\leqslant \|w - \xi\|(I_2) \leqslant ch^{d+q+1}\|D^{d+q+1}w\| \\ &\leqslant ch^{d+q+1}\|\psi\|_d(I_0'). \end{aligned} \tag{2.3.20}$$

From (2.3.19) there follows also

$$\|\omega\psi - \zeta\|_d(I_2) \leqslant ch\|\psi\|_d(I_0'). \tag{2.3.21}$$

But since the function $\omega\psi - \zeta$ is supported in I_2 we have by (2.3.20) and (2.3.21)

$$\|\omega\psi - \zeta\|_t \leqslant c\|\omega\psi - \zeta\|_t(I_2) \leqslant ch^{d+1-t}\|\psi\|_d(I_0')$$

for $t = -q$ and $t = d$. Interpolating, we obtain

(2.3.22) $$\|\omega\psi - \zeta\|_t \leqslant ch^{d+1-t}\|\psi\|_d(I_0')$$

for $-q \leqslant t \leqslant d$.

Take $d \leqslant t \leqslant t_0 < d + \frac{1}{2}$. If was proved in [2] that there exists an approximation $\hat{\eta} \in S^d$ such that

(2.3.23) $$\|\omega\psi - \hat{\eta}\|_t \leqslant ch^{d+1-t}\|\psi\|_d(I_0').$$

Thus we get from (2.3.2), (2.3.22), and (2.3.23),

$$\begin{aligned}\|\omega\psi - \zeta\|_t &\leqslant \|\omega\psi - \hat{\eta}\|_t + \|\hat{\eta} - \zeta\|_t \\ &\leqslant ch^{d+1-t}\|\psi\|_d(I_0') + ch^{d-t}\|\hat{\eta} - \zeta\|_d \\ &\leqslant ch^{d+1-t}\|\psi\|_d(I_0') + ch^{d-t}\|\omega\psi - \hat{\eta}\|_d + ch^{d-t}\|\omega\psi - \zeta\|_d \\ &\leqslant ch^{d+1-t}\|\psi\|_d(I_0').\end{aligned}$$

This, combined with (2.3.6), proves the assertion (2.3.16). The estimate (2.3.17) follows from (2.3.16). The proof of (b) is a minor modification of the case (a) and is omitted. □

We point out that Lemma 2.5 is very essential for obtaining local error estimates. Similar approximation properties have been used, e.g., in [11] and [12]. However, it is difficult to locate a proof in the literature which covers the smoothest splines of general order, even for nonnegative values of the indices s and t. The idea to regularize $\omega\psi$, as in the proof of (2.3.16) or (2.3.17), to obtain the "extra power" of h was already used in [7].

3. Petrov-Galerkin Method.

3.1. *Stability and Global Convergence.* We assume that the principal symbol $a(x, \xi)$ of the pseudodifferential operator A satisfies

(3.1.1) $$\operatorname{Re} a(x, \xi) \geqslant c > 0, \qquad x \in \Gamma, |\xi| = 1, \xi \in \mathbf{R}.$$

This assumption implies the validity of the Gårding inequality [15]. Thus we have the representation

(3.1.2) $$A = A_0 + K,$$

where A_0 satisfies

(3.1.3) $$\operatorname{Re}(A_0 u \mid u) \geqslant c\|u\|_\alpha^2, \qquad u \in H^\alpha,$$

and where K: $H^s \to H^{s-2\alpha}$ is compact. Since A is assumed to be invertible, we deduce from (3.1.2) and (3.1.3) that the Babuška stability condition,

(3.1.4) $$\inf\sup|(A\varphi \mid \psi)| \geqslant c > 0, \qquad \|\varphi\|_\alpha = 1, \|\psi\|_\alpha = 1, \varphi \in S^d, \psi \in S^d,$$

is satisfied for small $0 < h \leqslant h_0$. Here S^d is a spline space such that $S^d \subset H^\alpha$, i.e., $\alpha < d + \frac{1}{2}$.

To verify (3.1.4), one can choose $\psi = \varphi - P_h(A^*)^{-1}K^*\varphi$, where P_h: $H^\alpha \to S^d$ is the L_2-orthogonal projection and where A^* and K^* are L_2-adjoints; cf. [14]. The mapping T: $\varphi \to \psi$ satisfies

(3.1.5) $$c_1\|\varphi\|_\alpha \leqslant \|T\varphi\|_\alpha \leqslant c_2\|\varphi\|_\alpha, \qquad \varphi \in S^d.$$

We are concerned with more general Petrov-Galerkin methods, where we use $S^d = S^d(\Delta)$ as the trial subspace and $T^{d'} = S^{d'}(\tilde{\Delta})$ as the test subspace. We require that both meshes Δ and $\tilde{\Delta}$ be quasiuniform. Furthermore, it is assumed that these meshes have the same number of points in the unit interval, i.e., S^d and $T^{d'}$ have the same finite dimension. The Petrov-Galerkin solution $u_h = Gu \in S^d$ of (2.1.1) is defined by

$$(3.1.6) \qquad (Au_h \mid \varphi) = (Au \mid \varphi), \qquad \varphi \in T^{d'}.$$

We assume that the approximation is stable with respect to the norm $\|\cdot\|_x$. For this, we first require that the inequalities

$$(3.1.7) \qquad x < d + \tfrac{1}{2} \quad \text{and} \quad 2\alpha - x < d' + \tfrac{1}{2}$$

be valid. By assumption (3.1.7) we have $S^d \subset H^x$ and $T^{d'} \subset H^{2\alpha - x}$. Furthermore, we assume that the stability condition $(0 < h \leqslant h_0)$

$$(3.1.8) \quad \inf\sup |(A\varphi \mid \psi)| \geqslant c > 0, \qquad \|\varphi\|_x = 1,\ \|\psi\|_{2\alpha - x} = 1,\ \varphi \in S^d,\ \psi \in T^{d'},$$

is satisfied. Applications with different trial and test functions will be discussed in the next section and in Section 5, when collocation with even-order splines is considered. Since the operator $A\colon H^s \to H^{s-2\alpha}$ is bounded, we have

$$(3.1.9) \qquad |(Au \mid v)| \leqslant c\|u\|_x \|v\|_{2\alpha - x}$$

for all $u \in H^x$, $v \in H^{2\alpha - x}$. From (3.1.8) and (3.1.9) it follows that the Petrov-Galerkin method (3.1.6) is quasioptimal, i.e.,

$$(3.1.10) \qquad \|u - u_h\|_x \leqslant c \inf_{\varphi \in S^d} \|u - \varphi\|_x.$$

We have the following asymptotic error estimates.

THEOREM 3.1. *For the Petrov-Galerkin approximation there holds*

$$(3.1.11) \qquad \|u - u_h\|_t \leqslant ch^{s-t}\|u\|_s,$$

when $2\alpha - (d' + 1) \leqslant t \leqslant s \leqslant d + 1$, $t < d + \frac{1}{2}$, $x \leqslant s$. *For the values* $2\alpha - (d' + 1) \leqslant t \leqslant x$ *we also have*

$$(3.1.12) \qquad \|u - u_h\|_t \leqslant ch^{x-t}\|u - u_h\|_x.$$

Proof. If $t \leqslant x$ we take $w \in H^{-t}$ such that $\|u - u_h\|_t^2 = (u - u_h \mid w) = \|w\|_{-t}^2$. With the element $y = (A^*)^{-1}w \in H^{-t+2\alpha}$ we have

$$(3.1.13) \qquad \|u - u_h\|_t^2 = (u - u_h \mid A^*y) = (A(u - u_h) \mid y - \zeta)$$

for all $\zeta \in T^{d'}$. The approximation property (2.3.3), together with (3.1.13), gives for $2\alpha - (d' + 1) \leqslant t$

$$\begin{aligned} \|u - u_h\|_t^2 &\leqslant ch^{x-t}\|A(u - u_h)\|_{x-2\alpha} \cdot \|y\|_{2\alpha - x} \\ &\leqslant ch^{x-t}\|u - u_h\|_x \cdot \|u - u_h\|_t, \end{aligned}$$

which proves (3.1.12). Estimates (2.3.1), (3.1.10), and (3.1.12) imply (3.1.11) for $2\alpha - (d' + 1) \leqslant t \leqslant x$. The remaining cases with $x \leqslant t < d + \frac{1}{2}$ follow as usual by using the inverse property (2.3.2). □

3.2. *Decomposition of Local Error.* We write $e = u - u_h$. For the local estimates it is sufficient to bound the norms over any interval $I_0 \subset\subset I$ where I is some interval of unit length. We fix further subintervals such that

$$I_0 \subset\subset I_0' \subset\subset I_1' \subset\subset I_1'' \subset\subset I_1 \subset\subset I.$$

Moreover, let $\omega \in C_0^\infty(I_0')$ be such that $\omega(x) = 1$, $x \in I_0$. Further, take functions $\eta \in C_0^\infty(I_1'')$ and $\xi \in C_0^\infty(I_1)$ such that $\eta|_{I_1'} = 1$ and $\xi|_{I_1''} = 1$. For any function f we write $\tilde{f} = \omega f$. We decompose the local error $\tilde{e}$ as

$$\tilde{e} = (\tilde{u} - G\tilde{u}) + (G\tilde{u} - G\tilde{u}_h) - (\tilde{u}_h - G\tilde{u}_h) \tag{3.2.1}$$

and study each of the three terms separately. For any $u \in H^x$ the restriction $u|_{I_1}$ of the distribution u to the open interval I_1 is well defined. In what follows we write $u \in H^s(I_1)$, which more precisely means $u|_{I_1} \in H^s(I_1)$. Theorem 3.1 yields

LEMMA 3.2. *Assume $x \leqslant s \leqslant d + 1$, $u \in H^s(I_1) \cap H^x$. Then we have*

$$\|\tilde{u} - G\tilde{u}\|_x \leqslant ch^{s-x}\|\omega u\|_s \leqslant ch^{s-x}\|u\|_s(I_1). \tag{3.2.2}$$

For the second term of the decomposition (3.2.1) we obtain

LEMMA 3.3. *For any fixed $\beta \leqslant x$ there holds*

$$\|G(\tilde{u} - \tilde{u}_h)\|_x \leqslant c\left(h^{\tau'}\|e\|_x(I_1) + \|e\|_{x-1}(I_1) + \|e\|_\beta\right) \tag{3.2.3}$$

with $\tau' = \min(1, d' + 1 + x - 2\alpha)$.

Proof. By (3.1.8) we have

$$\|G(\tilde{u} - \tilde{u}_h)\|_x \leqslant c\sup|(AG(\tilde{u} - \tilde{u}_h)\,|\,\psi)|, \qquad \|\psi\|_{2\alpha-x} = 1,\ \psi \in T^{d'}. \tag{3.2.4}$$

The Petrov-Galerkin equation (3.1.6) yields for $\psi \in T^{d'}$

$$(AG\tilde{e}\,|\,\psi) = (A\tilde{e}\,|\,\psi) = (Ae\,|\,\omega\psi) + ([A,\omega]e\,|\,\psi) \tag{3.2.5}$$

with the commutator $[A,\omega] = A\omega - \omega A$. By using (2.3.17) we find with some $\zeta \in T^{d'}$, $\operatorname{supp}\zeta \subset I_1'$, for $\|\psi\|_{2\alpha-x} = 1$

$$\begin{aligned}(Ae\,|\,\omega\psi) &= (Ae\,|\,\omega\psi - \zeta) = (\eta Ae\,|\,\omega\psi - \zeta)\\ &\leqslant ch^{\tau'}\left(\|\eta A\xi e\|_{x-2\alpha} + \|\eta A(1-\xi)e\|_{x-2\alpha}\right).\end{aligned} \tag{3.2.6}$$

Since $\eta(1 - \xi) = 0$, and since from the theory of pseudodifferential operators it is known that $\eta A(1 - \xi)$ is a pseudodifferential operator of order $-\infty$, we have the estimate

$$\|\eta A(1 - \xi)e\|_{x-2\alpha} \leqslant c\|e\|_\beta.$$

Therefore, by (3.2.6) it follows that

$$(Ae\,|\,\omega\psi) \leqslant ch^{\tau'}\left(\|e\|_x(I_1) + \|e\|_\beta\right). \tag{3.2.7}$$

The term involving the commutator can be handled as follows,

$$\begin{aligned}([A,\omega]e\,|\,\psi) &= ([A,\omega]\eta e\,|\,\psi) - (\omega A(1-\eta)e\,|\,\psi)\\ &\leqslant c\left(\|e\|_{x-1}(I_1) + \|e\|_\beta\right).\end{aligned} \tag{3.2.8}$$

Summing up (3.2.4), (3.2.5), (3.2.7), and (3.2.8), we get the desired assertion,

$$\|G(\tilde{u} - \tilde{u}_h)\|_x \leqslant c\left(h^{\tau'}\|e\|_x(I_1) + \|e\|_{x-1}(I_1) + \|e\|_\beta\right). \quad \square$$

For the last term in (3.2.1) we prove

LEMMA 3.4. *Assume that* $u \in H^s(I_1) \cap H^x$, *where* $x \leqslant s \leqslant d+1$. *Then we have*

$$\|G\tilde{u}_h - \tilde{u}_h\|_x \leqslant c\big(h^{s-x}\|u\|_s(I_1) + h^\tau\|e\|_x(I_1)\big), \tag{3.2.9}$$

where $\tau = \min(1, d+1-x)$.

Proof. We first assume that $x \leqslant s \leqslant d$. By Lemma 2.5,

$$\|G\tilde{u}_h - \tilde{u}_h\|_x \leqslant ch^{s+1-x}\|u_h\|_s(I_0'). \tag{3.2.10}$$

With the smooth functions η and ξ defined earlier in this section we have $\eta = \eta\xi$ and thus

$$\|u_h\|_s(I_0') \leqslant c\|\eta u_h\|_s \leqslant c\big(\|\eta(u_h - \varphi)\|_s + \|\eta\varphi\|_s\big), \tag{3.2.11}$$

where

$$\|\eta\varphi\|_s \leqslant c\big(\|\xi u - \varphi\|_s + \|\xi u\|_s\big) \tag{3.2.12}$$

for every $\varphi \in S^d$. By (2.3.1) we can choose the function φ such that

$$\|\xi u - \varphi\|_s \leqslant c\|\xi u\|_s \leqslant c\|u\|_s(I_1) \tag{3.2.13}$$

and

$$\|\xi u - \varphi\|_x \leqslant ch^{s-x}\|\xi u\|_s \leqslant ch^{s-x}\|u\|_s(I_1). \tag{3.2.14}$$

By (3.2.12) and (3.2.13) there follows

$$\|\eta\varphi\|_s \leqslant c\|u\|_s(I_1). \tag{3.2.15}$$

Furthermore, Lemma 2.4 gives, since $\eta \in C_0^\infty(I_1'')$,

$$\begin{aligned} \|\eta(u_h - \varphi)\|_s &\leqslant ch^{x-s}\|u_h - \varphi\|_x(I_1'') \\ &\leqslant ch^{x-s}\big(\|u - u_h\|_x(I_1'') + \|u - \varphi\|_x(I_1'')\big) \\ &\leqslant ch^{x-s}\big(\|u - u_h\|_x(I_1'') + \|\xi u - \varphi\|_x\big), \end{aligned} \tag{3.2.16}$$

where we have used the fact that $\xi \in C_0^\infty(I_1)$, $\xi|_{I_1''} = 1$. Combining (3.2.10), (3.2.11), (3.2.14), (3.2.15), and (3.2.16), we have

$$\|G\tilde{u}_h - \tilde{u}_h\|_x \leqslant c\big(h^{s+1-x}\|u\|_s(I_1) + h\|e\|_x(I_1)\big). \tag{3.2.17}$$

Assume now that $x \leqslant d \leqslant s \leqslant d+1$. Then we have $u \in H^d(I_1)$, and the previous result (3.2.17) yields

$$\begin{aligned} \|G\tilde{u}_h - \tilde{u}_h\|_x &\leqslant c\big(h^{d+1-x}\|u\|_d(I_1) + h\|e\|_x(I_1)\big) \\ &\leqslant c\big(h^{s-x}\|u\|_s(I_1) + h\|e\|_x(I_1)\big). \end{aligned}$$

Finally, if $d \leqslant x \leqslant s \leqslant d+1$, $x < d + \frac{1}{2}$, we have by Lemma 2.5

$$\begin{aligned} \|G\tilde{u}_h - \tilde{u}_h\|_x &\leqslant ch^{d+1-x}\|u_h\|_d(I_0') \\ &\leqslant ch^{d+1-x}\big(\|u\|_d(I_0') + \|u - u_h\|_d(I_0')\big) \\ &\leqslant c\big(h^{s-x}\|u\|_d(I_1) + h^\tau\|e\|_x(I_1)\big). \end{aligned}$$

Thus the lemma is proven. □

Summing up the estimates of the previous lemmas, we arrive at

THEOREM 3.5. *Assume that $u \in H^s(I_1) \cap H^x$ with $x \leqslant s \leqslant d+1$. For any $\beta \leqslant x$ there holds*

(3.2.18) $\|e\|_x(I_0) \leqslant c\left[h^{s-x}\|u\|_s(I_1) + h^{\tau_0}\|e\|_x(I_1) + \|e\|_{x-1}(I_1) + \|e\|_\beta\right],$

where $\tau_0 = \min(1, d+1-x, d'+1+x-2\alpha)$.

4. Local Convergence of the Petrov-Galerkin Method.

4.1. *General Result.* To utilize Theorem 3.5, we estimate the lower-order local norms $\|\cdot\|_t$ for $t \leqslant x$. For $\omega e \in H^t$ there exists $w \in H^{-t}$ such that

$$(4.1.1) \qquad (\omega e \mid w) = \|\omega e\|_t^2 = \|w\|_{-t}^2.$$

Taking $y = (A^*)^{-1}w$ we have $y \in H^{-t+2\alpha}$ and

$$(4.1.2) \qquad \|y\|_{-t+2\alpha} \leqslant c\|w\|_{-t} = c\|\omega e\|_t.$$

We first prove

LEMMA 4.1. *Let $2\alpha - (d'+1) \leqslant t \leqslant x$ and let $\beta \leqslant \alpha$ be fixed. Then*

$$(4.1.3) \qquad \|e\|_t(I_0) \leqslant c\left(h^{x-t}\|e\|_x(I_1) + \|e\|_{t-1}(I_1) + \|e\|_\beta\right).$$

Proof. We proceed similarly as in the proof of Lemma 3.2. We have

$$(4.1.4) \qquad \|\omega e\|_t^2 = (\omega e \mid A^* y) = (A\omega e \mid y) = (\omega A e \mid y) + ([A,\omega]e \mid y).$$

According to Lemma 2.5, we can estimate by using an element $\zeta \in T^{d'}$, $\operatorname{supp}\zeta \subset I_1$,

$$(4.1.5) \qquad \begin{aligned} (\omega A e \mid y) &= (\eta A e \mid \omega y - \zeta) \\ &\leqslant c\left(\|\eta A \xi e\|_{x-2\alpha} + \|\eta A(1-\xi)e\|_{x-2\alpha}\right)h^{x-t}\|y\|_{2\alpha-t} \\ &\leqslant c h^{x-t}\left(\|e\|_x(I_1) + \|e\|_\beta\right)\cdot\|\omega e\|_t, \end{aligned}$$

if $2\alpha - t \leqslant d'+1$. The term involving the commutator in (4.1.4) has the estimate

$$(4.1.6) \qquad ([A,\omega]e \mid y) \leqslant c\left(\|e\|_{t-1}(I_1) + \|e\|_\beta\right)\cdot\|\omega e\|_t.$$

By using (4.1.4)–(4.1.6) we have the assertion. □

THEOREM 4.2. *Assume that $u \in H^s(I_1) \cap H^r$, where $x \leqslant s$, $r \leqslant d+1$. Then we have*

$$(4.1.7) \qquad \|e\|_x(I_0) \leqslant c\left(h^{s-x}\|u\|_s(I_1) + h^{d'+1+r-2\alpha}\|u\|_r\right).$$

Proof. Given the intervals $I_0 \subset\subset I_1$, we choose a sequence of intervals J_k, $0 \leqslant k \leqslant m$, such that

$$I_0 = J_0 \subset\subset J_1 \subset\subset \cdots \subset\subset J_{m-1} \subset\subset J_m = I_1.$$

By Theorem 3.5 we have

$$(4.1.8) \qquad \|e\|_x(J_0) \leqslant c\left(h^{s-x}\|u\|_s(J_1) + h^{\tau_0}\|e\|_x(J_1) + \|e\|_{x-1}(J_1) + \|e\|_\beta\right),$$

and Lemma 4.1 yields

$$(4.1.9) \qquad \|e\|_{x-1}(J_1) \leqslant c\left(h^{\tau_0}\|e\|_{x-1}(J_2) + \|e\|_{x-2}(J_2) + \|e\|_\beta\right).$$

By (4.1.8) and (4.1.9) we obtain

$$\|e\|_x(J_0) \leqslant c\left(h^{s-x}\|u\|_s(I_1) + h^{\tau_0}\|e\|_x(J_2) + \|e\|_{x-2}(J_2) + \|e\|_\beta\right).$$

Continuing this process, we finally achieve

$$\|e\|_x(J_0) \leqslant c\big(h^{s-x}\|u\|_s(I_1) + h^{\tau_0}\|e\|_x(J_m) + \|e\|_{x-m}(J_m) + \|e\|_\beta\big).$$

Taking m so large that $x - m \leqslant \beta$, we have

$$\|e\|_x(I_0) \leqslant c\big(h^{s-x}\|u\|_s(I_1) + h^{\tau_0}\|e\|_x(I_1) + \|e\|_\beta\big). \tag{4.1.10}$$

A further application of (4.1.10) to sufficiently many subintervals eliminates the term $h^{\tau_0}\|e\|_x(I_1)$ and yields

$$\|e\|_x(I_0) \leqslant c\big(h^{s-x}\|u\|_s(I_1) + \|e\|_\beta\big). \tag{4.1.11}$$

Choosing $\beta = 2\alpha - (d' + 1)$, we have by Theorem 3.1

$$\|e\|_\beta \leqslant ch^{d'+1+r-2\alpha}\|u\|_r,$$

which together with (4.1.11) proves the assertion. □

We are now ready to state the main result of this section. First, we observe that repeated use of the inequality (4.1.3) gives the estimate

$$\|e\|_t(I_0) \leqslant c\big(h^{x-t}\|e\|_x(I_1) + \|e\|_{2\alpha-d'-1}\big) \tag{4.1.12}$$

for $2\alpha - d' - 1 \leqslant t \leqslant x$. We now have

THEOREM 4.3. *Assume that $u \in H^s(I_1) \cap H^r$, where $x \leqslant s$, $r \leqslant d + 1$. Then for $2\alpha - d' - 1 \leqslant t \leqslant x$ there holds*

$$\|e\|_t(I_0) \leqslant c\big(h^{s-t}\|u\|_s(I_1) + h^{d'+1+r-2\alpha}\|u\|_r\big). \tag{4.1.13}$$

For $x < t < d + \frac{1}{2}$ we have

$$\|e\|_t(I_0) \leqslant c\big(h^{s-t}\|u\|_s(I_1) + h^{d'+1+r+x-2\alpha-t}\|u\|_r\big). \tag{4.1.14}$$

Proof. The first estimate (4.1.13) follows by combining (4.1.12) and (4.1.11) with the global estimate (3.1.11). Assume that $x < t < d + \frac{1}{2}$. We continue using the notation in the proof of Lemma 3.4 with the smooth functions η and ξ and with $\varphi \in S^d$ satisfying (3.2.13), (3.2.14).

Thus, we have by Lemma 2.4,

$$\begin{aligned} \|e\|_t(I_0) &\leqslant c\big(\|\xi u - \varphi\|_t + \|\eta(u_h - \varphi)\|_t\big) \\ &\leqslant c\big(\|\xi u - \varphi\|_t + h^{x-t}\|u_h - \varphi\|_x(I_1'')\big) \\ &\leqslant c\big(\|\xi u - \varphi\|_t + h^{x-t}\big(\|u - u_h\|_x(I_1'') + \|\xi u - \varphi\|_x\big)\big) \\ &\leqslant c\big(h^{s-t}\|u\|_s(I_1) + h^{x-t}\|e\|_x(I_1'')\big). \end{aligned} \tag{4.1.15}$$

Estimates (4.1.13) and (4.1.15) yield the assertion (4.1.14). □

4.2. *Galerkin Method.* Let us consider the special case of the Galerkin method where $x = \alpha$ and $d = d'$, $\alpha < d + \frac{1}{2}$. Then for $\alpha \leqslant t < d + \frac{1}{2}$ it suffices to take $r = \alpha$. We have

THEOREM 4.4. *For the Galerkin method with $u \in H^s(I_1) \cap H^r$, $\alpha \leqslant s$, $r \leqslant d + 1$, there holds*

$$\|e\|_\alpha(I_0) \leqslant ch^{s-t}\big(\|u\|_s(I_1) + \|u\|_\alpha\big) \tag{4.2.1}$$

and for $2\alpha - d - 1 \leqslant t \leqslant \alpha$,

$$\|e\|_t(I_0) \leqslant c\big(h^{s-t}\|u\|_s(I_1) + h^{d+1+r-2\alpha}\|u\|_r\big). \tag{4.2.2}$$

For $\alpha < t < d + \frac{1}{2}$, $t \leqslant s$, *we have*

(4.2.3) $$\|e\|_t(I_0) \leqslant ch^{s-t}\big(\|u\|_s(I_1) + \|u\|_\alpha\big).$$

As a conclusion from Theorem 4.4 we observe that for the Sobolev norms $\|\cdot\|_t(I_0)$ with $t \geqslant \alpha$, local convergence is not affected by the lack of smoothness outside I_1, $I_0 \subset\subset I_1$, if the minimal requirement $u \in H^\alpha$ is satisfied. On the other hand, for the lower-order norms with $t \leqslant \alpha$ we need additional global regularity to obtain the optimal-order local convergence, even if the solution is smooth in a neighborhood of I_0.

As a concrete example we choose the Galerkin method for the second-order differential operator, taking $\alpha = 1$. We assume that piecewise cubic splines are used in the approximation. Then we have for the local energy norm the optimal-order convergence,

$$\|e\|_1(I_0) \leqslant ch^{s-1}\big(\|u\|_s(I_1) + \|u\|_1\big), \qquad 1 \leqslant s \leqslant 4.$$

For the local L_2-norm we have by (4.2.2), for the values $1 \leqslant r \leqslant s \leqslant 4$,

$$\|e\|_0(I_0) \leqslant c\big(h^s\|u\|_s(I_1) + h^{2+r}\|u\|_r\big).$$

Thus the global regularity $u \in H^2$ is needed to obtain the optimal order $O(h^4)$ for locally smooth solutions. Behavior of the above kind was derived already in [12].

4.3. *Effect of Different Trial and Test Functions.* In order to illustrate the effect of using different-order splines as trial and test functions, we consider the following family of Petrov-Galerkin methods. We assume that the mesh Δ is the same for the trial and for the test splines, but the order of the test functions is higher than the order of the trial functions. Let $S^d = S^d(\Delta)$ be the trial subspace and $S^{d'} = S^{d'}(\Delta)$ be the test subspace such that $d' = d + 2l$, $l \in \mathbf{N}$, $l \geqslant 1$.

The method: Find $u_h \in S^d$ such that

(4.3.1) $$(Au_h \mid \psi) = (Au \mid \psi), \qquad \psi \in S^{d'},$$

is equivalent to a Galerkin method. Let, indeed, M: $H^s \to H^{s-2}$ be the operator $Mu = (J - D^2)u$. Then M^l defines a bijective mapping $S^{d'} \to S^d$. Writing $v_h = M^{-l}u_h \in S^{d'}$ and $v = M^{-l}u$, Eq. (4.3.1) is equivalent to the problem: Find $v_h \in S^{d'}$ such that

(4.3.2) $$\big(AM^l v_h \mid \psi\big) = \big(AM^l v \mid \psi\big), \qquad \psi \in S^{d'}.$$

The operator AM^l decomposes as $AM^l = AJ - AD^{2l}$, where AD^{2l} is a strongly elliptic pseudodifferential operator of order $2(\alpha + l)$ and where AJ: $H^s \to H^{s-2(\alpha+l)}$ is compact. We assume that $\alpha - l < d + \frac{1}{2}$. Since we have

$$c_1\|M^l\psi\|_{\alpha-l} \leqslant \|\psi\|_{\alpha+l} \leqslant c_2\|M^l\psi\|_{\alpha-l},$$

one deduces from the stability (3.1.4),

$$\inf\sup\big|\big(AM^l\varphi \mid \psi\big)\big| \geqslant c > 0, \qquad \|\varphi\|_{\alpha+l} = 1,\ \|\psi\|_{\alpha+l} = 1,\ \varphi, \psi \in S^{d'},$$

that the requirement (3.1.8) is valid. There, $x = \alpha - l$, $2\alpha - x = \alpha + l$. Inequality (3.1.9) also holds.

By Theorem 3.1 we have the global estimate

(4.3.3) $$\|u - u_h\|_t \leqslant ch^{s-t}\|u\|_s$$

with $2\alpha - d - 2l - 1 \leqslant t \leqslant s \leqslant d + 1$, $t < d + \frac{1}{2}$, $\alpha - l \leqslant s$. Thus, the effect of using the higher-order test functions, in contrast to the Galerkin method, is to gain convergence for a larger range of the index t.

For local convergence with the values $2\alpha - d - 2l - 1 \leqslant t \leqslant \alpha - l$ there holds

$$\|u - u_h\|_t(I_0) \leqslant c\left(h^{s-t}\|u\|_s(I_1) + h^{d+2l+1+r-2\alpha}\|u\|_r\right)$$

by (4.1.13). For the values $\alpha - l \leqslant t < d + \frac{1}{2}$, the first term on the right-hand side of (4.1.14) is dominating, and we have

$$\|u - u_h\|_t(I_0) \leqslant ch^{s-t}\left(\|u\|_s(I_1) + \|u\|_{\alpha-l}\right), \tag{4.3.4}$$

$t \leqslant s \leqslant d + 1$, $\alpha - l \leqslant t < d + \frac{1}{2}$. Thus, if we increase the degree of the test functions, the range of locality increases, as we see from (4.3.4). In particular, the convergence can be made completely local with respect to any fixed norm $\|\cdot\|_t$ by choosing sufficiently smooth test splines. By this we mean that

$$\|u - u_h\|_t \leqslant ch^{s-t}\left(\|u\|_s(I_1) + \|u\|_t\right)$$

if $d' = d + 2l$ is large enough.

For example, in the previous example of second-order differential operators, it suffices for $d = 3$ to choose $d' = 5$ in order to have a local estimate with respect to the L_2-norm. With this choice, we have

$$\|u - u_h\|_t(I_0) \leqslant ch^{s-t}\left(\|u\|_s(I_1) + \|u\|_0\right)$$

for $0 \leqslant t \leqslant s \leqslant 4$, $t < 7/2$.

Finally, we point out that the use of lower-order splines as test functions decreases the range of convergence and reduces locality.

We have taken as an example differential operators. Let us briefly mention some other applications which are covered by our analysis, with different choices of α. The case $\alpha = 0$, with $A = I$, covers L_2-orthogonal projections and with $A = I + K$, K compact, second-kind Fredholm integral equations. Also Cauchy type singular integral equations belong to this case. With $\alpha = -\frac{1}{2}$ we have results for first-kind integral equations with logarithmic kernel.

5. Local Convergence of Some Collocation Methods.

5.1. *Collocation Equations*. Our key point in deriving the local results for the collocation method is to employ the approaches in [1] and [14]. The collocation equations for the odd-order splines as trial functions were analyzed by means of a modified Galerkin method in [1]. Correspondingly, for the even-order splines a modified Petrov-Galerkin method was used to describe the collocation equations. In the case of the odd-order splines, the collocation equations for the problem (2.1.1) are given by

$$(Au_\Delta)(x_k) = (Au)(x_k), \tag{5.1.1}$$

where x_k are the break points of the mesh Δ and where $u_\Delta \in S^d$ is the sought collocation solution. In [1] it was utilized that the equations (5.1.1) are equivalent to the following Galerkin type problem: Find $u_\Delta \in S^d$ such that

$$(A_\Delta u_\Delta \mid \varphi)_j = (A_\Delta u \mid \varphi)_j \quad \text{for all } \varphi \in S^d, \tag{5.1.2}$$

where $d = 2j - 1$, $A_\Delta = (I - J + J_\Delta)A$, and where J_Δ is given by the trapezoidal approximation of J,

$$J_\Delta u = \sum_{k=1}^{N} \frac{1}{2}(x_{k+1} - x_{k-1})u(x_k).$$

In the case where the collocation solution u_Δ is sought among even-order splines $u_\Delta \in S^d$, $d = 2j - 2$, $j \geqslant 1$, we use as the collocation points the midpoints

$$t_k = x_k + \tfrac{1}{2}(x_{k+1} - x_k)$$

of the mesh Δ. The collocation equations

$$(5.1.3) \qquad (Au_\Delta)(t_k) = (Au)(t_k)$$

were analyzed in [14] by using the equivalent Petrov-Galerkin type formulation

$$(5.1.4) \qquad (A_\Delta u_\Delta \mid \varphi)_j = (A_\Delta u \mid \varphi)_j, \qquad \varphi \in S^{d+1}(\tilde{\Delta}),$$

where $\tilde{\Delta}$ is the mesh $\tilde{\Delta} = \{t_k\}$, and where in the formula $A_\Delta = (I - J + J_\Delta)A$ the trapezoidal approximation J_Δ of J uses the points t_k instead of x_k. The analysis in [14] is based on Fourier analysis and assumes that the mesh is uniform and the principal symbol of A has constant coefficients.

Let us for convenience recall the convergence results proven in [1], [14]. Under the assumption

$$(5.1.5) \qquad d > 2\alpha,$$

the optimal-order convergence result

$$\|u - u_\Delta\|_t \leqslant ch^{s-t}\|u\|_s,$$

with $2\alpha \leqslant t \leqslant s \leqslant d + 1$, $t < d + \frac{1}{2}$, $\frac{1}{2}(d + 1) + \alpha \leqslant s$, was proved in [1], [14]. Further, the methods (5.1.1) and (5.1.3) are stable with respect to the norm $\|\cdot\|_{\frac{1}{2}(d+1)+\alpha}$.

5.2. *Odd-Order Splines.* We consider the collocation method (5.1.1) with odd-order splines as trial functions. Let P: $H^{j+\alpha} \to S^d$ be the collocation projection defined by $Pu = u_\Delta$. In order to use the previous considerations for the Galerkin methods, we introduce the projection Q: $H^{j+\alpha} \to S^d$ by means of $Qu = u_h$,

$$(5.2.1) \qquad (Au_h \mid \varphi)_j = (Au \mid \varphi)_j, \qquad \varphi \in S^d.$$

Using integration by parts, we find that the equation (5.2.1) is equivalent to

$$\left(\left[(-1)^j D^{2j} + 1\right] Au_h \mid \varphi\right) = \left(\left[(-1)^j D^{2j} + 1\right] Au \mid \varphi\right)$$

for all $\varphi \in S^d$. Thus, Q is the Galerkin projection corresponding to the pseudodifferential operator

$$(5.2.2) \qquad B = \left[(-1)^j D^{2j} + 1\right] A,$$

which is of order $2(\alpha + j)$. Moreover, the assumption (5.1.5) used for the collocation method implies that the assumption $\alpha + j < d + \frac{1}{2}$, which is needed for the Galerkin method, is also valid. Further, since A is an isomorphism, B is an isomorphism, too. Thus the projection Q satisfies the global results of Theorem 3.1 and the local result of Theorem 4.3 when α is replaced by $\alpha + j$.

We shall obtain the local convergence results for the collocation projection P by showing that the difference $P - Q$, even globally, is small enough with respect to any norm to be considered.

First we prove

LEMMA 5.1. *For sufficiently small* $0 < h \leqslant h_0$ *there holds*

$$\|u_\Delta - u_h\|_{j+\alpha} \leqslant ch^{j-\alpha}\|u_h - u\|_{j+\alpha} \leqslant ch^{r-2\alpha}\|u\|_r \tag{5.2.3}$$

for $u \in H^r$, $j + \alpha \leqslant r \leqslant d + 1$.

Proof. Equations (5.1.2) and (5.2.1) imply

$$(A(u_\Delta - u_h)\,|\,\varphi)_j = (J - J_\Delta)A(u_\Delta - u)\cdot J\varphi \tag{5.2.4}$$

for all $\varphi \in S^d$. Choosing $\varphi = T(u_\Delta - u_h)$, where T: $S^d \to S^d$ is the mapping used in Subsection 3.1, satisfying the estimate (3.1.5), we obtain

$$\begin{aligned}\|u_\Delta - u_h\|_{j+\alpha}^2 &\leqslant c\left|\left(\left((-1)^j D^{2j} + 1\right)A(u_\Delta - u_h)\,|\,T(u_\Delta - u_h)\right)\right| \\ &= c\left|(A(u_\Delta - u_h)\,|\,T(u_\Delta - u_h))_j\right| \\ &= c\left|(J - J_\Delta)A(u_\Delta - u)\cdot JT(u_\Delta - u_h)\right|.\end{aligned} \tag{5.2.5}$$

Define $\delta = \min(j - \alpha, 2)$. From the error term of the trapezoidal rule we have

$$|(J - J_\Delta)A(u_\Delta - u_h)| \leqslant ch^\delta\|A(u_\Delta - u_h)\|_{j-\alpha} \leqslant ch^\delta\|u_\Delta - u_h\|_{j+\alpha}. \tag{5.2.6}$$

Since by (3.1.5),

$$|JT(u_\Delta - u_h)| \leqslant c\|u_\Delta - u_h\|_{j+\alpha}, \tag{5.2.7}$$

we obtain from (5.2.5)–(5.2.7)

$$\|u_\Delta - u_h\|_{j+\alpha} \leqslant c\left(h^\delta\|u_\Delta - u_h\|_{j+\alpha} + |(J - J_\Delta)A(u_h - u)|\right),$$

which for small $0 < h \leqslant h_0$ yields

$$\|u_\Delta - u_h\|_{j+\alpha} \leqslant c|(J - J_\Delta)A(u_h - u)|. \tag{5.2.8}$$

Since $j - \alpha > \frac{1}{2}$, we can choose a number ε such that $\frac{1}{2} < \varepsilon < \min(j - \alpha, 2)$. By (5.2.8) it then follows that

$$\|u_\Delta - u_h\|_{j+\alpha} \leqslant ch^\varepsilon\|A(u_h - u)\|_\varepsilon \leqslant ch^\varepsilon\|u_h - u\|_{\varepsilon+2\alpha}. \tag{5.2.9}$$

By (3.1.12) we have

$$\|u_h - u\|_{\varepsilon+2\alpha} \leqslant ch^{j-\alpha-\varepsilon}\|u_h - u\|_{j+\alpha}, \tag{5.2.10}$$

since $2\alpha \leqslant \varepsilon + 2\alpha \leqslant j + \alpha$. Now (5.2.9) and (5.2.10), together with Theorem 3.1, imply the assertion. □

Finally, for the collocation method (5.1.1) we have local error estimates with a full range of the indices.

THEOREM 5.2. *Assume that* $u \in H^s(I_1) \cap H^r$, *where* $j + \alpha \leqslant s$, $r \leqslant d + 1$. *Then for* $2\alpha \leqslant t \leqslant j + \alpha$ *there holds*

$$\|u - u_\Delta\|_t(I_0) \leqslant c\left\{h^{s-t}\|u\|_s(I_1) + h^{r-2\alpha}\|u\|_r\right\}. \tag{5.2.11}$$

For $j + \alpha < t < d + \frac{1}{2}$, $t \leqslant s$, *we obtain*

$$\|u - u_\Delta\|_t(I_0) \leqslant ch^{s-t}\left(\|u\|_s(I_1) + \|u\|_{j+\alpha}\right). \tag{5.2.12}$$

Proof. If $2\alpha \leqslant t \leqslant j + \alpha$ we have, combining estimate (4.2.2) with Lemma 5.1,

$$\begin{aligned}\|u - u_\Delta\|_t(I_0) &\leqslant \|u - u_h\|_t(I_0) + \|u_h - u_\Delta\|_{j+\alpha} \\ &\leqslant c\left(h^{s-t}\|u\|_s(I_1) + h^{d+1+r-2(\alpha+j)}\|u\|_r\right) + ch^{r-2\alpha}\|u\|_r \\ &= c\left(h^{s-t}\|u\|_s(I_1) + h^{r-2\alpha}\|u\|_r\right).\end{aligned}$$

For the higher-order norms with $j + \alpha < t < d + \frac{1}{2}$, an inverse estimate yields, by (4.2.3),

$$\begin{aligned}\|u - u_\Delta\|_t(I_0) &\leqslant \|u - u_h\|_t(I_0) + \|u_h - u_\Delta\|_t \\ &\leqslant \|u - u_h\|_t(I_0) + ch^{j+\alpha-t}\|u_h - u_\Delta\|_{j+\alpha} \\ &\leqslant ch^{s-t}\left(\|u\|_s(I_1) + \|u\|_{j+\alpha}\right) + ch^{2j-t}\|u\|_{j+\alpha} \\ &\leqslant ch^{s-t}\left(\|u\|_s(I_1) + \|u\|_{j+\alpha}\right),\end{aligned}$$

which proves (5.2.12). □

5.3. *Even-Order Splines*. We briefly mention the collocation method (5.1.3) using even-order splines as trial functions and collocating at midpoints of the original mesh. The equivalent formulation (5.1.4) can be viewed as a perturbation of the Petrov-Galerkin method: Find $u_h \in S^d$ such that

(5.3.1) $$(Bu_h \mid \varphi) = (Bu \mid \varphi), \qquad \varphi \in T^{d'} = S^{d+1}(\tilde{\Delta}),$$

where B is the operator given in (5.2.2). Here we apply the results in Section 4, replacing α by $\alpha + j$. We now have $d' = d + 1$, and the norm preserving stability is obtained with $x = j + \alpha - \frac{1}{2}$, [14]. The condition (3.1.7) is equivalent to the assumption $d > 2\alpha$ used for the collocation method. By Theorem 4.3 we have for the Petrov-Galerkin method (5.3.1), with $u \in H^s(I_1) \cap H^r$,

(5.3.2) $$\|u - u_h\|_t(I_0) \leqslant c\left(h^{s-t}\|u\|_s(I_1) + h^{r-2\alpha}\|u\|_r\right),$$

when $j + \alpha - \frac{1}{2} \leqslant s, r \leqslant d + 1$, $2\alpha \leqslant t \leqslant j + \alpha - \frac{1}{2}$, and

(5.3.3) $$\|u - u_h\|_t(I_0) \leqslant ch^{s-t}\left(\|u\|_s(I_1) + \|u\|_{j+\alpha-1/2}\right)$$

for $j + \alpha - \frac{1}{2} < t < d + \frac{1}{2}$. Proceeding exactly as in the case of the odd-order splines, we finally obtain

THEOREM 5.3. *Let* $u_\Delta \in S^d(\Delta)$ *be the collocation solution of* (5.1.3). *Then the local error estimates* (5.3.2) *and* (5.3.3), *where* u_h *is replaced by* u_Δ, *are valid.*

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