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Computation of Character Decompositions of Class Functions on Compact Semisimple Lie Groups*

By R. V. Moody** and J. Patera

Abstract. A new algorithm is described for splitting class functions of an arbitrary semisimple compact Lie group K into sums of irreducible characters. The method is based on the use of elements of finite order (EFO) in K and is applicable to a number of problems, including decompositions of tensor products and various symmetry classes of tensors, as well as branching rules in group-subgroup reductions. The main feature is the construction of a decomposition matrix D, computed once and for all for a given range of problems and for a given K, which then reduces any particular splitting to a simple matrix multiplication. Determination of D requires selection of a suitable set S of conjugacy classes of EFO representing a finite subgroup of a maximal torus T of K and the evaluation of (Weyl group) orbit sums on S. In fact, the evaluation of D can be coupled with the evaluation of the orbit sums in such a way as to greatly enhance the efficiency of the latter. The use of the method is illustrated by some extensive examples of tensor product decompositions in E_6. Modular arithmetic allows all computations to be performed exactly.

1. Introduction. In the study of compact Lie groups, both in theory and application, the representation theory is fundamental. Numerous computational problems arise in this connection which, in general, pose significant difficulties for all but the lowest rank groups. In this paper we are primarily concerned with a new algorithm for determining the splitting of class functions on a simple or semisimple compact Lie group K into finite sums of irreducible characters of K. The solution to this rather general problem can be applied to a number of well-known problems arising in applications of group theory.

For instance, consider the standard problem of determining the decomposition or branching of a unitary representation of a simple compact group $\tilde{K}$ relative to a subgroup K. The given representation ρ: $\tilde{K} \to SU(V)$ determines a character $\tilde{\chi}_V$: $\tilde{K} \to C$ which is a class function on K. Restricting $\tilde{\chi}$ to K, we have

$$\tilde{\chi}|_K = \sum \chi_i, \tag{1.1}$$

where $V = \oplus V_i$ is the decomposition of V into irreducible K-modules and χ_i is the character of K on V_i. The problem is to determine the right-hand side of (1.1).

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**On leave from: Department of Mathematics, University of Saskatchewan, Saskatoon, Saskatchewan, Canada

Again, consider the problem of decomposing the tensor product of two irreducible representations $\rho_i\colon K \to SU(V_i)$, $i = 1,2$. Then

$$(1.2) \qquad V_1 \otimes V_2 = \sum_{i=3}^{r} V_i,$$

where the V_i are irreducible. Correspondingly for the characters, we have

$$(1.3) \qquad \chi_{V_1 \otimes V_2} = \chi_{V_1}\chi_{V_2} = \sum_{i=3}^{r} \chi_{V_i}.$$

Here the problem is to determine the decomposition on the right-hand side of (1.3).

Another example occurs in the problem of finding the irreducible constituents of some symmetry class $V^{\mathbf{Y}}$ of tensors determined by a representation $\rho\colon K \to SU(V)$ and some Young tableau $\mathbf{Y}$. There is an explicit way of writing the character $\chi^{\mathbf{Y}}$ of $V^{\mathbf{Y}}$ in terms of the character χ of ρ and the characters of the symmetric group [25, §12], [20]. We are left with the determination of the splitting of $\chi^{\mathbf{Y}}$.

The splitting of class functions is always possible by a close examination of the weights involved, and for isolated examples of low rank this is probably the most efficient approach. We note, for instance, the tables of branching rules [1] for rank $\leqslant 8$ and dimensions less than 5000 (less than 10,000 for the exceptional groups). The approach here is directed to more extensive computations where the initial investment in time is compensated by the resulting efficiency of the splittings, and for higher-rank groups where practically no other methods exist.

The method proposed here has as its central feature the construction of a certain complex matrix D called a *decomposition matrix*. For a given set of weights which encompasses all those which may appear in the envisioned decompositions, and for a given simple or semisimple Lie group K, a matrix D may be computed once and for all. This is the most laborious part of the procedure. After that, all decompositions are determined by a single matrix multiplication.

The determination of decomposition matrices depends on the selection of suitable sets of elements of finite order (**EFO**) from K and the computation of their character values (or more precisely orbit sum values) on various irreducible representations of K. Fortunately, by a bootstrapping procedure it is possible to combine the construction of D and the evaluation of the orbit sum values, thereby greatly alleviating the amount of computing required. The introduction of real and complex arithmetic can be avoided by the use of modular arithmetic. Apart from being more elegant in lower-rank cases, modular arithmetic becomes indispensable for avoiding round-off problems in higher ranks.

Let us describe in a little more detail the ideas involved. The general problem is to determine the splitting of a set

$$(1.4) \qquad f^{(k)} = \sum a_i^{(k)} \chi_i, \qquad k = 1, \ldots, t,$$

of class functions $f^{(k)}$ on a compact Lie group K, where the χ_i are the characters of certain irreducible representations ρ_i of K. We will assume, as is usually possible in applications (in particular in the cases above), that we have some prior knowledge as to which χ_i can possibly occur, so that our task is to determine the coefficients $a_i^{(k)}$ (which may, of course, be zero). For simplicity of notation we will assume that we have only one class function f and suppress the superscripts (k).

Since f and the χ_i are class functions, they are completely determined by their restriction to a given maximal torus T of K. Let us assume that such a torus is fixed once and for all. Now if $x_1, \ldots, x_g \in T$ are some arbitrary elements then

$$f(x_j) = \sum a_i \chi_i(x_j), \qquad j = 1, \ldots, g, \tag{1.5}$$

determines a system of linear equations.

Assuming that the x_j are suitably independent, the a_i are determined by the solution of (1.5). Of course, we can do much better than this by choosing the right elements x_j. However, before we do this it is useful to introduce the *orbit sums* ϕ_i. These are simply sums of exponential functions on T corresponding to weight orbits of the Weyl group, and are related to characters by equations of the form

$$\chi_k = \sum_i m_k^i \phi_i, \tag{1.6}$$

where, assuming appropriate indexing, $M = (m_k^i)$ is a certain integral unipotent matrix called the *dominant weight multiplicity matrix* (see Section 3). The determination of M is in any case essential for any computing in semisimple Lie groups. Our algorithm for computing M was described in [17], [3] and extensive tables appear in [4]. Instead of decomposing f according to (1.5) it is advantageous to determine the unknown coefficients b_i in

$$f(x_j) = \sum b_i \phi_i(x_j), \qquad j = 1, \ldots, g. \tag{1.7}$$

Now suppose that $x_1, \ldots, x_g$ are elements of a finite Abelian group A contained in T. Then some Fourier analysis leads to certain orthogonality-like relations, and the inversion of (1.7) becomes trivial. This is the origin of the decomposition matrix. The solution to (1.5) is obtained by back substitution.

This then is the first ingredient in our algorithm. However, using Weyl group symmetry, a second enormous simplification occurs. Recall that if N is the normalizer of T in K then we have the Weyl group $W := N/T$ ($:=$ means that the right-hand side defines the left). W is a finite group whose size grows exponentially with the rank. It is well known that the W-conjugacy classes of T are a cross section of the conjugacy classes of K. Since the functions appearing in (1.4) are dependent only on the K-conjugacy classes, it is sufficient to take our Abelian group A to be W-stable (for all $w \in W$, $wAw^{-1} = A$) and to take $x_1, \ldots, x_h$ to be representatives of the W-conjugacy classes of A together with their multiplicities in A. Usually we take A to be

$$T_n := \{x \in T \mid x^n = 1\}. \tag{1.8}$$

Then $x_1, \ldots, x_h$ are required to be representatives for the W-conjugacy classes of elements of T satisfying $x^n = 1$. For these EFO there is a very precise and simply computable classification (see Section 2). The table in Section 7 illustrates the relation between $h = h(n)$ and $|T_n| = n^6$ for K of type E_6. The use of these classes obviously makes a huge difference in the number of elements we have to handle.

Briefly, the contents of the paper are as follows. In Section 2 we describe the classification of the conjugacy classes of EFO in semisimple compact Lie groups. In Proposition 1 we determine the sizes of the conjugacy classes in T_n. Section 3 is devoted to the algorithm for splitting class functions and describing the decomposition matrix D. In Section 4 we discuss the process of bootstrapping the construction of D and orbit sum evaluation. Section 5 introduces modular arithmetic and Section 6 collects together some additional comments and remarks. Finally, Section 7 presents some E_6 tensor product decompositions and some discussion of their computation. The present paper is an independent continuation of the general study of EFO begun in [18]; related and more particular problems may be found in [5], [19], [20], [23], [24]. Much of the program development of this project has been carried out by Wendy McKay whose tireless energy has been an enormous encouragement to us. Two extensive samples of computations carried out with this algorithm are the E_8 tables of characters and decompositions of plethysms and tensor products appearing in [13], [14].

2. Elements of finite order. In this section we establish the notation and briefly describe the classification of conjugacy classes of EFO in a simply connected semisimple compact Lie group. The classification is due to V. G. Kac [12]. (For a further description of the theory of these elements, their computation, and the determination of their values on characters and orbit sums the reader is referred to [19].) We then go on to determine the sizes of the various conjugacy classes in T_n (Proposition 1).

Suppose that K is a simply connected semisimple compact Lie group of rank l. Then $K \simeq K_1 \times \cdots \times K_r$ where $K_1, \ldots, K_r$ are simply connected simple compact Lie groups.

Conjugacy classes of EFO in K are determined by piecing together the corresponding classes of EFO in the various factors. If necessary then, we may assume that K is simple. For the present we do *not* make this assumption.

Let $\mathfrak{k}$ be the Lie algebra of K and let $\mathfrak{g}$ be the complexification of $\mathfrak{k}$. Fix a maximal torus T of K once and for all and let $i\mathfrak{t} \subset \mathfrak{k}$ be its Lie algebra (thus $\mathfrak{t}$ is a certain real subspace of $\mathfrak{g}$ which is a Euclidean space under the Killing form). We have the usual accoutrements relative to $\mathfrak{t}$ and some fixed (but otherwise arbitrary) ordering of the dual space $\mathfrak{t}^*$ of $\mathfrak{t}$:

(2.1)

$\Delta \subset \mathfrak{t}^*$ root system

$Q \subset \mathfrak{t}^*$ root lattice

$P \subset \mathfrak{t}^*$ weight lattice

$Q^\wedge \subset \mathfrak{t}$ coroot lattice ($\mathbb{Z}$-dual of P)

$P^\wedge \subset \mathfrak{t}$ coweight lattice ($\mathbb{Z}$-dual of Q)

$\Pi = \{\alpha_1, \ldots, \alpha_l\} \subset \Delta$ base of Δ

$\{\omega_1, \ldots, \omega_l\} \subset P$ basis of fundamental weights

$\{\alpha_1^\wedge, \ldots, \alpha_l^\wedge\} \subset Q^\wedge$ dual basis to $\{\omega_1, \ldots, \omega_l\}$

Δ^+ positive roots

W Weyl group acting on $\mathfrak{t}$ and $\mathfrak{t}^*$.

Here $\mathbb{Z}$-duals are taken relative to the natural pairing $\langle \cdot, \cdot \rangle$: $\mathfrak{t}^* \times \mathfrak{t} \to \mathbb{R}$. An alternative and important characterization of $Q^\wedge$ is as the kernel of the exponential mapping:

$$0 \to Q^\wedge \to \mathfrak{t} \xrightarrow{\exp 2\pi i(\cdot)} T \to 1. \tag{2.2}$$

The set $\{\alpha_1^\wedge, \dots, \alpha_l^\wedge\}$ is a base of the system of "dual" roots which may be considered as the system of roots of another Lie group $K^\wedge$.

According to (2.2) the subgroup

$$T_n := \{x \in T \mid x^n = 1\}$$

of T is in 1-1 correspondence with $\frac{1}{n}Q^\wedge/Q^\wedge$:

$$T_n \simeq \tfrac{1}{n}Q^\wedge/Q^\wedge. \tag{2.3}$$

Clearly, $|T_n| = n^l$.

As we pointed out in the introduction, these are precisely the groups in which we are interested for character decompositions. As is well known, two elements $X, Y \in \mathfrak{t}$ determine K-conjugate elements $\exp 2\pi i X$ and $\exp 2\pi i Y$ in K if and only if there exists $w \in W$ such that $wX \equiv Y \bmod Q^\wedge$. Alternatively we need a $\tilde{w} \in \tilde{W} := Q^\wedge \rtimes W$ with $\tilde{w}X = Y$. In this case we write $X \sim Y$.

Since class functions do not distinguish conjugate elements, it is only necessary to determine

CC(i) a cross section of the equivalence relation ~

CC(ii) the size of each of these equivalence classes in $\mathfrak{t}$ when they are reduced modulo $Q^\wedge$.

We begin in the case when K is *simple*. Let $-\alpha_0$ denote the highest root $\sum_{i=1}^{l} n_i\alpha_i$ of Δ relative to Π. Set $n_0 = 1$, so that

$$\sum_{i=0}^{l} n_i\alpha_i = 0. \tag{2.4}$$

We call $n_0, n_1, \dots, n_l$ the *numerical marks* of K. The matrix

$$A = (A_{ij})_{0 \leqslant i,j \leqslant l} := \big(2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)\big), \tag{2.5}$$

where $(\cdot,\cdot)$ is the standard positive definite form on $\mathfrak{t}^*$, is the *extended* or *affine* Cartan matrix associated with K. Then **CC**(i) is handled using the well-known fundamental region $\mathbf{F}$ of the action of $\tilde{W}$ on $\mathfrak{t}$:

$\mathbf{F}$ is the set of points $X \in \mathfrak{t}$ satisfying

F1: $\langle \alpha_i, X \rangle \geqslant 0$, $i = 1, \dots, l$,

F2: $\langle -\alpha_0, X \rangle \leqslant 1$.

Then $\mathfrak{t} = \tilde{W}\mathbf{F} := \{wX \mid w \in \tilde{W},\ X \in \mathbf{F}\}$, and for $X, Y \in \mathbf{F}$ and $w \in \tilde{W}$, $wX = Y$ if and only if $X = Y$. The conjugacy classes of EFO in T_n are specified by the points of $\frac{1}{n}Q^\wedge \cap \mathbf{F}$. From the point of view of computation these are most easily determined as certain $(l+1)$-tuples $\mathbf{s} = [s_0, \dots, s_l]$ of nonnegative integers (*Kac coordinates* [12], [18]) as follows. Let $X \in \frac{1}{n}Q^\wedge \cap \mathbf{F}$ and let $x = \exp 2\pi i X$. Then $\mathrm{Ad}(x)$ (image of x in the adjoint group of K) has some finite order which is in fact the least positive integer M such that $MX \in P^\wedge$. Note that the order of x is the least positive integer N such that $NX \in Q^\wedge$. One has $M \mid N$ and $N \mid n$. Define $s_0, s_1, \dots, s_l$ by

K1: $\langle \alpha_i, X \rangle = s_i/M$, $i = 1, \dots, l$;

K2: $\sum_{i=0}^{l} n_i s_i = M$;

and note that

K3: $\gcd(s_0, \dots, s_l) = 1$.

Conversely, every $(l+1)$-tuple of nonnegative integers $\mathbf{s} = [s_0, \dots, s_l]$ satisfying **K1–3** for some $M \mid N$ specifies a point X of $\mathbf{F}$ and, provided that $X \in \frac{1}{n}Q^\wedge$, X defines a conjugacy class of T_n. The precise relation between M and N is given in

[18]. We note that $\mathbf{s}$ has two interpretations: first as a point of $\mathbf{F}$, second as the label for the conjugacy class of elements determined by $\exp(2\pi i\mathbf{s})$. These are understood by the context.

The second problem, CC(ii), is answered by Proposition 1 below, for which we need more notation. We recall that W is generated by the root reflections $r_1, \dots, r_l$ in the simple roots $\alpha_1, \dots, \alpha_l$. We define r_0 to be the reflection in α_0. Given $\mathbf{s} = [s_0, \dots, s_l]$, an $(l+1)$-tuple of nonnegative integers, we define $W_{\mathbf{s}}$ to be the group generated by the r_i for which $s_i = 0$, $i = 0, \dots, l$.

PROPOSITION 1. *Let* $X \in \frac{1}{n}Q^{\wedge} \cap \mathbf{F}$ *have coordinates* $\mathbf{s}$. *Then the number of elements of* T *conjugate to* $\exp 2\pi iX$ *is the index* $[W : W_{\mathbf{s}}]$ *of* $W_{\mathbf{s}}$ *in* W.

Proof. (Partially based on a proof of T. A. Springer [27].) The conjugates of $\exp 2\pi iX$ in T are given by $\exp 2\pi iwX$ as w runs through W. The number of such conjugates is the index of the subgroup

$$S := \{ w \in W \mid \exp 2\pi iwX = \exp 2\pi iX \}$$

in W. Now $w \in S$ if and only if $wX \equiv X \bmod Q^{\wedge}$, which happens if and only if there is a $\tilde{w} \in \tilde{W}$ such that $\tilde{w}X = X$. The stabilizer of a point in $\mathfrak{t}$ under $\tilde{W}$ is generated by the reflecting hyperplanes through it [2, Chapter V, Section 3.3]. These hyperplanes are of the form

$$H_{\tilde{\alpha}} = \{ Y \in \mathfrak{t} \mid \langle \alpha, Y \rangle = k \}, \qquad \alpha \in \Delta,\ k \in \mathbf{Z}.$$

Thus w is a product of some of the corresponding affine reflections, $\tilde{w} = r_{\beta_1, k_1}, \dots, r_{\beta_s, k_s}$, and w is the corresponding product $r_{\beta_1} \cdots r_{\beta_s}$ in W. Thus S is generated by the reflections r_β such that

$$\langle \beta, X \rangle \in \mathbf{Z}. \tag{2.6}$$

Let $\beta = \sum b_i \alpha_i$ be a root satisfying (2.6). We can assume that $\beta \in \Delta^+$. Let $\mathbf{s} = [s_0, \dots, s_l]$. Then

$$\langle \beta, X \rangle = (1/M) \sum_{i=1}^{l} b_i s_i \in \mathbf{Z}_{\geqslant 0},$$

whence

$$\sum_{i=1}^{l} b_i s_i \equiv 0 \pmod{M}.$$

Since $0 \leqslant \sum_{i=1}^{l} b_i s_i \leqslant \sum_{i=0}^{l} n_i s_i = M$ and $b_i \leqslant n_i$ for each i, we have either

(i) $\sum_{i=1}^{l} b_i s_i = 0$, or

(ii) $\beta = -\alpha_0 = \sum_{i=1}^{l} n_i \alpha_i$ and $s_0 = 0$.

In either case,

$$\beta \in \Delta_{\mathbf{s}} := \Delta \cap \left\{ \alpha = \sum_{i=0}^{l} c_i \alpha_i \mid c_i \in \mathbf{Z},\ c_i = 0 \text{ if } s_i \neq 0 \right\}.$$

Conversely, we can see that if β is an element of Δ_s written as $\sum_{i=0}^{l} c_i\alpha_i$ with $c_i = 0$ if $s_i \neq 0$, then β satisfies (2.6). Indeed we have

$$\langle \beta, X\rangle = \left\langle \sum_{i=0}^{l} c_i\alpha_i, X\right\rangle = c_0\langle \alpha_0, X\rangle$$

$$= \begin{cases} -c_0\left\langle \sum_{i=1}^{l} n_i\alpha_i, X\right\rangle = -c_0 \in \mathbb{Z} & \text{if } s_0 = 0, \\ 0 \quad \text{if } s_0 \neq 0. \end{cases}$$

Now Δ_s is a subroot system of Δ and has a base $\Pi_s := \{\alpha_i \mid 0 \leqslant i \leqslant l,\ s_i = 0\}$. One way to see this is to consider the affine root system $\tilde{\Delta}$ based on the affine Cartan matrix $\tilde{A}$ (2.5) with base $\{\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_l\}$ corresponding to $\alpha_0, \alpha_1, \ldots, \alpha_l$ [16]. Then $\tilde{\Delta}_s := \{\sum_{i=0}^{l} c_i\tilde{\alpha}_i \mid c_i = 0 \text{ if } s_i \neq 0\} \subset \sum_{i=0}^{l} \mathbb{Z}\tilde{\alpha}_i$ is evidently a finite subroot system of $\tilde{\Delta}$ with base $\tilde{\Pi}_s := \{\tilde{\alpha}_i \mid 0 \leqslant i \leqslant l,\ s_i = 0\}$. The reduction $\sum_{i=0}^{l} \mathbb{Z}\tilde{\alpha}_i \to Q$ with $\tilde{\alpha}_i \to \alpha_i$ has kernel $\mathbb{Z}(\sum_{i=0}^{l} n_i\alpha_i)$ and so is injective on Δ_s. Thus Π_s is a base and W_s is its Weyl group. Finally, then, S is generated by the r_β, $\beta \in \Delta_s$ and $S = W_s$. □

In the case when K is only assumed to be semisimple, we have, corresponding to the decomposition $K = K_1 \times \cdots \times K_r$, decompositions

$$\Delta = \Delta_1 \cup \cdots \cup \Delta_r, \qquad \Pi = \Pi_1 \cup \cdots \cup \Pi_r,$$
$$W = W^1 \times \cdots \times W^r, \qquad \mathbf{F} = \mathbf{F}_1 \times \cdots \times \mathbf{F}_r \quad \text{etc.}$$

Each Δ_i produces its own numerical marks $n_j^{(i)}$, $j = 1, \ldots, l^{(i)}$, and each conjugacy class of EFO in K is specified by an r-tuple $[\mathbf{s}^{(1)}, \ldots, \mathbf{s}^{(r)}]$, where each $\mathbf{s}^{(i)} = [s_0^{(i)}, \ldots, s_{l^{(i)}}^{(i)}]$ accords with **K1–3**. The common order of the elements of the conjugacy class is the least common multiple of the constituent EFO determined by the $\mathbf{s}^{(i)}$.

For a given $\mathbf{s} = [\mathbf{s}^{(1)}, \ldots, s^{(r)}]$, $W_s = W_{s^{(1)}}^1 \times \cdots \times W_{s^{(r)}}^r$, and the number of EFO in T conjugate to $\exp 2\pi i\mathbf{s}$ is

$$[W^1 : W_{s^{(1)}}^1] \times \cdots \times [W^r : W_{s^{(r)}}^r] = [W : W_{s^{(1)}}^1 \times \cdots \times W_{s^{(r)}}^r] = [W : W_s].$$

Thus Proposition 1 holds also when K is only assumed to be semisimple.

3. Decomposing Class Functions. Let K be a semisimple simply connected compact group as before. We consider now the integral representation ring $R(K)$ of K, that is, the Grothendieck ring formed out of the isomorphism classes of unitary representations of K, with addition and multiplication derived from the formation of direct sums and tensor products. For each unitary representation ψ on a space V we have the character $\chi_V = \chi_\psi$: $K \to \mathbb{C}$. If V has the weight space decomposition

$$V = \bigoplus_{\lambda \in \Omega} V^\lambda, \tag{3.1}$$

where $\Omega \subset \mathfrak{t}^*$ is the weight system of V relative to T, then χ_ψ restricted to T is given explicitly by

$$\chi_\psi|_T = \sum_{\lambda \in \Omega} \left(\dim_{\mathbb{C}} V^\lambda\right) e^{2\pi i\lambda}. \tag{3.2}$$

This acts on $x = \exp 2\pi i\mathbf{X} \in T$ by

$$\chi_\psi(x) = \sum \left(\dim_{\mathbb{C}} V^\lambda\right) e^{2\pi i\langle \lambda, \mathbf{X}\rangle}. \tag{3.3}$$

For each of the fundamental weights $\omega_1, \dots, \omega_l$ of (2.1) denote by ϕ_i the unitary representation of K with highest weight ω_i and let χ_{ϕ_i} denote the corresponding character. Let $X(K)$ be the ring generated by all the characters χ_ψ as ψ runs over all the unitary representations of K. The following facts are well known [1]:

RR1: the set of χ_ψ as ψ runs over the irreducible representations of K is a $\mathbb{Z}$-basis of $X(K)$;

RR2: $R(K) \simeq X(K)$ via $[V] \to \chi_V$;

RR3: $X(K) = \mathbb{Z}[\chi_{\phi_1}, \dots, \chi_{\phi_l}]$ and $\chi_{\phi_1}, \dots, \chi_{\phi_l}$ are algebraically independent over $\mathbb{Z}$.

Furthermore, define

$$\mathbb{Z}[P] = \left\{ \sum a_\lambda e^{2\pi i \lambda} \mid \lambda \in P,\ a_\lambda \in \mathbb{Z}, \text{ finite sums} \right\}.$$

Then $\mathbb{Z}[P]$ admits an action of W through $we^{2\pi i\lambda} := e^{2\pi i w\lambda}$, and for the subring of W-invariants, $\mathbb{Z}[P]^W$, we have [1, 6.19].

RR4: $X(K) \simeq \mathbb{Z}[P]^W$.

We denote by $X_{\mathbb{C}}(K)$ the complexification $\mathbb{C} \otimes_{\mathbb{Z}} X(K)$ of $X(K)$.

Our point of view is that we are presented with an element f of $X_{\mathbb{C}}(K)$ which we know as a function (at least on sufficiently many EFO). The object is to compute the decomposition

$$f = \sum a_\psi \chi_\psi, \qquad a_\psi \in \mathbb{C},$$

guaranteed by **RR1**. It turns out to be better to compute in terms of the orbit sums. For each $\mu \in P$, the *orbit sum* defined by μ is

$$\phi_\mu := \sum_{\lambda \in W\mu} e^{2\pi i\lambda} \in \mathbb{Z}[P]^W, \tag{3.4}$$

where $W\mu := \{ w\mu \mid w \in W \}$. Clearly, $\phi_{w\mu} = \phi_\mu$ for all $w \in W$, $\mu \in P$, so we restrict our attention to ϕ_μ for dominant μ. We recall that the set of *dominant* elements of P, P^{++}, is defined by

$$\mu \in P^{++} \Leftrightarrow (\mu, \alpha_i) \geqslant 0 \quad \text{for each } i = 1, \dots, l.$$

Every W-orbit of weights contains exactly one dominant element.

If ψ is a representation of K on the space V with weight space decomposition (3.1) and if $\Omega^{++} := \Omega \cap P^{++}$, then

$$\chi_\psi = \sum_{\lambda \in \Omega^{++}} (\dim_{\mathbb{C}} V^\lambda) \phi_\lambda. \tag{3.5}$$

The weight multiplicities $\dim_{\mathbb{C}} V^\lambda$ for dominant λ are fundamental quantities in the computational theory of simple Lie groups. The reader is referred to [3], [17], [18] for more details. Extensive details of dominant weight multiplicities appear in [4].

We introduce the *level vector* $\mathbf{l} \in \mathfrak{t}$, uniquely specified by $\langle \alpha_i, \mathbf{l} \rangle = 2$ for $i = 1, \dots, l$ (cf. [4, Table 1]). Using $\mathbf{l}$, take any partial ordering $\leqslant$ on P such that $\lambda < \mu$ if $\langle \lambda, \mathbf{l} \rangle < \langle \mu, \mathbf{l} \rangle$. In particular, if $\mu - \lambda = \sum c_i \alpha_i$ with $c_i \in \mathbb{N}$ then $\lambda \leqslant \mu$. For each $\mu \in P^{++}$ the set of $\lambda \in P^{++}$ such that $\lambda \leqslant \mu$ is finite. Furthermore, all the weights of the representation ψ^μ with highest weight μ satisfy $\lambda \leqslant \mu$. Thus, if $m_\mu^\lambda := \dim_{\mathbb{C}} V^\lambda$ in the representation afforded by $\psi^\mu (\mu \in P^{++})$ and χ_μ is the character of ψ^μ, then the system of equations

$$\chi_\nu = \sum_{\lambda \leqslant \nu} m_\nu^\lambda \phi_\lambda, \qquad \nu \in P^{++},\ \nu \leqslant \mu, \tag{3.6}$$

determines ϕ_λ in terms of the χ_ν by means of the unipotent matrix (m_ν^λ). These matrices, called *dominant weight multiplicity matrices*, are precisely the tables of [4].

A list $\{\lambda_1, \dots, \lambda_r\}$ of dominant weights is said to be *consistent* if for each λ_i all the weights λ occurring in the decomposition (3.6) of χ_{λ_i} also appear in the list. Normally, we wish to work with consistent lists of weights.

It follows from [2, Chapter VI, Section 3.4] that the set of orbit sums ϕ_λ as λ runs over P^{++} is a $\mathbb{Z}$-basis of $X(K)$ and that $\phi_{\omega_1}, \dots, \phi_{\omega_l}$ is a set of algebraically independent ring generators of $X(K)$. We define

$$X^+(K) = \left\{ \sum c_\lambda \phi_\lambda \mid \lambda \in P^{++},\ c_\lambda \in \mathbb{N} \right\}.$$

Definition. We say that a subset A of T *separates* two subsets S_1, S_2 of P if for each pair $(\lambda_1, \lambda_2) \in S_1 \times S_2$ with $\lambda_1 \neq \lambda_2$ there is an $x \in A$ such that $\exp(2\pi i \lambda_1)(x) \neq \exp(2\pi i \lambda_2)(x)$. If $S_1 = S_2$ we simply say that A separates S_1. If $f = \sum a_\lambda e^{2\pi i \lambda} \in X(K)$, we say that A separates f if A separates the weights which actually appear in the sum ($a_\lambda \neq 0$).

Let A be a finite Abelian subgroup of T. We define

$$\langle \cdot , \cdot \rangle_A : \mathbb{Z}[P] \times \mathbb{Z}[P] \to \mathbb{C}$$

by

$$\langle f_1, f_2 \rangle_A = \sum_{x \in A} f_1(x) \overline{f_2(x)}, \tag{3.7}$$

where the overbar denotes complex conjugation.

Below we assume that A is *W-stable*: $wAw^{-1} \subset A$ for all $w \in W$.

PROPOSITION 2. *Let A be a finite W-stable (Abelian) subgroup of T of order g. Let $\lambda, \mu \in P^{++}$, $\lambda \neq \mu$.*

(i) *If A separates $W\lambda$ and $W\mu$, then*

$$\langle \phi_\lambda, \phi_\mu \rangle_A = \delta_{\lambda\mu} \cdot g|W\lambda|$$

where $\delta_{\lambda\mu}$ is the Kronecker δ-function.

(ii) *In any case $\langle \phi_\lambda, \phi_\mu \rangle_A$ is a nonnegative integral multiple of $g|W\lambda|$.*

Proof. We have

$$\langle \phi_\lambda, \phi_\mu \rangle_A = \sum_{x \in A} \sum_{\sigma \in W\lambda} \sum_{\tau \in W\mu} e^{2\pi i(\sigma - \tau)}(x) = \sum_{(\sigma,\tau) \in W\lambda \times W\mu} \sum_{x \in A} e^{2\pi i(\sigma - \tau)}(x).$$

Suppose that A separates $W\lambda$ and $W\mu$. Then none of the nonzero differences $\sigma - \tau$ in this sum vanishes on A. Thus

$$\sum_{x \in A} e^{2\pi i(\sigma - \tau)}(x) = \delta_{\sigma\tau} g, \tag{3.8}$$

and (i) follows. Even without separation, the sum in (3.8) can only be 0 or g. If it is g, so that $\{\sigma\}$ and $\{\tau\}$ are not separated by A, then neither are $\{w\sigma\}$ and $\{w\tau\}$, $w \in W$, and hence we obtain a contribution of $g|W\lambda|$ to $\langle \phi_\lambda, \phi_\mu \rangle_A$. Thus (ii) follows. □

In the applications described in Section 1 the class functions to be decomposed are sums of terms $e^{2\pi i \lambda}$, $\lambda \in P$, and hence lie in $X^+(K)$. Because of its importance, we prefer now to restrict ourselves to this situation and to make some comments about the general case at the end of the section.

Thus let

$$f = \sum_{\lambda \in P^{++}} a_\lambda \phi_\lambda \in X^+(K). \tag{3.9}$$

The $a_\lambda \in \mathbb{N}$, but are otherwise unknown. Let A be a finite W-stable subgroup of T of order g. Then

$$b_\lambda := (g|W\lambda|)^{-1}\langle f, \phi_\lambda\rangle_A \tag{3.10}$$

is an *integer* and

$$b_\lambda \geqslant a_\lambda \tag{3.11}$$

with equality if A separates f. Since

$$f(1) = \sum a_\lambda |W\lambda|, \tag{3.12}$$

it is easy to check when $\{a_\lambda\}$ is in fact a solution to (3.9).

To diminish the work in summing involved in (3.10) we now assume that $A = T_n = \{x \in T \mid x^n = 1\}$ and use the results of Section 2. Thus let $\mathrm{s}_1, \dots, \mathrm{s}_h$ denote the points of $\mathbf{F} \cap \frac{1}{n}Q^\wedge$ and let $x_j = \exp 2\pi i \mathrm{s}_j$, $j = 1, \dots, h$. Let

$$S_j = |W_{\mathrm{s}_j}|, \qquad j = 1, \dots, h, \tag{3.13}$$

so that there are in T precisely $|W|/S_j$ elements conjugate to x_j.

In addition, we assume that the dominant weights appearing in (3.9) are amongst the set $\{\lambda_1, \dots, \lambda_r\}$, where $\lambda_1 \leqslant \cdots \leqslant \lambda_r$. For each $j = 1, \dots, r$ let L_j be the order of the stabilizer of λ_j in W. Thus $|W\lambda_j| = |W|/L_j$. Then Eq. (3.10) becomes

$$b_{\lambda_j} = n^{-l} L_j \sum_{i=1}^{h} S_i^{-1} f(x_i)\overline{\phi_j(x_i)}, \qquad j = 1, \dots, r, \tag{3.14}$$

where we have written ϕ_i for ϕ_{λ_i} and the overbar denotes complex conjugation.

The equations (3.14) suggest that we define the $r \times h$ matrix $U = U^{[n]}$ by

$$U_{ji} = n^{-l/2}\sqrt{L_j/S_i}\,\phi_j(x_i). \tag{3.15}$$

If T_n separates the weights of $W\lambda_1 \cup \cdots \cup W\lambda_r$, then, replacing f of (3.9) by ϕ_k, we have from (3.14)

$$\delta_{kj} = n^{-l} L_j \sum_{i=1}^{h} S_i^{-1}\phi_k(x_i)\overline{\phi_j(x_i)} = \sqrt{L_j/L_k}\sum_{i=1}^{h} U_{ki}\overline{U}_{ji}, \qquad 1 \leqslant k, j \leqslant r.$$

Thus

$$U\overline{U}^T = \mathbf{1}_{r\times r} \quad (r \times r \text{ identity matrix}). \tag{3.16}$$

In terms of U, (3.10) reads

$$(b_{\lambda_1}, \dots, b_{\lambda_r})^T = n^{-l/2}\sqrt{L}\,\overline{U}\sqrt{S^{-1}}\,(f(x_1), \dots, f(x_h))^T, \tag{3.17}$$

where $L = \operatorname{diag}\{L_1, \dots, L_r\}$ and $S = \operatorname{diag}\{S_1, \dots, S_h\}$.

The $r \times h$ matrix

$$\left(D_{ji}^{[n]}\right) = n^{-l/2}\sqrt{L}\,\overline{U}\sqrt{S^{-1}} = n^{-l}L\left(\overline{\phi_j(x_i)}\right)S \tag{3.18}$$

is called the *decomposition matrix at torsion n*. Of course, $D^{[n]}$ depends on the choice of weights $\{\lambda_1, \dots, \lambda_r\}$. There are fairly natural choices for these—for example all the dominant weights up to a given level, or all the dominant weights of a given congruence class (see Section 4) up to a given level. With this in mind, it makes sense to compute the decomposition matrices for suitable n once and for all. In Section 4 we will show how this can be done in conjunction with the computing of the orbit sum values $\phi_j(x_i)$. The question of knowing how large n needs to be to separate the required weights is not easy. In Section 6 we give a reasonable upper bound on n. In practice, we have been experimentally determining suitable n somewhat lower than this bound.

On the basis of (3.10), (3.11) and (3.12) it appears that in principle the a_λ might be determined by minimizing trial solutions $b_\lambda^{(n)}$ for various small n. Our experience is that this is not particularly effective. Once nonseparation becomes prevalent, the $b_\lambda^{(n)}$ become badly wrong and several poor overestimates are of little use.

The development through (3.10)–(3.18) is unchanged if f in (3.9) is replaced by an element of $X_{\mathbf{C}}(K)$, except that b_λ in (3.10) is no longer necessarily an integer and (3.12) is no longer a decisive test for a correct solution. As long as one knows that $\phi_{\lambda_1}, \ldots, \phi_{\lambda_r}$ are the only orbit sums in the decomposition of f and (3.16) holds, this is not essential.

4. Bootstrapping. Up to this point we have been concentrating on a technique for decomposing class sums, given prior knowledge of the orbit sum values on suitable sets of EFO. In [18] we devoted much attention to the problem of computing character values, and although the method advocated there is indeed practical for ranks say $\leqslant 10$, it still can become fairly laborious when large numbers of EFO are involved. In the process of decomposing class functions it becomes possible to use decompositions available at any moment to compute unknown orbit sum values which in turn allow further decompositions. This leads to a bootstrap approach to computing both decomposition matrices and orbit sum values in which the orbit sums need only be evaluated by summing at the so-called *elementary* dominant weights [19]. The elementary dominant weights are the fundamental weights corresponding to the *ends* of the Coxeter-Dynkin diagram (see below). The orbit sum values for other dominant weights are computed by using various tensor and alternating products as we now explain.

Let us assume that $\{\lambda_1, \ldots, \lambda_r\} \in P^{++}$ is *ordered* and *complete with respect to level* in the sense that

$$(4.1)\qquad \begin{aligned} &\text{(i)}\quad i < j \Rightarrow \langle \lambda_i, \mathbf{l} \rangle \leqslant \langle \lambda_j, \mathbf{l} \rangle \\ &\text{(ii)}\quad \text{if } \mu \in P^{++} \text{ and } \langle \mu, \mathbf{l} \rangle < \langle \lambda_j, \mathbf{l} \rangle \\ &\qquad\ \text{for some } j \text{ then } \mu \in \{\lambda_1, \ldots, \lambda_r\}. \end{aligned}$$

In particular, such a set is consistent. We also assume that we have a set $x_j = \exp 2\pi i \mathbf{s}_j$, $j = 1, \ldots, h$, of EFO which represent the conjugacy classes of a finite W-stable Abelian group A which separates $\{\lambda_1, \ldots, \lambda_r\}$. For simplicity we will actually assume that $A = T_n$ as defined in Section 3.

Our object is to compute both the orbit sum values $\phi_{\lambda_i}(x_j)$ and the decomposition matrix $D^{[n]}$ (3.18). This is accomplished by an inductive process on the level. If the values $\phi_{\lambda_i}(x_j)$, $j = 1, \ldots, h$, are known for $\lambda_1, \ldots, \lambda_{p-1}$ and if we can write

$$(4.2)\qquad \lambda_p = \lambda_r + \lambda_s, \qquad \lambda_r, \lambda_s \neq 0$$

(where necessarily $r, s < p$), then the class sum $\phi_{\lambda_r} \cdot \phi_{\lambda_s}$ decomposes as

$$(4.3) \qquad \phi_{\lambda_r} \cdot \phi_{\lambda_s} = \phi_{\lambda_p} + \sum_{k=1}^{p-1} a_k \phi_{\lambda_k},$$

where the quantities $a_k \in \mathbb{N}$. By assumption, the values $\phi_{\lambda_k}(x_j)$, $k < p$, are known and hence so are the first $p - 1$ rows of the decomposition matrix $D^{[n]}$. That is sufficient to determine $a_1, \dots, a_{p-1}$ by direct matrix multiplication using (3.17), whence we have $\phi_{\lambda_p}(x_j)$ from (4.3).

If no decomposition (4.2) is available it is because $\lambda_p = \omega$, where ω is one of the fundamental weights (2.1). Provided that ω does not belong to one of the *ends* of the Coxeter-Dynkin diagram—that is, a node attached to only one other node—we can use the method of alternating tensors to compute the orbit sum values. There is always an A-type string of nodes from some end to the node belonging to ω—say

$$(4.4) \qquad \overset{\circ}{t_1} \text{---} \overset{\circ}{t_2} \cdots \overset{\circ}{t_{p-1}} \text{---} \overset{\circ}{t_p}$$

where $\omega = \omega_{t_p}$. Let V be the irreducible representation with highest weight ω_{t_1}. We will denote characters by the symbol χ subscripted by either the name of the representation space or the highest weight (for an irreducible representation), whichever is convenient. Then we have

$$(4.5) \qquad \chi_{(\Lambda^p V)} = \phi_\omega + \sum_{\lambda < \omega} a_\lambda \phi_\lambda$$

This is well known [10], although it is usually expressed with characters rather than orbit sums on the right-hand side of the equation.

Now the values of $\chi_{(\Lambda^p V)}$ on EFO are computable directly as long as the so-called *power maps* are available. These are the mappings which describe for each EFO the conjugacy classes to which each of its powers belong. Precisely, they are the mappings

$$(4.6) \qquad p_j : \{1, \dots, m_j\} \to \{1, \dots, h\}, \qquad j = 1, \dots, h,$$

which provide for each of the EFO x_j (with order m_j) the unique element $x_{p_j(k)}$ which is conjugate to x_j^k, $k = 1, \dots, m_j$. This type of information is not hard to obtain by methods described already in [18, Section 5].

Assuming that the power maps are in place, we have the formula [25, §12], [20]

$$(4.7) \quad \chi_{(\Lambda^p V)}(x) = \frac{1}{p!} \sum_{[d]=[d_1,\dots,d_p]} h^{[d]} \sigma([d]) \left(\chi_V(x^1)\right)^{d_1} \cdots \left(\chi_V(x^p)\right)^{d_p},$$

where $[d] = [d_1, \dots, d_p]$ runs over all partitions

$$1^{d_1} 2^{d_2} \cdots p^{d_p} \text{ of } \{1, \dots, p\};$$

$$h^{[d]} = \frac{p!}{1^{d_1} d_1! \cdots p^{d_p} d_p!}$$

is the size of the conjugacy class of the symmetric group S_p with cycle type $[d]$; and

$$\sigma : S_p \to \{\pm 1\} \text{ is the alternating character,}$$

$$\sigma([d]) = (-1)^{d_2 + d_4 + \cdots}.$$

Again the a_λ in (4.5) are obtained from the part of $D^{[n]}$ already available, and the values $\phi_\omega(x_j)$ are obtained from (4.7) and (4.5).

In this way, the pth row of $D^{[n]}$ is constructed, thus completing the induction step. Needless to say, the decompositions (4.3) and (4.5) may be used for the computing of arbitrary character values. The idea of using various classes of tensors to compute character values is not new. Indeed, it is the approach of J. Conway and L. Queen in [5]. However, their computations are very much special to E_8 and there are no bootstrapping ideas to determine their tensor decompositions.

The additional complexity involved in (4.7) has to be weighted against the direct computation of the corresponding orbit sum. For higher-rank algebras there is no question that (4.7) is more efficient, as a simple example shows. In D_8, the fundamental representation corresponding to the trivalent node involves computing $2^7 8!/2^2 6! = 1792$ cosets before any summing is begun, whereas (4.7) can be utilized with a fork node and $p = 2$ to give

$$\chi_{\Lambda^2 V}(x) = \tfrac{1}{2}\{\chi_V(x)^2 - \chi_V(x^2)\}. \tag{4.8}$$

5. Modular Arithmetic. If one examines the entire collection of algorithms involved in computing weights, weight space multiplicities, and so on [3], [4], [17], [18] one sees that it is only in the evaluation of the orbit sums and character sums that real arithmetic enters. Its presence brings more than just loss of aesthetic appeal: round-off errors become a very acute problem in the high-rank/high-dimension cases, even when only integer answers are sought. Thus, in E_8 we found in using the 63 conjugacy classes of elements of order 8 that the round-off errors on a Cyber 835 forced us to stop long before separation became a problem. Such problems are completely eliminated by using mappings of cyclotomic integers into suitable prime fields.

Let p be a prime and let n be a positive integer dividing $p - 1$. Let O_n be the ring of the nth cyclotomic field L_n. Then in the prime field $F_p = \mathbb{Z}/p\mathbb{Z}$ there is a primitive nth root ξ of 1. Since the minimum polynomial of ξ over F_p is the nth cyclotomic polynomial $\Phi_n(x)$ (reduced modulo p), there is a ring homomorphism

$$\phi\colon O_n = \mathbb{Z}[e^{2\pi i/n}] \to F_p \tag{5.1}$$

such that $\phi|_{\mathbb{Z}}$ is reduction mod p and $\phi(e^{2\pi i/n}) = \xi$. We define the "conjugate map" $\bar{\phi}\colon O_n \to F_p$ through $\bar{\phi}(e^{2\pi i/n}) = \xi^{-1}$. Thus, $\phi(\bar{z}) = \bar{\phi}(z)$ for all $z \in O_n$. Most importantly, the kernel of ϕ in $\mathbb{Z}$ is $p\mathbb{Z}$.

Now in the orbit sum and bootstrapping methods of Sections 3 and 4, we may perform all the calculations in F_p rather than in $\mathbb{C}$, provided that we choose a prime p such that the torsion n and the Weyl group order $|W|$ satisfy

$$n \mid p - 1, \quad \gcd(|W|, p) = 1. \tag{5.2}$$

The resulting modular decomposition matrix

$$\phi\left(D_{ji}^{[n]}\right) = \phi\left(n^{-1} L_j \overline{\phi_j(x_i)}\, S_i\right) \tag{5.3}$$

can be used to determine modular decompositions

$$\left(\phi b_1, \ldots, \phi b_{\lambda_r}\right)^T = \left(\phi D^{[n]}\right)\left(\phi f(x_1), \ldots, \phi f(x_h)\right)^T \tag{5.4}$$

of *integral* class sums. The decomposition $(b_1, \ldots, b_{\lambda_r})^T$ then can be recovered provided that we begin with a suitably large prime p, or we use several primes and the Chinese remainder theorem.

The idea of number theoretical transforms is not new. J. D. Dixon advocated the modular calculation of characters of finite groups in [6], and their use in convolution algorithms is well established [21]. The situation here does however seem particularly suitable for their application, since the intermediate complex quantities are very large whereas the final answers are both integral and relatively small.

As an example we have used the bootstrapping in the modular setting to compute the decompositions (4.3) and (4.5) for the first 38 (by level) weights of E_8. For this we used the elements of order $n = 8$ and the prime $2^{28} - 119$.

One should note that the orbit decomposition (4.3) and (4.5) tend to involve much larger integers than their corresponding reformulations in terms of characters. Thus it is preferable to perform the conversion back to $\mathbb{Z}$ only after such a reformulation (using the triangular system of equations (3.6)).

6. Additional Remarks. (i) We begin with an estimate on the size of n required for T_n to separate the weights of $\Lambda = W\lambda_1 \cup \cdots \cup W\lambda_r$, where $\lambda_1, \dots, \lambda_r$ are some given dominant weights. By definition we require that for each pair λ, $\mu \in \Lambda$, $\lambda \neq \mu$, there is a point X of $(\frac{1}{n})Q^\wedge$ such that $\langle \lambda - \mu, X \rangle \notin \mathbb{Z}$. Since the $\mathbb{Z}$-dual of $Q^\wedge$ is P, this is equivalent to requiring that $\lambda - \mu \notin nP$. We assume that K is *simple*.

Let $\omega_1, \dots, \omega_l$ be the fundamental basis of P and let $\alpha_1^\wedge, \dots, \alpha_l^\wedge$ be the basis $\mathbb{Z}$-dual to it in $\mathfrak{t}$ (see Section 2). Let $\lambda_j = \sum_{i=1}^{l} c_{ij}\omega_i$, $j = 1, \dots, r$, and define

$$C = \max\{ c_{ij} \mid 1 \leqslant i \leqslant l, 1 \leqslant j \leqslant r \},$$

$$(6.1) \qquad M = \max\left\{ \sum_{i=1}^{l} n_i^\wedge c_{ij} \mid 1 \leqslant j \leqslant r \right\},$$

where $n_1^\wedge, \dots, n_l^\wedge$ are the numerical marks of the "dual" group $K^\wedge$ of K (see Section 2).

PROPOSITION 3 (K simple). *If $n > C + M$ then T_n separates Λ.*

Proof. Take $n > C + M$. Let $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$. We show that $\lambda - \mu \notin nP$. By the W-invariance of P and nP we may assume that μ is dominant—say $\mu = \lambda_k$. Let $\lambda = w\lambda_j$. The coefficient of ω_i in $w\lambda_j$ is

$$\left\langle w\lambda_j, \alpha_i^\wedge \right\rangle = \left\langle \lambda_j, w^{-1}\alpha_i^\wedge \right\rangle.$$

Now $w^{-1}\alpha_i^\wedge$ is a coroot (that is a root of the root system $\Delta^\wedge := W\{\alpha_1^\wedge, \dots, \alpha_l^\wedge\}$ of $K^\wedge$). Thus $w^{-1}\alpha_i^\wedge = \sum_{k=1}^{l} d_k \alpha_k^\wedge$ with $|d_k| \leqslant n_k^\wedge$, $j = 1, \dots, l$, and $|\langle \lambda_j, w^{-1}\alpha_i^\wedge \rangle| \leqslant \sum n_k^\wedge c_{kj} \leqslant M$. Thus the coefficient of ω_i in $w\lambda_j - \lambda_k = \lambda - \mu$ is bounded in absolute value by $M + C$. This proves the result. □

Formulas for the number $h(n)$ of conjugacy classes of EFO in T_n are known. The semisimple case follows directly from the simple case, and for K simple, generating functions for $h(n)$ appear in [18]. Explicit formulas are given by Djoković in [7]. When n and $|W|$ are relatively prime (or even under slightly more relaxed conditions [8]), Djoković has given the elegant formula

$$(6.2) \qquad h(n) = \prod_{i=1}^{l} \left(\frac{m_i + n}{m_i + 1} \right),$$

where $m_1, \dots, m_l$ are the exponents of K.

(ii) The *congruence classes* of weights are the Q-cosets of P. Any Weyl group orbit and any weight system of an irreducible representation of K lies entirely in such one congruence class. Thus we may speak of the congruence class of an orbit sum or an irreducible character. The center Z of K may be identified with the character group $X(P/Q)$ of P/Q by

$$\theta \in X(P/Q) \leftrightarrow z^\theta \in Z \tag{6.3}$$

with z^θ acting on weight spaces of weight $\lambda \in P$ by

$$z^\theta|_{V^\lambda} = \theta(\lambda + Q).$$

Let $\mathbf{z}^\theta = [z_0^\theta, \ldots, z_l^\theta]$ be the corresponding point in $\mathbf{F}$ (actually the $\mathbf{z}^\theta$ are the vertices of $\mathbf{F}$ [18]) so that

$$z^\theta|_{V^\lambda} = e^{2\pi i\langle\lambda, \mathbf{z}^\theta\rangle}. \tag{6.4}$$

Now if $\lambda, \mu \in P$, then

$$\sum_{\theta \in X(P/Q)} e^{2\pi i\langle\lambda-\mu, \mathbf{z}^\theta\rangle} = |Z|\delta_{\bar\lambda,\bar\mu}, \tag{6.5}$$

where $\bar\lambda = \lambda + Q$, $\bar\mu = \mu + Q$. Thus we see that

$$\langle\phi_\lambda, \phi_\mu\rangle_Z = |Z|\,|W\lambda|\,|W\mu|\delta_{\bar\lambda,-\bar\mu}, \tag{6.6}$$

so that Z, and hence any group $A \subset T^n$ which contains Z, can separate the congrunce classes.

Let $\{\bar\gamma_1, \ldots, \bar\gamma_{|Z|}\}$ be the elements of P/Q and let $\gamma_1, \ldots, \gamma_{|Z|}$ be some arbitrary representatives of the $\bar\gamma_i$ in P. Then for any class function $f = \sum a_\lambda e^{2\pi i\lambda}$ on T we have the decomposition

$$f = \sum_{k=1}^{|Z|} f_k, \quad \text{where } f_k = \sum_{\lambda \in \bar\gamma_k} a_\lambda e^{2\pi i\lambda},$$

and from (6.5) we have

$$f_k(x) = \frac{1}{|Z|} \sum_{\theta \in X(P/Q)} e^{-2\pi i\langle\gamma_k, \mathbf{z}^\theta\rangle} f(z^\theta x). \tag{6.7}$$

In many problems, for instance those involving tensor products, it is possible to keep the congruence classes separate and there is no need to involve the center. However, in problems like group-subgroup reductions, mixing of classes is unavoidable and (6.7) provides a useful way to separate them.

(iii) As we have suggested above, the use of orbit sums has several advantages over the direct use of characters. We briefly consider here the situation with characters. Consider first the problem of numerical integration of a class function f on G. Assuming Haar integrals $\int_G$ and $\int_T$ on G and T, both normalized so that $\int_G 1 = 1 = \int_T 1$, then there is a well-known formula of H. Weyl,

$$\int_G f = |W|^{-1} \int_T f d\bar d,$$

where $d = \prod_{\alpha \in \Delta^+} (e^{\pi i\alpha} - e^{-\pi i\alpha})$ is the discriminant function on T [1, Chapter 6]. If $f = \sum_{\lambda \in P} a_\lambda e^{2\pi i\lambda}$, $a_\lambda \in \mathbf{C}$, then

$$\int_T f = \int_{\mathfrak{t}/Q^\wedge} \sum_\lambda a_\lambda e^{2\pi i\langle\lambda, X\rangle} = a_0.$$

On the other hand,

$$\sum_{X \in \frac{1}{n}Q^\wedge/Q^\wedge} e^{2\pi i\langle\lambda, X\rangle} = \begin{cases} n^l & \text{if } \langle\lambda, Q^\wedge\rangle \subset n\mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\int_G f = |W|^{-1}\int_T f d\bar{d} = |W|^{-1}n^{-l} \sum_{X \in \frac{1}{n}Q^\wedge/Q^\wedge} (fd\bar{d})(x), \tag{6.8}$$

where $x = \exp 2\pi i X$, provided that the weights appearing in $fd\bar{d}$ are separated from $\{0\}$. In view of the orthogonality relations for characters [1, Chapter 3], we have for irreducible characters χ and χ' of K

$$|W|^{-1}n^{-l} \sum_{x \in \frac{1}{n}Q^\wedge/Q^\wedge} \left(\chi d\left(\overline{\chi' d}\right)\right)(x) = \delta_{\chi\chi'} \tag{6.9}$$

provided that T_n separates the weights appearing in the function $\chi\chi' d\bar{d}$. This certainly allows numerical decompositions of character sums in principle. However, separating the weights of $d\bar{d}$ is an additional expense and, unlike the case of the orbit sums, we do not have well-defined useful information (like (3.11)) in the failure of separation.

(iv) We have always assumed that an advance knowledge of the weights which may occur in the character decomposition of a given $f \in X(K)$ is at our disposal. A few examples of this might be helpful. Consider the problem of decomposing the tensor product (1.2) or, what amounts to the same thing, decomposing the product (1.3) of $\chi_{V_1}\chi_{V_2} \in X^+(K)$. If the highest weight of V_i is λ_i, $i = 1, 2, \ldots, r$, then $\lambda_i \leqslant \lambda_1 + \lambda_2$, $i = 3, \ldots, r$. In fact, it can be shown that $\lambda_3, \ldots, \lambda_r$ lie in the set Ω of dominant weights of the irreducible module of highest weight $\lambda_1 + \lambda_2$ (Ω is complete). Although it is irrelevant to the orbit sum method, it is interesting to note that there is a "least" λ_i, say λ_3, with $\lambda_3 \leqslant \lambda_i$ for $i = 3, \ldots, r$. This λ_3 is the dominant weight in the W-orbit of $\lambda_1 - \lambda_2$ [24].

(v) Consider the problem of subgroup reduction. Here we have $K \subset \tilde{K}$ and the task is to perform the character decomposition or *branching* of a function $f \in X^+(\tilde{K})$ when it is restricted to K. If T and $\tilde{T}$ are maximal tori of K and $\tilde{K}$, respectively, then we may always assume that $T \subset \tilde{T}$. Then, under restriction $\tilde{\lambda} \mapsto \lambda$, weight systems of $\tilde{K}$-modules project onto weight systems of K-modules. It is always possible to choose total orderings $\preccurlyeq$ and $\tilde{\preccurlyeq}$ on the weight lattices P and $\tilde{P}$ (relative to T and $\tilde{T}$) such that for all $\tilde{\lambda}$, $\tilde{\mu} \in \tilde{P}$, $\lambda \prec \mu$ implies $\tilde{\lambda} \mathrel{\tilde{\prec}} \tilde{\mu}$ [9, Chapter 1]. Use these to order the corresponding root systems Δ and $\tilde{\Delta}$, and let Π and $\tilde{\Pi}$ be the corresponding bases. Then for

$$\tilde{\alpha} \in \tilde{\Delta}^+, \qquad 0 \mathrel{\tilde{\prec}} \tilde{\alpha} \Rightarrow 0 \prec \alpha.$$

If $\tilde{\mu}$ is a dominant weight in $\tilde{P}$ then the $\tilde{W}$-orbit $\tilde{W}\tilde{\mu}$ lies in $\{\tilde{\mu} - \sum c_{\tilde{\alpha}}\tilde{\alpha} \mid \tilde{\alpha} \in \tilde{\Pi}, c_{\tilde{\alpha}} \in \mathbb{N}\}$, and hence for all $\tilde{\lambda} \in \tilde{W}\tilde{\mu}$, $\lambda \preccurlyeq \mu$. Thus we have strict control over the weights appearing in the reduction of orbit sums.

7. E_6 Example. The Tables 1 and 2 below contain the dominant weight matrices (m_ν^λ) for the E_6-congruence classes 0 and 1. The matrix for class 2 can be obtained directly from that of class 1 by permuting the labels according to the E_6 diagram symmetry. These tables are examples of the form of the main set of tables of [4]. In addition to the dominant weight multiplicities, Tables 1 and 2 provide the following auxiliary information about the representations.

S.P.: Scalar product (λ, λ) of the dominant weight λ, so normalized that $(\alpha, \alpha) = 2\det(\text{Cartan matrix})$ for short roots α.

O.S.: Orbit size, the number of weights on the Weyl group orbit of λ.

LEVEL: The number of levels of the weight system $\Omega(\lambda)$.

DIM'N: The dimension of the representations with the highest weight λ.

WEIGHT: The weight λ given by its coordinates in terms of the fundamental weights, arranged into a Dynkin diagram.

NUMBER: Numbering of dominant weights of the class (this has no canonical meaning).

The Tables 3–6 below are examples of E_6 tensor product decompositions which were obtained by the method outlined in Section 3. We took advantage of the congruence classes and computed one decomposition matrix D for each class. For E_6 we have the following values for $h(n)$ = number of conjugacy classes of EFO in T_n:

n	2	3	4	5	6	7	8	9	10	11	12
$h(n)$	3	8	14	26	49	77	124	197	287	418	603

For the purposes of the example we chose the 77 conjugacy classes of EFO in T_7 and constructed the corresponding decomposition matrices, referring to the ordered list of E_6 dominant weights (cf. Tables 1 and 2). The elements of T_7 separate all the weights in at least the first 30 weights of each congruence class. For instance, for class 0 the matrix UU^T (3.16) is $I_{52\times 52} + E_{15,34} + E_{34,15} + E_{32,32} + E_{37,37} + E_{38,38} + E_{42,42} + E_{43,43} + E_{51,51}$, where $I_{52\times 52}$ is the 52×52 identity matrix and E_{ij} is the (i, j) matrix unit with 1 in the (i, j)th position and 0's elsewhere. Thus one sees for example that the orbits of weight #15 and #34 are 'aliased' by T_7. It is worthwhile noting however that the elements of order 7 do better than one might anticipate from Proposition 3. There, for class 0 and the first 30 representations, C and M of Section 6(i) are 5 and 6, respectively, with corresponding value of 12 for n in Proposition 3.

After removing the obvious symmetries, there are four types of tensor products that one needs to consider:

class 0 ⊗ class 0 → class 0 (Table 3)
class 0 ⊗ class 1 → class 1 (Table 4)
class 1 ⊗ class 1 → class 2 (Table 5)
class 1 ⊗ class 2 → class 0 (Table 6)

The resulting tables were obtained by using the decomposition matrices on all the mutual products of the first 10 nontrivial representations of each class and retaining those which passed the (definitive) test (3.12). We have denoted the irreducible representations by number-letter pairs where the number refers to the numbering of the highest weight (cf. Tables 1 and 2) and the letter to the class by the convention $A \leftrightarrow$ class 0, $B \leftrightarrow$ class 1, C $\leftrightarrow$ class 2. Thus a tensor product of representations 2B and 3C is denoted by (2B, 3C).

There are other E_6 tensor product decomposition tables in the literature [11], [26], [28], calculated by entirely different methods. We have made no attempt to compare these methods with the one here.

TABLE 1

ALGEBRA E6, REPRESENTATIONS OF CLASS 0

NUMBER	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
S.P.	0	6	12	18	24	24	24	36	36	30	36	42	42	42	48	54	48	48	60	60	54	60	60	60	66	66
O.S.	1	72	270	720	72	432	432	27	27	2 160	2 160	1 080	1 080	1 440	270	72	2 160	2 160	432	432	6 480	1 080	1 080	2 160	2 160	2 160
LEVEL	1	23	33	43	45	47	47	49	49	55	61	63	63	65	65	67	69	69	71	71	75	77	77	77	79	79
DIM'N	1	78	650	2 925	2 430	5 824	5 824	3 003	3 003	34 749	70 070	78 975	78 975	105 600	85 293	43 758	252 252	252 252	146 432	146 432	852 930	600 600	600 600	812 175	972 972	972 972
WEIGHT	0 0 0 0 0 0	0 0 1 0 0 0	1 0 0 0 0 1	0 0 0 1 0 0	0 0 2 0 0 0	1 1 0 0 0 0	0 0 0 0 1 1	3 0 0 0 0 0	0 0 0 0 0 3	1 0 1 0 0 1	0 1 0 0 1 0	0 1 0 0 0 2	2 0 0 0 1 0	0 0 1 1 0 0	2 0 0 0 0 2	0 0 3 0 0 0	0 0 1 0 1 1	1 1 1 0 0 0	3 0 1 0 0 0	0 0 1 0 0 3	1 0 0 1 0 1	1 0 0 0 2 0	0 2 0 0 0 1	1 0 2 0 0 1	2 1 0 0 0 1	1 0 0 0 1 2
0	1	6	20	45	36	64	64	24	24	261	374	360	360	480	351	180	900	900	416	416	2 376	1 470	1 470	2 085	2 124	2 124
1		1	5	15	11	24	24	10	10	106	169	170	170	220	165	80	445	445	220	220	1 236	790	790	1 085	1 165	1 165
2			1	4	3	8	8	4	4	40	72	76	76	96	76	35	211	211	112	112	622	412	412	552	623	623
3				1	1	2	2	1	1	14	28	31	31	42	33	17	95	95	52	52	303	205	205	278	320	320
4					1	.	.	.	.	5	9	10	10	20	15	11	40	40	20	20	141	95	95	145	155	155
5						1	.	1	.	4	10	10	15	15	14	6	36	45	31	20	140	100	100	130	174	151
6							1	.	1	4	10	15	10	15	14	6	45	36	20	31	140	100	100	130	151	174
7								1	.	.	.	.	5	.	4	.	.	10	16	.	25	20	20	20	64	26
8									1	.	.	5	.	.	4	.	10	.	.	16	25	20	20	20	26	64
9										1	3	4	4	6	5	3	16	16	10	10	62	45	45	62	76	76
10											1	1	1	2	1	1	6	6	4	4	25	20	20	27	33	33
11												1	.	.	1	.	3	.	.	4	10	9	7	9	11	24
12													1	.	1	.	.	3	4	.	10	7	9	9	24	11
13														1	.	1	2	2	1	1	9	7	7	14	12	12
14															1	.	.	.	.	.	4	3	3	3	8	8
15																1	.	.	.	.	.	.	.	5	.	.

NUMBER	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51
S.P.	84	84	66	72	72	72	78	78	96	78	78	84	84	90	90	90	90	96	96	96	96	84	96	96	108	90
O.S.	270	270	6 480	720	4 320	4 320	1 440	2 160	72	4 320	4 320	2 160	2 160	216	2 160	2 160	216	432	432	432	432	8 640	2 160	2 160	270	12 960
LEVEL	81	81	83	85	85	85	87	87	89	89	89	91	91	91	91	91	91	93	93	93	93	93	95	95	97	97
DIM'N	371 800	371 800	2 453 814	1 337 050	2 977 975	2 977 975	1 911 195	3 490 695	537 966	4 752 384	4 752 384	5 054 400	5 054 400	1 559 376	3 309 696	3 309 696	1 559 376	3 162 159	3 162 159	3 007 368	3 007 368	12 514 788	7 779 200	7 779 200	4 548 180	23 795 200
WEIGHT	1 0 0 0 0 4	4 0 0 0 0 1	0 1 1 0 1 0	0 0 0 2 0 0	0 1 1 0 0 2	2 0 1 0 1 0	0 0 2 1 0 0	2 0 1 0 0 2	0 0 4 0 0 0	1 1 0 1 0 0	0 0 0 1 1 1	0 0 2 0 1 1	1 1 2 0 0 0	0 3 0 0 0 0	0 0 0 1 0 3	3 0 0 1 0 0	0 0 0 0 3 0	3 0 2 0 0 0	0 0 2 0 0 3	0 0 0 0 2 2	2 2 0 0 0 0	1 1 0 0 1 1	3 0 0 0 1 1	1 1 0 0 0 3	3 0 0 0 0 3	1 0 1 1 0 1
0	640	640	4 914	2 440	5 320	5 320	3 249	5 679	810	7 296	7 296	7 200	7 200	2 052	4 320	4 320	2 052	3 816	3 816	3 600	3 600	16 596	8 960	8 960	4 656	28 000
1	370	370	2 724	1 365	3 020	3 020	1 819	3 254	446	4 256	4 256	4 220	4 220	1 235	2 610	2 610	1 235	2 326	2 326	2 195	2 195	9 925	5 480	5 480	2 871	16 960
2	210	210	1 472	750	1 676	1 676	1 004	1 836	248	2 432	2 432	2 432	2 432	728	1 544	1 544	728	1 392	1 392	1 318	1 318	5 832	3 304	3 304	1 761	10 128
3	112	112	777	408	904	904	559	1 018	145	1 358	1 358	1 382	1 382	418	885	885	418	808	808	772	772	3 352	1 950	1 950	1 067	5 978
4	55	55	399	225	470	470	324	559	96	736	736	780	780	230	480	480	230	450	450	439	439	1 870	1 120	1 120	646	3 500
5	55	70	394	210	455	500	290	550	76	761	720	740	785	240	474	540	234	501	439	429	468	1 896	1 180	1 105	640	3 440
6	70	55	394	210	500	455	290	550	76	720	761	785	740	234	540	474	240	439	501	468	429	1 896	1 105	1 180	640	3 440
7	10	40	80	45	85	160	60	160	16	240	160	160	240	80	104	240	64	216	95	110	200	576	480	320	240	1 040
8	40	10	80	45	160	85	60	160	16	160	240	240	160	64	240	104	80	95	216	200	110	576	320	480	240	1 040
9	30	30	196	109	242	242	158	289	45	392	392	418	418	129	274	274	129	260	260	249	249	1 038	644	644	374	1 964
10	14	14	95	54	119	119	81	142	24	202	202	224	224	69	145	145	69	144	144	134	134	554	352	352	209	1 092
11	14	4	38	23	68	45	33	76	10	84	116	120	92	31	108	58	42	58	98	98	59	296	176	236	128	576
12	4	14	38	23	45	68	33	76	10	116	84	92	120	42	58	108	31	98	58	59	98	296	236	176	128	576
13	5	5	45	27	55	55	48	69	17	98	98	123	123	34	69	69	34	75	75	66	66	278	182	182	116	607
14	4	4	14	12	24	24	15	40	6	48	48	48	48	18	40	40	18	36	36	44	44	152	112	112	76	304
15	.	.	9	5	10	10	20	15	11	16	16	40	40	5	10	10	5	20	20	9	9	55	40	40	35	180

1	.	.	1	3	4	2	4	3	8	5	1	18	11	28	22	18	32	6	40	55	62	52	14	48	28	22	34	47	47	27	144	88	112	64	310	16
	1	1	.	3	2	4	4	8	3	1	5	18	11	22	28	18	32	6	55	40	52	62	22	28	48	14	47	34	27	47	144	112	88	64	310	17
		1	.	.	.	.	.	4	.	.	5	.	.	.	10	.	10	.	15	.	.	15	6	.	30	.	26	.	.	25	36	60	20	30	65	18
			1	.	.	.	.	.	4	5	.	.	.	10	.	.	10	.	.	15	15	.	.	30	.	6	.	26	25	.	36	20	60	30	65	19
				1	1	1	1	2	2	1	1	7	5	10	10	8	14	3	22	22	26	26	9	18	18	9	19	19	19	19	70	50	50	33	158	20
					1	.	.	.	1	1	.	3	2	3	4	3	4	1	8	12	12	12	3	10	6	7	10	9	13	6	32	20	24	14	76	21
						1	.	1	.	.	1	3	2	4	3	3	4	1	12	8	12	12	7	6	10	3	9	10	6	13	32	24	20	14	76	22
							1	.	.	.	.	3	2	4	4	6	5	3	8	8	16	16	3	6	6	3	10	10	6	6	27	20	20	15	84	23
								1	.	.	1	.	.	.	3	.	4	.	6	.	.	6	3	.	10	.	9	.	.	10	16	24	10	14	32	24
									1	1	.	.	.	3	.	.	4	.	.	6	6	.	.	10	.	3	.	9	10	.	16	10	24	14	32	25
										1	.	.	.	.	.	.	.	.	.	.	.	.	.	4	.	.	.	3	3	.	.	.	8	4	.	26
											1	.	.	.	.	.	.	.	.	.	.	.	.	.	4	.	3	.	.	3	.	8	.	4	.	27
												1	1	1	1	2	1	1	4	4	6	6	2	3	3	2	4	4	4	4	12	8	8	5	38	28
													1	.	.	1	.	1	2	2	2	2	1	1	1	1	1	1	3	3	4	2	2	1	18	29
														1	.	.	1	.	.	2	3	.	.	3	.	1	.	4	3	.	6	4	8	5	15	30
															1	.	1	.	2	.	.	3	1	.	3	.	4	.	.	3	6	8	4	5	15	31
																1	.	1	.	.	2	2	.	.	.	.	1	1	.	.	.	.	.	.	9	32
																	1	.	.	.	.	.	.	.	.	.	.	.	.	3	4	4	5	6	33
																		1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	34
																			1	.	.	1	1	.	1	.	1	.	.	2	2	2	1	1	6	35
																				1	1	.	.	1	.	1	.	1	2	.	2	1	2	1	6	36
																					1	.	.	.	.	.	.	1	.	.	.	.	.	.	3	37
																						1	.	.	.	.	1	.	.	.	.	.	.	.	3	38
																							1	.	.	.	.	.	.	1	.	.	.	.	.	39
																								1	.	.	.	1	1	.	.	.	2	1	.	40
																									1	.	1	.	.	1	.	2	.	1	.	41
																										1	.	.	1	.	.	.	.	.	.	42
																											1	.	.	.	.	.	.	.	.	43
																												1	.	.	.	.	.	.	.	44
																													1	.	.	.	.	.	.	45
																														1	.	.	.	.	.	46
																															1	1	1	1	2	47
																																1	.	1	.	48
																																	1	1	.	49
																																		1	.	50
																																			1	51

TABLE 2

ALGEBRA E6, REPRESENTATIONS OF CLASS 1

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
S.P.	4	10	16	16	22	28	28	34	34	40	40	40	46	64	52	46	52	52	58	58	64	64	58	76	70	64
O.S.	27	216	27	432	1 080	270	1 080	432	2 160	432	216	2 160	432	27	270	4 320	2 160	2 160	1 080	2 160	432	432	4 320	432	1 080	2 160
LEVEL	17	31	33	39	47	49	53	55	59	61	61	63	63	65	65	69	71	73	75	75	77	77	77	79	79	79
DIM'N	27	351	351	1 728	7 371	7 722	17 550	19 305	51 975	46 332	34 398	112 320	54 054	19 305	61 425	314 496	359 424	386 100	393 822	494 208	412 776	459 459	967 680	359 424	579 150	1 123 200
	1	0	0	1	0	2	0	0	1	1	0	1	2	4	1	0	2	0	0	0	0	0	1	0	3	1
	0	0	0	0	1	0	0	0	0	0	2	0	1	0	0	1	0	0	0	0	0	0	1	0	0	1
	0 0	0 0	0 0	1 0	0 0	0 0	1 0	1 0	0 1	2 0	0 0	0 0	0 0	0 0	0 0	1 0	1 0	0 1	2 0	0 1	0 0	2 0	0 0	0 0	0 0	0 0
	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	1	1	0	2	0	1	1	1	0
WEIGHT	0	0	2	0	1	1	0	2	0	0	0	1	0	0	3	1	1	0	0	2	1	2	0	3	0	2
NUMBER	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
0	1	5	4	16	44	40	80	75	180	146	104	320	144	40	145	720	736	750	704	856	646	731	1 600	480	820	1 680
1		1	1	4	15	14	30	31	75	61	46	144	66	20	70	344	360	375	355	446	344	386	836	268	445	900
2			1	.	5	4	10	16	25	20	20	64	26	10	40	160	160	180	170	240	184	216	416	160	225	480
3				1	4	5	10	10	30	26	19	60	30	10	30	155	175	180	174	216	175	190	425	140	240	465
4					1	1	3	4	10	9	7	24	11	4	14	68	76	83	83	108	88	98	204	76	118	236
5						1	.	.	4	3	3	8	8	4	4	24	40	33	32	40	38	36	104	32	76	112
6							1	1	3	4	2	8	3	1	5	28	32	37	41	48	41	47	92	36	54	112
7								1	.	.	.	4	.	.	5	10	10	15	15	30	25	26	36	30	20	60
8									1	1	1	2	2	1	1	10	14	14	16	18	16	19	44	14	31	50
9										1	.	.	.	.	.	4	5	5	10	6	5	10	15	4	10	20
10											1	.	1	1	.	4	4	5	6	6	5	10	20	4	15	20
11												1	.	.	1	3	4	6	6	10	10	9	16	11	10	24
12													1	1	.	.	4	.	.	.	.	.	10	.	15	10
13														1	.	.	.	.	.	.	.	.	.	.	5	.
14															1	.	.	.	.	4	3	3	.	8	.	8
15																1	1	2	3	3	3	4	6	3	4	8

	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51
S.P.	76	100	64	70	76	70	76	82	76	82	82	82	88	88	88	88	88	94	106	112	94	100	100	94	106	100
O.S.	270	27	4 320	1 080	432	8 640	2 160	2 160	6 480	6 480	2 160	4 320	2 160	2 160	4 320	1 080	4 320	2 160	1 080	270	4 320	1 080	2 160	12 960	432	8 640
LEVEL	81	81	81	83	83	85	85	87	89	91	91	91	93	93	93	93	95	95	95	97	97	97	99	99	99	101
DIM'N	638 820	100 386	1 640 925	1 253 070	741 312	4 200 768	2 088 450	2 559 843	5 776 056	7 601 958	4 582 656	6 243 237	7 528 950	6 747 300	7 757 100	5 553 900	10 378 368	9 189 180	3 281 850	3 675 672	14 017 536	5 501 925	13 478 400	30 115 800	6 675 669	37 459 422
	3	0	1	0	1	1	2	1	0	2	0	0	2	1	0	2	0	2	0	2	0	0	0	1	0	1
	0	0	0	2	0	0	1	0	1	0	1	1	0	2	1	0	0	0	1	0	0	0	0	1	0	1
	0 0	0 0	1 1	1 0	3 0	1 0	1 0	1 0	0 1	0 1	0 0	2 0	2 0	0 0	0 0	0 0	1 1	0 0	0 0	0 0	1 1	3 0	1 0	1 0	3 0	1 0
	0	0	0	0	0	1	0	0	0	0	2	0	0	0	1	2	1	1	0	0	0	1	2	1	0	0
WEIGHT	2	5	0	0	0	1	0	3	1	1	0	1	1	1	2	0	0	2	4	4	2	0	1	0	2	2
NUMBER	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51
0	830	105	2 380	1 640	960	5 248	2 480	2 769	6 306	7 596	4 576	6 340	7 000	6 180	7 090	5 090	9 264	7 760	2 525	2 636	11 456	4 445	10 160	23 690	4 932	27 204
1	455	62	1 270	904	516	2 920	1 395	1 601	3 615	4 425	2 676	3 663	4 110	3 675	4 230	3 025	5 500	4 680	1 565	1 644	6 956	2 660	6 260	14 440	3 030	16 839
2	240	40	635	484	256	1 600	740	960	2 040	2 485	1 536	2 083	2 320	2 140	2 540	1 760	3 200	2 810	1 000	1 060	4 256	1 560	3 840	8 600	1 891	10 345
3	250	35	670	485	281	1 580	780	884	2 021	2 550	1 530	2 074	2 405	2 150	2 450	1 770	3 214	2 770	935	996	4 106	1 580	3 780	8 695	1 812	10 254
4	128	20	334	252	144	840	407	500	1 111	1 414	858	1 161	1 352	1 223	1 420	1 006	1 848	1 622	569	613	2 440	928	2 268	5 114	1 102	6 184
5	76	10	168	131	72	416	248	240	578	820	464	604	788	715	751	588	1 008	920	306	344	1 312	508	1 280	3 028	596	3 608
6	64	10	168	126	80	432	206	263	593	768	466	645	762	674	787	555	1 056	921	324	359	1 404	556	1 328	2 965	657	3 653
7	30	10	65	54	30	224	85	175	310	386	256	330	380	355	495	295	570	550	240	250	860	295	820	1 615	396	2 180
8	33	4	80	65	38	208	116	126	304	423	245	337	423	381	409	312	570	507	171	199	762	306	738	1 715	368	2 100
9	15	1	40	30	26	100	50	60	151	210	120	194	235	190	205	155	324	270	85	110	426	200	410	950	230	1 194
10	14	1	35	35	16	96	60	60	155	215	124	179	220	210	205	170	304	266	85	110	420	170	400	970	219	1 185
11	14	4	32	26	16	104	46	76	154	206	132	170	212	191	246	160	309	288	118	128	449	167	448	921	214	1 218
12	14	.	15	15	6	36	45	20	65	140	54	69	130	130	85	100	130	151	35	55	170	70	180	550	85	645
13	4	.	.	.	.	.	10	.	.	25	.	.	20	20	.	20	.	26	.	10	.	.	.	80	.	85
14	4	4	.	.	.	24	.	40	33	40	32	32	36	38	104	36	72	112	76	76	168	37	168	205	72	416
15	5	1	15	13	9	46	22	32	74	101	64	92	113	97	115	81	167	143	54	64	237	100	237	511	126	666

16	1	.	.	.	.	.	3	.	4	4	5	.	6	6	3	16	16	10	32	62	26	36	62	61	46	45	71	76	20	30	98	42	103	283	52	356
17		1	1	1	2	1	2	2	1	2	1	1	6	5	4	20	9	14	34	45	34	43	56	45	60	40	88	69	31	33	124	53	136	270	66	355
18			1	.	.	1	.	.	.	.	.	.	3	2	4	8	3	5	13	18	12	28	32	17	21	16	48	30	10	15	64	41	64	140	47	184
19				1	1	1	.	2	.	2	1	1	.	.	.	10	.	14	14	18	16	16	19	16	44	19	38	50	31	33	80	21	86	113	38	208
20					1	.	.	1	.	.	.	1	.	.	.	4	.	4	5	6	10	6	10	5	20	10	20	20	15	14	35	10	50	60	16	96
21						1	.	.	.	.	.	.	.	.	.	4	.	5	5	6	6	10	10	5	15	10	20	20	10	15	40	15	40	60	26	100
22							1	.	1	1	1	.	2	3	1	6	6	4	15	25	13	18	27	29	22	20	37	33	10	14	53	23	56	146	31	185
23								1	.	.	.	1	.	.	.	.	.	4	.	.	.	.	.	.	10	.	.	10	15	14	15	.	15	.	6	36
24									1	.	1	.	.	.	.	.	3	.	.	10	.	.	9	9	.	7	.	11	.	4	.	.	.	38	.	45
25										1	1	.	.	.	.	3	.	4	6	10	6	6	9	10	16	9	17	24	10	14	32	10	36	59	16	104
26											1	.	.	.	.	.	.	.	.	4	.	.	3	3	.	3	.	8	.	4	.	.	.	14	.	24
27												1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	5	4	.	.	.	.	.	.	.
28													1	1	1	2	2	1	6	9	5	10	14	11	9	7	20	12	4	5	27	16	29	72	19	92
29														1	.	.	1	.	3	3	2	4	4	8	3	2	8	3	1	1	12	6	12	36	10	44
30															1	.	.	.	.	.	.	4	5	.	.	.	5	.	.	.	6	10	5	15	10	20
31																1	.	1	2	3	3	3	4	3	6	4	9	8	4	5	15	6	19	30	9	46
32																	1	.	.	3	.	.	4	4	.	2	.	3	.	1	.	.	.	18	.	22
33																		1	.	.	.	.	.	.	3	.	.	4	4	5	6	.	6	.	3	16
34																			1	1	1	1	1	2	2	1	4	2	1	1	6	3	8	14	4	20
35																				1	.	.	1	1	.	1	.	2	.	1	.	.	.	7	.	10
36																					1	.	.	.	1	1	2	1	1	1	2	1	6	6	1	6
37																						1	1	.	.	.	2	.	.	.	3	3	3	6	4	8
38																							1	.	.	.	.	.	.	.	.	.	.	3	.	4
39																								1	.	.	.	.	.	.	.	.	.	3	.	4
40																									1	.	.	1	1	1	2	.	3	.	1	6
41																										1	.	1	.	1	.	.	.	3	.	3
42																											1	.	.	.	1	1	2	2	1	2
43																												1	.	1	.	.	.	.	.	3
44																													1	1	.	.	.	.	.	.
45																														1	.	.	.	.	.	.
46																															1	.	1	.	1	2
47																																1	.	.	1	.
48																																	1	.	.	.
49																																		1	.	1
50																																			1	.
51																																				1

TABLE 3

E6: TENSOR PRODUCT DECOMPOSITIONS INTO IRREDUCIBLE CHARACTERS   CLASS 0 X CLASS 0 = CLASS 0

	0A	1A	2A	3A	4A	5A	6A	7A	8A	9A	10A	11A	12A	13A	14A	15A	16A	17A	18A	19A	20A	21A	22A	23A	24A	25A	26A	27A	28A	29A	30A	31A	32A	33A
(1A, 1A)	1	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1A, 2A)	.	1	2	1	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1A, 3A)	.	1	1	1	1	1	1	.	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1A, 4A)	.	1	.	1	1	.	.	.	.	1	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1A, 5A)	.	.	1	1	.	2	.	1	.	1	1	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1A, 6A)	.	.	1	1	.	.	2	.	1	1	1	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1A, 7A)	.	.	.	.	.	1	.	1	.	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1A, 8A)	.	.	.	.	.	.	1	.	1	.	.	1	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1A, 9A)	.	.	1	1	1	1	1	.	.	3	1	1	1	1	1	.	1	1	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.
(1A,10A)	.	.	.	1	.	1	1	.	.	1	2	1	1	1	.	.	1	1	.	.	1	1	1	.	.	.	.	.	1	.	.	.	.	.
(2A, 2A)	1	2	3	2	1	2	2	1	1	2	1	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(2A, 3A)	.	1	2	3	1	2	2	.	.	3	2	1	1	1	.	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
(2A, 4A)	.	.	1	1	1	1	1	.	.	2	1	.	.	1	.	.	1	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
(2A, 5A)	.	1	2	2	1	3	1	1	.	3	2	1	2	1	1	.	.	2	1	.	1	.	1	.	1	.	.	.	.	.	.	.	.	.
(2A, 6A)	.	1	2	2	1	1	3	.	1	3	2	2	1	1	1	.	2	.	.	1	1	1	.	.	.	1	.	.	.	.	.	.	.	.
(2A, 7A)	.	.	1	.	.	1	.	1	.	1	.	.	1	.	1	.	.	1	1	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.
(2A, 8A)	.	.	1	.	.	.	1	.	1	1	.	1	.	.	1	.	1	.	.	1	.	.	.	.	.	1	1	.	.	.	.	.	.	.
(2A, 9A)	.	1	2	3	2	3	3	1	1	6	4	3	3	3	2	1	3	3	1	1	3	1	1	2	1	1	.	.	1	.	1	1	.	1
(3A, 3A)	1	1	3	2	2	2	2	1	1	4	3	1	1	2	1	1	2	2	.	.	1	1	1	1	.	.	.	.	1	1	.	.	.	.

TABLE 4

E6: TENSOR PRODUCT DECOMPOSITIONS INTO IRREDUCIBLE CHARACTERS CLASS 0 X CLASS 1 = CLASS 1

	0B	1B	2B	3B	4B	5B	6B	7B	8B	9B	10B	11B	12B	13B	14B	15B	16B	17B	18B	19B	20B	21B	22B
(1A, 0B)	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1A, 1B)	1	1	1	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1A, 2B)	.	1	1	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1A, 3B)	1	1	.	2	1	1	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.
(1A, 4B)	.	1	1	1	2	1	1	1	1	.	1	1	.	.	.	1	.	.	.	.	.	.	.
(1A, 5B)	.	.	.	1	1	2	.	.	1	.	.	1	1	.	.	.	1	.	.	.	.	.	.
(1A, 6B)	.	1	.	1	1	.	2	1	1	1	.	1	.	.	.	1	.	1	1	.	.	.	.
(1A, 7B)	.	.	1	.	1	.	1	2	.	.	.	1	.	.	1	1	.	.	.	1	.	1	.
(1A, 8B)	.	.	.	1	1	1	1	.	2	1	1	1	1	.	.	1	1	1	.	.	.	.	1
(1A,10B)	.	.	.	.	1	.	.	.	1	.	1	.	1	.	.	1	.	.	.	.	.	.	1
(2A, 0B)	1	1	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(2A, 1B)	1	2	1	2	2	1	1	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.
(2A, 2B)	1	1	1	1	1	1	1	1	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.
(2A, 3B)	1	2	1	3	3	2	2	1	2	1	1	1	1	.	.	1	1	.	.	.	.	.	.
(2A, 4B)	1	2	1	3	4	2	3	2	3	1	1	3	1	.	1	2	1	1	.	1	.	.	1
(2A, 5B)	1	1	1	2	2	3	1	1	2	1	1	2	2	1	1	1	2	.	.	.	.	.	1
(2A, 6B)	.	1	1	2	3	1	4	2	3	2	1	3	.	.	.	3	1	2	1	1	1	1	1
(2A, 7B)	.	1	1	1	2	1	2	3	1	1	.	3	.	.	2	2	1	1	1	2	1	1	.
(2A, 8B)	.	1	.	2	3	2	3	1	5	2	2	3	2	.	.	4	3	2	1	1	.	.	3
(2A,10B)	.	.	.	1	1	1	1	.	2	1	2	1	1	.	.	2	1	1	.	.	.	.	2
(3A, 0B)	.	1	.	1	1	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(3A, 1B)	1	1	1	2	2	1	2	1	1	1	1	1	.	.	.	1	.	1	.	.	.	.	.
(3A, 2B)	.	1	.	1	2	.	1	1	1	.	.	1	.	.	.	1	.	.	.	1	.	.	.
(3A, 3B)	1	2	1	3	3	2	3	1	3	2	1	2	1	.	.	2	1	1	1	.	.	.	1
(3A, 4B)	1	2	2	3	4	3	4	3	4	2	2	4	1	.	1	4	2	2	1	1	1	1	2
(3A, 5B)	.	1	.	2	3	2	2	1	4	1	1	3	2	.	.	2	3	1	.	1	.	.	2

	23B	24B	25B	26B	27B	28B	29B	30B	31B	32B	33B	34B	35B	36B	37B	38B	39B	40B	41B	42B	43B	44B	45B
(1A, 0B)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1A, 1B)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1A, 2B)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1A, 3B)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1A, 4B)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1A, 5B)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1A, 6B)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1A, 7B)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1A, 8B)	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1A,10B)	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(2A, 0B)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(2A, 1B)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(2A, 2B)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(2A, 3B)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(2A, 4B)	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(2A, 5B)	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(2A, 6B)	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(2A, 7B)	1	.	1	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.
(2A, 8B)	.	1	1	.	.	2	1	.	1	1	.	1	1	.	.	.	.	.	.	.	.	.	.
(2A,10B)	.	.	1	.	.	1	1	.	.	1	.	1	.	.	.	.	1	.	.	.	.	.	.
(3A, 0B)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(3A, 1B)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(3A, 2B)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(3A, 3B)	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(3A, 4B)	.	.	1	.	.	1	1	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.
(3A, 5B)	.	1	1	.	.	1	.	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.

TABLE 4 (*continued*)

(4A, 0B)	.	.	.	1	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(4A, 1B)	.	1	.	1	1	.	1	1	1	1	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(4A, 2B)	.	.	1	.	1	.	1	1	.	.	1	.	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(5A, 0B)	.	.	.	1	1	1	.	.	1	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(5A, 1B)	1	1	.	2	2	2	1	.	2	1	1	1	1	.	.	1	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(5A, 2B)	1	1	.	2	1	1	1	.	1	1	.	1	.	.	.	1	1	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(5A, 3B)	.	1	1	2	3	3	2	1	4	1	2	2	3	1	.	2	2	1	.	.	.	.	2	.	1	.	.	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
(5A, 4B)	1	2	1	4	4	3	4	2	5	3	2	4	2	.	1	4	4	2	1	1	.	.	3	.	1	2	1	.	2	1	.	1	1	.	1	1	.	.	.	1	.	.	.	.	.	.
(6A, 0B)	.	1	1	.	1	.	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(6A, 1B)	.	1	1	1	2	1	2	2	1	.	.	2	.	.	1	1	.	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(6A, 2B)	.	.	.	1	1	1	1	1	1	.	.	2	.	.	1	.	.	1	.	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(6A, 3B)	1	2	2	2	4	1	3	3	2	1	1	3	.	.	1	3	1	1	1	1	.	1	1	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(6A, 4B)	1	2	1	3	4	3	4	3	4	2	1	6	1	.	2	4	2	3	1	3	2	1	2	1	.	2	.	.	1	.	.	2	.	1	1	.	1	.	.	.	1	.	.	.	.	.
(6A, 5B)	1	2	1	3	3	2	3	2	3	2	1	3	1	.	1	3	3	1	1	1	.	1	2	.	1	2	1	.	1	.	.	2	.	1	.	1	.	.	.	.	.	1	.	1	.	.
(7A, 0B)	.	.	.	.	.	1	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(7A, 1B)	.	.	.	1	.	1	.	.	1	.	.	.	1	.	.	.	1	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(7A, 2B)	1	.	.	1	.	1	.	.	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(8A, 0B)	.	.	1	.	.	.	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(8A, 1B)	.	.	.	.	1	.	.	1	.	.	.	1	.	.	1	.	.	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(8A, 2B)	.	.	.	.	.	1	.	.	.	.	.	1	.	.	1	.	.	.	.	.	1	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(8A, 3B)	.	1	1	.	1	.	1	2	.	.	.	1	.	.	1	1	.	.	.	1	.	1	.	.	.	1	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
(8A, 4B)	.	.	.	1	1	1	1	1	1	.	.	2	.	.	1	1	1	1	.	2	1	.	.	1	.	1	.	.	.	.	.	1	.	1	.	.	.	.	.	.	1	.	.	.	1	.
(8A, 5B)	1	1	1	1	1	1	1	1	.	1	.	1	.	.	1	1	1	.	1	.	.	1	.	.	.	1	1	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	1	.	1
(9A, 0B)	.	.	.	1	1	1	1	1	1	1	.	1	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(9A, 1B)	.	1	1	2	3	2	3	2	3	2	1	3	1	.	1	3	2	1	1	1	.	1	1	.	.	1	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(9A, 2B)	.	1	1	1	2	1	2	2	2	1	1	2	1	.	1	2	1	1	1	1	.	1	1	.	.	1	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.
(10A, 0B)	.	.	.	.	1	.	1	.	1	.	1	1	.	.	.	1	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(10A, 1B)	.	1	.	1	2	1	2	1	3	1	1	3	1	.	.	3	1	2	1	1	1	.	2	.	.	1	.	.	1	1	.	1	.	.	1	.	1	.	.	.	.	.	.	.	.	.
(10A, 2B)	.	.	.	1	1	1	2	.	2	1	1	2	.	.	.	2	1	2	.	1	1	.	1	.	.	1	.	.	1	.	.	1	.	.	1	.	.	.	.	.	1	.	.	.	.	.

TABLE 5

E6: TENSOR PRODUCT DECOMPOSITIONS INTO IRREDUCIBLE CHARACTERS CLASS 1 X CLASS 1 = CLASS 2

	0C	1C	2C	3C	4C	5C	6C	7C	8C	9C	10C	11C	12C	13C	14C	15C	16C	17C	18C	19C	20C	21C	22C	23C	24C	25C	26C	27C	28C	29C	30C	31C	32C	33C	34C	35C	36C	37C	38C	39C	40C	41C	42C	43C	44C	45C
(0B, 0B)	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(0B, 1B)	1	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(0B, 2B)	1	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(0B, 3B)	.	1	1	1	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(0B, 4B)	.	1	.	1	1	1	1	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(0B, 5B)	.	.	1	.	1	1	.	1	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(0B, 6B)	.	.	.	1	1	.	1	.	1	1	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(0B, 7B)	.	.	.	1	.	1	.	.	1	1	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(0B, 8B)	.	.	.	.	1	.	1	1	1	.	.	1	.	.	.	1	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(0B, 9B)	.	.	.	.	.	.	1	1	.	1	.	.	.	.	.	1	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(0B,10B)	.	.	.	.	.	.	1	.	.	.	.	1	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1B, 1B)	1	1	1	2	1	1	1	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1B, 2B)	.	1	.	1	1	1	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1B, 3B)	1	2	1	2	2	1	2	1	1	1	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1B, 4B)	1	1	1	2	3	2	2	1	2	1	1	2	1	.	.	1	1	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1B, 5B)	.	1	1	1	2	1	1	2	1	.	.	2	.	.	1	1	1	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1B, 6B)	.	1	.	2	2	1	2	1	3	2	1	1	1	.	.	2	1	1	1	.	.	.	1	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1B, 7B)	.	.	.	1	1	2	1	.	2	1	1	1	2	1	.	1	2	.	.	.	.	.	1	.	1	.	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
(1B, 8B)	.	1	1	1	2	1	3	2	2	1	1	3	.	.	1	3	1	2	1	1	1	1	1	.	.	1	.	.	1	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.
(1B,10B)	.	1	.	.	1	.	1	1	1	.	.	2	.	.	.	1	.	1	1	1	1	.	1	.	.	.	.	.	.	.	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.
(2B, 2B)	.	.	1	.	1	1	.	.	.	.	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(2B, 3B)	1	1	.	2	1	1	1	.	1	1	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(2B, 4B)	.	1	1	1	2	1	1	1	2	.	1	1	1	.	.	1	1	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(2B, 5B)	1	1	1	1	1	1	1	1	.	1	.	1	.	.	1	1	1	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(2B, 6B)	.	.	.	1	1	1	2	.	2	1	1	1	1	.	.	1	1	1	.	.	.	.	1	.	.	.	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
(2B, 8B)	.	1	.	1	2	.	2	1	2	1	.	2	.	.	.	2	1	1	1	1	.	.	1	.	.	1	.	.	1	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.
(2B,10B)	.	.	1	.	1	.	1	1	.	.	1	1	.	.	.	1	.	1	.	.	.	1	1	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	1	.	.	.	.
(3B, 3B)	1	2	2	2	4	1	3	3	2	1	1	2	.	.	1	2	.	1	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(3B, 4B)	1	2	1	4	4	3	4	2	4	2	1	4	1	.	1	3	2	2	1	1	1	.	1	.	.	1	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(3B, 5B)	.	1	1	1	3	2	2	3	2	.	1	4	1	.	2	2	1	1	.	2	1	1	1	1	.	1	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.
(4B, 4B)	1	3	2	4	6	4	5	4	6	3	3	6	3	1	1	6	4	3	1	2	1	1	4	.	1	2	1	.	2	1	.	2	1	.	1	1	1	.	.	.	.	1	.	.	.	.
(4B, 5B)	1	2	1	3	4	2	4	3	4	2	1	5	1	.	2	4	3	2	1	3	1	1	2	1	1	3	1	.	1	.	.	2	.	1	1	1	.	.	.	.	1	.	.	1	.	.
(5B, 5B)	1	1	1	2	2	3	2	2	2	1	1	4	2	1	3	2	2	2	1	2	2	1	2	2	1	2	1	1	.	.	.	2	.	2	.	.	1	.	.	.	1	1	.	1	1	1

TABLE 6

E6: TENSOR PRODUCT DECOMPOSITIONS INTO IRREDUCIBLE CHARACTERS CLASS 1 X CLASS 2 = CLASS 0

	0A	1A	2A	3A	4A	5A	6A	7A	8A	9A	10A	11A	12A	13A	14A	15A	16A	17A	18A	19A	20A	21A	22A	23A	24A	25A	26A	27A	28A	29A	30A	31A	32A	33A
(0B, 0C)	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(0B, 1C)	.	1	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(0B, 2C)	.	.	1	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(0B, 3C)	.	1	1	1	1	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(0B, 4C)	.	.	1	1	.	1	1	.	.	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(0B, 5C)	.	.	1	.	.	.	1	.	1	1	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(0B, 6C)	.	.	.	1	1	1	.	.	.	1	1	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(0B, 7C)	.	.	.	.	.	1	.	1	.	1	.	.	1	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(0B, 8C)	.	.	.	1	.	.	1	.	.	1	1	1	.	1	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
(0B, 9C)	.	.	.	.	1	.	.	.	.	1	.	.	.	1	.	1	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
(0B,10C)	.	.	.	.	.	.	1	.	.	.	1	.	.	.	.	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
(1B, 0C)	.	1	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1B, 1C)	1	1	2	1	1	1	1	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1B, 2C)	.	1	1	1	.	1	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1B, 3C)	.	1	2	2	1	1	2	.	1	2	1	1	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1B, 4C)	.	1	2	2	1	2	2	1	.	3	2	1	1	1	1	.	1	1	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
(1B, 5C)	.	.	1	1	.	1	2	.	1	2	1	2	1	.	1	.	1	.	.	1	1	.	.	.	.	1	.	.	.	.	.	.	.	.
(1B, 6C)	.	1	1	2	1	2	1	.	.	3	2	1	1	2	.	1	1	2	.	.	1	.	1	1	.	.	.	.	1	.	.	.	.	.
(1B, 7C)	.	.	1	1	1	2	.	1	.	2	1	.	2	1	1	.	.	2	1	.	1	.	.	1	1	.	.	.	.	.	.	1	.	.
(2B, 0C)	.	.	1	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(2B, 1C)	.	1	1	1	.	.	1	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(2B, 2C)	1	1	1	.	1	.	.	.	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(2B, 3C)	.	.	1	1	.	1	2	.	1	1	1	1	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.

(2B, 4C)	.	1	1	2	1	1	1	.	.	2	1	1	1	1	1	.	1	.	.	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.
(2B, 5C)	.	.	1	.	.	1	1	1	1	1	1	1	1	.	1	.	1	.	.	1	.	1	.	.	.	1	1	.	.	.	.	.	.	.
(2B, 6C)	.	.	1	1	1	1	1	.	.	2	2	1	.	1	.	.	1	1	.	.	1	.	1	1	.	.	.	.	.	.	1	.	.	.
(2B, 7C)	.	1	1	1	1	1	.	.	.	2	.	.	1	1	1	1	.	1	.	.	1	.	.	1	1	.	.	.	.	.	.	.	.	1
(3B, 0C)	.	1	1	1	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(3B, 1C)	.	1	2	2	1	2	1	1	.	2	1	.	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(3B, 2C)	.	.	1	1	.	2	1	1	.	1	1	.	1	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(3B, 3C)	1	2	3	3	2	2	2	.	.	4	2	1	1	2	1	1	1	1	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.
(3B, 4C)	.	1	3	3	1	4	3	1	1	5	4	2	3	2	1	.	2	3	1	.	2	1	1	1	1	.	.	.	1	.	.	1	.	.
(3B, 5C)	.	1	2	2	1	1	3	.	1	4	2	3	1	1	2	.	2	1	.	1	2	.	1	1	1	1	.	.	.	.	1	.	.	1
(4B, 0C)	.	.	1	1	.	1	1	.	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(4B, 1C)	.	1	2	2	1	2	2	.	1	3	2	1	1	1	1	.	1	1	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.
(4B, 2C)	.	1	1	2	1	1	1	.	.	2	1	1	1	1	1	.	.	1	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.
(4B, 3C)	.	1	3	3	1	3	4	1	1	5	4	3	2	2	1	.	3	2	.	1	2	1	1	1	.	1	.	.	1	.	1	.	.	.
(5B, 0C)	.	.	1	.	.	1	.	1	.	1	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(5B, 1C)	.	.	1	1	.	2	1	1	.	2	1	1	2	.	1	.	.	1	1	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.
(5B, 2C)	.	.	1	.	.	1	1	1	1	1	1	1	1	.	1	.	.	1	1	.	.	.	1	.	1	.	.	1	.	.	.	.	.	.
(5B, 3C)	.	1	2	2	1	3	1	1	.	4	2	1	3	1	2	.	1	2	1	.	2	1	.	1	1	1	.	.	.	.	.	1	.	1
(6B, 0C)	.	.	.	1	1	.	1	.	.	1	1	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(6B, 1C)	.	1	1	2	1	1	2	.	.	3	2	1	1	2	.	1	2	1	.	.	1	1	.	1	.	.	.	.	1	.	.	.	.	.
(6B, 2C)	.	.	1	1	1	1	1	.	.	2	2	.	1	1	.	.	1	1	.	.	1	1	.	1	.	.	.	.	.	.	.	1	.	.
(7B, 0C)	.	.	.	.	.	.	1	.	1	1	.	1	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(7B, 1C)	.	.	1	1	1	.	2	.	1	2	1	2	.	1	1	.	2	.	.	1	1	.	.	1	.	1	.	.	.	.	1	.	.	.
(7B, 2C)	.	1	1	1	1	.	1	.	.	2	.	1	.	1	1	1	1	.	.	.	1	.	.	1	.	1	.	.	.	.	.	.	.	1
(8B, 0C)	.	.	.	1	.	1	.	.	.	1	1	.	1	1	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
(9B, 0C)	.	.	.	.	1	.	.	.	.	1	.	.	.	1	.	1	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
(10B, 0C)	.	.	.	.	.	1	.	.	.	.	1	.	.	.	.	.	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.

Finally, let us recall that the unique feature of the decomposition matrix approach is in its applicability to any class function (providing that the weight systems of the irreducible components are separated by the T_n in question). The present example is a result of our attempt to determine by direct computation how far the separation extends using T_7. The decompositions were computed in a couple of minutes on a CDC Cyber 835.

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Department of Mathematics
Concordia University
Montreal, Quebec, Canada

Centre de Recherches, Mathématiques
Université de Montréal
Montreal, Quebec, Canada

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