14[34-02, 34C15, 34C25, 34C29, 34E05].—E. A. GREBENIKOV & YU. A. RYABOV, *Constructive Methods in the Analysis of Nonlinear Systems* (Translated from the Russian by Ram S. Wadhwa), "Mir", Moscow, 1983, 328 pp., 22 cm. Price $9.95.

These authors represent a very strong Soviet specialty, asymptotic analysis of nonlinear oscillatory systems. They have published extensively in Russian, especially concerning the detailed development of asymptotic methods for problems of celestial mechanics. This translation of a 1979 book makes no attempt to relate its material to Western research or textbooks. This is a pity, because minor changes in terminology and references to standard English presentations could increase its usefulness substantially. The writing is quite satisfactory, though neither colloquial nor easy-reading mathematics. C. L. Siegel is a victim of double translation, having become K. L. Zigel. There is, indeed, much to be learned, by amateurs and experts, from reading this monograph. *Mathematics of Computation* readers will, however, find it quite analytical (rather than computational).

The first half of this book deals with the method of averaging. This method dates back to Lagrange and Laplace, but was largely developed and applied by Soviet mathematicians beginning about fifty years ago. Grebenikov and Ryabov first emphasize that coordinate transformations are basic to this Krylov-Bogoliubov theory for dealing with "multifrequency" systems of ordinary differential equations with slow and fast variables. The simplest averaging possibility relates to approximating solutions to

$$\frac{dz}{dt} = \mu Z(z, t, \mu),$$

for μ small, by solutions to

$$\frac{d\bar{z}}{dt} = \mu \lim_{T\to\infty} \frac{1}{T}\int_0^T Z(\bar{z}, t, 0)\, dt.$$

Such averaging theories and many generalizations to higher-order methods are presented in detail. For complete proofs, however, the reader must refer to the Russian literature. Resonance enters, once one considers systems of van der Pol equations

$$\frac{d^2x_k}{dt^2} + \omega_k^2 x_k = \mu f_k(x_1, \ldots, x_s, \dot{x}_1, \ldots, \dot{x}_s)$$

where the frequencies $\omega_1, \ldots, \omega_s$ become rationally commensurable. Illuminating examples are given, and the small-divisor problem is encountered through Fourier analysis. Among several applications, a very readable discussion of the bounded three-body problem is included.

Although we generally associate the names of Cauchy and Picard with iteration, these authors give Lyapunov major credit for introducing majorants. Many uncommon and useful techniques for solving operator equations are presented, including use of the trigonometric norm, estimates of the domain of existence and uniqueness, and the relation to (non)contraction. The authors apply such methods to their special interest of obtaining periodic solutions. They treat differential equations involving regular and singular perturbations in both resonant and nonresonant situations. Unlike most authors, they worry and obtain convergent expansions, in contrast to asymptotic approximations (which could provide a starting guess for a numerical algorithm). Special attention is given to obtaining the resonance curve for

Duffing's equation and to calculating eigenvalues for Mathieu's equation. New iterative methods with quadratic convergence properties are then used as successive approximation schemes to effectively obtain periodic solutions to differential equations.

The authors finally present their approach to numerical-analytic solutions, which emphasizes the use of a computer for intermediate steps in obtaining analytical solutions, as in using iterative methods to solve algebraic equations. Much of such necessary algebraic details might also benefit from symbolic computation. I cannot report that this monograph is preferable to all others available in English (such as Arnold [1], Guckenheimer and Holmes [2], and Sanders and Verhulst [3]). It does, however, present valuable material and a unique perspective on an important, though specialized, class of problems.

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1. V. I. ARNOLD, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, New York, 1983.

2. J. GUCKENHEIMER & P. HOLMES, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1983.

3. J. A. SANDERS & F. VERHULST, *Averaging Methods in Nonlinear Dynamical Systems*, Springer-Verlag, New York, 1985.

15[45D05, 65R20].—PETER LINZ, *Analytical and Numerical Methods for Volterra Equations*, SIAM Studies in Appl. Math., SIAM, Philadelphia, Pa., 1985, xiii + 227 pp., $23\frac{1}{2}$ cm. Price $32.50.

This book contains an elementary but thorough and self-contained introduction to the theory and the numerical solution of Volterra integral equations; as stated in the preface, "The audience for which this book is intended is a practical one with an immediate need to solve real-world problems." Thus, the chosen mathematical setting is that of (continuous) real-valued functions of one or several real variables, and proofs are often either just sketched or omitted entirely (the reader is then directed to an appropriate reference).

The first part of the book (six chapters, covering some 90 pages) deals with the classical quantitative theory of linear and nonlinear Volterra equations. It includes a brief chapter on some typical applications of Volterra integral and integro-differential equations, and it introduces some elementary results on the asymptotic behavior of solutions to certain second-kind integral equations. These chapters are particularly valuable, as most books on integral equations focus on Fredholm equations and treat Volterra equations only in a passing manner.

Numerical methods are discussed in the second part of the book, comprising about 110 pages. Chapter 7 covers direct quadrature methods (including a convergence analysis, asymptotic error estimates, and numerical stability), block-by-block methods, and explicit Runge-Kutta methods for second-kind equations with bounded kernels. Various product integration methods for second-kind equations with unbounded (or otherwise poorly behaved) kernels form the contents of Chapter 8. The