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Supplement to Uniform High-Order Difference Schemes for a Singularly Perturbed Two-Point Boundary Value Problem

By Eugene C. Gartland, Jr.

Appendix

Proof of Theorem 3.1. For simplicity, we consider the local system on an interval $[-h,h]$ (suppress the subscript i) and define

$$E(x) \equiv \exp\left(\frac{1}{\varepsilon}\int_0^x a\right) .$$

The auxiliary evaluation points are assumed to be given by

$$\text{(A.1)}\qquad \begin{aligned} &\xi_1 = 0 \ , \quad p = 1 , \\ &\xi_j = -h + \frac{j-1}{p-1} h \ , \qquad j = 1, \ldots, 2p-1, \ p = 2,3 \ldots \end{aligned}$$

Ordering and scaling the conditions of exactness on $\{1, x, \ldots, x^p, E(x), xE(x), \ldots, x^{p-1}E(x)\}$ appropriately gives rise to the local system (with $\sum \equiv \sum_{j=1}^{2p-1}$)

$$\text{(A.2)}\qquad \begin{aligned}
&\alpha_{-1} + \alpha_0 + \alpha_1 - \sum \beta_j b(\xi_j) = 0 \\
&-\alpha_{-1} + \quad + \alpha_1 - \frac{1}{h}\sum \beta_j[(a(\xi_j) - a_0) + b(\xi_j)\xi_j] = \frac{a_0}{h} \\
&E(-h)\alpha_{-1} + \alpha_0 + E(h)\alpha_1 - \sum \beta_j E(\xi_j)(b - a')(\xi_j) = 0 \\
&\sum \beta_j = 1 \\
&\frac{(-1)^k}{k} h\alpha_{-1} + \quad + \frac{1}{k} h\alpha_1 - \sum \beta_j \left\{ \left[\frac{-\varepsilon(k-1) + a(\xi_j)\xi_j}{h}\right]\left(\frac{\xi_j}{h}\right)^{k-2} + O(h) \right\} = 0 \ , \\
&\qquad\qquad k = 2, \ldots, p , \\
&\frac{(-1)^\ell}{\ell} h E(-h)\alpha_{-1} + \quad + \frac{1}{\ell} h E(h)\alpha_1 - \sum \beta_j E(\xi_j) \left\{ \left[\frac{-\varepsilon(\ell-1) - a(\xi_j)\xi_j}{h}\right]\left(\frac{\xi_j}{h}\right)^{\ell-2} + O(h) \right\} = 0 \ , \\
&\qquad\qquad \ell = 1, \ldots, p-1 \ ,
\end{aligned}$$

where $a_0 = a(0)$.

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We show that this linear system uniquely determines $\alpha_{-1}, \alpha_0, \alpha_1, \beta_1, \ldots, \beta_{2p-1}$ for all h sufficiently small, uniformly in ε, i.e., for any ratio $\rho \equiv h/\varepsilon$, $0 < \rho < \infty$. We break down the analysis into three parts: the limit $\rho \to 0$ $(h \ll \varepsilon)$, compact ranges $0 < \underline{\rho} \le \rho \le \bar{\rho} < \infty$, and the limit $\rho \to \infty$ $(\varepsilon \ll h)$.

First, for compact ranges of ρ, $0 < \underline{\rho} \le \rho \le \bar{\rho} < \infty$, we write the system (A.2) in the form

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \underline{\alpha} \\ \underline{\beta} \end{bmatrix} = \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \end{bmatrix}$$

where $\underline{\alpha} \equiv (\alpha_{-1}, \alpha_0, \alpha_1)^T$, $\underline{\beta} = (\beta_1, \ldots, \beta_{2p-1})^T$, $\underline{b}_1 \equiv (0, a_0/h, 0)^T$ and $\underline{b}_2 = (1, 0, \ldots, 0)^T$. For ρ bracketed in this way, we can establish that for $-h \le \xi \le h$,

$$\text{(A.3)} \qquad E(\xi) = \exp\left(\frac{1}{\varepsilon} a_0 \xi\right)(1 + O(h)) \quad .$$

It can then be shown that M_{11} has a bounded inverse for h sufficiently small, M_{12} is $O(1)$, M_{21} is $O(h)$, and M_{22} is an $O(h)$ perturbation of a matrix that is row equivalent to an interpolation matrix for the functions $\{1, x, \ldots, x^{p-1}, \exp(a_0 x/\varepsilon), x \exp(a_0 x/\varepsilon), \ldots, x^{p-2} \exp(a_0 x/\varepsilon)\}$, which form a Chebyshev system ([31], §8.2). It follows that M_{22} has a bounded inverse and the system is nonsingular for all h sufficiently small.

To consider the limiting case $\rho \to \infty$, it is convenient to define some rescaled coefficient and weight functions:

$$\underline{\alpha}' = (\alpha'_{-1}, \alpha'_0, \alpha'_1)^T \equiv (\alpha_{-1}, \alpha_0, E(h)\alpha_1)^T \quad ,$$

$$\underline{\beta}'_1 = (\beta'_1, \ldots, \beta'_p)^T \equiv (\beta_1, \ldots, \beta_p)^T \quad ,$$

$$\underline{\beta}'_2 = (\beta'_{p+1}, \ldots, \beta'_{2p-1})^T \equiv (E(\xi_{p+1})\beta_{p+1}, \ldots, E(\xi_{2p-1})\beta_{2p-1})^T \quad .$$

We can rewrite our system in terms of these variables and take the limit as $\rho \to \infty$ (using the distribution requirement (A.1) to conclude that $E(\xi_p) = 1$ and $E(\xi_1), \ldots, E(\xi_{p-1}), E(\xi_{p+1})^{-1}, \ldots, E(\xi_{2p-1})^{-1} \to 0$) to get

$$\begin{aligned}
\alpha'_{-1} + \alpha'_0 \quad & - \sum_{j=1}^{p} \beta'_j\, b(\xi_j) && = 0 \\
- \alpha'_{-1} \quad & - \frac{1}{h} \sum_{j=1}^{p} \beta'_j \left[(a(\xi_j) - a_0) + b(\xi_j)\xi_j\right] && = \frac{1}{h} a_0 \\
\alpha'_0 + \alpha'_1 \quad & - \sum_{j=p}^{2p-1} \beta'_j\, (b - a')(\xi_j) && = 0 \\
& \sum_{j=1}^{p} \beta'_j && = 1 \\
\frac{(-1)^k}{k} h\, \alpha'_{-1} \quad & - \sum_{j=1}^{p} \beta'_j\, a(\xi_j) \left(\frac{\xi_j}{h}\right)^{k-1} + O(h) && = 0 \quad , \quad k = 2, \ldots, p, \\
\frac{1}{\ell} h \alpha'_1 \quad & + \sum_{j=p}^{2p-1} \beta'_j\, a(\xi_j) \left(\frac{\xi_j}{h}\right)^{\ell-1} + O(h) && = 0 \quad , \quad \ell = 1, \ldots, p-1.
\end{aligned}$$

Here, Equations 1, 2, and 4 through $p+3$ give a system for $\alpha'_{-1}, \alpha'_0, \beta'_1, \ldots, \beta'_p$ that is easily seen to be nonsingular for h sufficiently small: The leading 2×2 diagonal block has a bounded inverse, and the lower $p \times p$ diagonal block is closely related to a Vandermonde matrix. This is, in fact, the same local system that

that one gets by seeking an approximation to the reduced differential equation

$$a(x)u' + b(x)u = f(x)$$

of the form

$$\alpha'_{-1}u^h_{i-1} + \alpha'_0 u^h_i = \beta'_1 f(\xi_1) + \ldots + \beta'_p f(\xi_p)$$

that is exact on $\{1, x, \ldots, x^p\}$ subject to $\beta'_1 + \ldots + \beta'_p = 1$. In a similar way, the remaining equations serve to uniquely determine $\alpha'_1, \beta'_{p+1}, \ldots, \beta'_{2p-1}$. Also it is a consequence of our scaling here that $\alpha_1 = O(E(h)^{-1})$, $\beta_{p+1} = O(E(\xi_{p+1})^{-1}), \ldots, \beta_{2p-1} = O(E(\xi_{2p-1})^{-1})$, as $\rho \to \infty$.

Lastly, we consider the limiting case when $\rho \to 0$. We cannot simply take the limit as $h/\varepsilon \to 0$ in (A.2) because the resulting system is singular, though consistent. It is, however, still the case that the coefficients and weights have finite limits. The easiest way to see this is to utilize a different basis for the functions upon which the local system is built. By taking appropriate linear combinations and using Taylor expansions for the $E(x)$ functions, one can produce an equivalent basis of the form $\{1, x, \ldots, x^p, x^{p+1}(1 + \psi_1(x)), \ldots, x^{2p}(1 + \psi_p(x))\}$, where $\psi_1, \ldots, \psi_p$ are smooth functions, provided a is sufficiently smooth, that satisfy $\psi_\ell(0) = 0$, $\psi'_\ell(0) = O(\varepsilon^{-1})$, and $\psi''_\ell(0) = O(\varepsilon^{-2})$, $\ell = 1, \ldots, p$. The resulting system is, to leading order in ρ, the standard HODIE system for (1.1) exact on $\{1, x, \ldots, x^{2p}\}$ and is nonsingular for all h sufficiently small (see [11] or [21]); the coefficients are $O(h)$ perturbations of standard central differences. It follows that our local determination of the α's and β's is well posed for all h sufficiently small, uniformly in ε.

The inequalities (3.4a,b) follow readily from

$$|\alpha_{-1} + \alpha_0 + \alpha_1| = |\textstyle\sum \beta_j b(\xi_j)| \le C\,\|b\|_\infty$$

and

$$h(\alpha_1 - \alpha_{-1}) = \textstyle\sum \beta_j [a(\xi_j) + b(\xi_j)\xi_j] = a_0(1 + O(h)) \quad .$$

Here we have used the facts that for h sufficiently small the β_j are well defined and $O(1)$ and satisfy $\sum \beta_j = 1$. The inequalities (3.4c) require a careful examination of the leading order behavior of α_{-1} and α_1. These satisfy

$$\alpha_{-1} + \alpha_0 + \alpha_1 = \textstyle\sum \beta_j b(\xi_j) \equiv b_h \;,$$

$$-h\alpha_{-1} + \qquad + h\alpha_1 = \textstyle\sum \beta_j [a(\xi_j) + b(\xi_j)\xi_j] \equiv a_h \;,$$

$$E(-h)\alpha_{-1} + \alpha_0 + E(h)\alpha_1 = \textstyle\sum \beta_j E(\xi_j)(b - a')(\xi_j) \equiv c_h \;,$$

and are given by

$$\alpha_{-1} = -\frac{a_h(E(h) - 1) + (b_h - c_h)h}{h(E(h) - 2 + E(-h))} \;,$$

(A.4)

$$\alpha_1 = -\frac{a_h(1 - E(-h)) + (b_h - c_h)h}{h(E(h) - 2 + E(-h))} \;.$$

For $\rho << 1$ (i.e., $h << \varepsilon$), we get from Taylor's formula

$$E(\xi) = 1 + \frac{a_0}{\varepsilon}\xi + \frac{(a_0^2 + \varepsilon a_0')}{2\varepsilon^2}\xi^2 + O(\rho^3) \quad .$$

Using this, we expand the formula of α_{-1} for ρ and h small ,

$$\alpha_{-1} = - \frac{(a_0 + O(h))(\frac{ha_0}{\varepsilon} + \frac{h^2}{2\varepsilon^2}(a_0^2 + \varepsilon a_0') + O(\rho^3))}{h \cdot \frac{h^2}{\varepsilon^2}(a_0^2 + \varepsilon a_0')(1 + O(\rho^2))}$$

$$= -\frac{\varepsilon}{h^2}(1 + \frac{ha_0}{2\varepsilon} + O(h))(1 + O(\rho^2))$$

$$= -\frac{\varepsilon}{h^2}(1 + \frac{1}{2}(a_0 + O(\varepsilon))\cdot\rho)(1 + O(\rho^2)) \quad .$$

Similarly,

$$\alpha_1 = -\frac{\varepsilon}{h^2}(1 - \frac{1}{2}(a_0 + O(\varepsilon))\rho)(1 + O(\rho^2)) \quad .$$

The inequalities (3.4c) follow directly from these estimates, for ε and ρ sufficiently small.

For any compact range $0 < \underline{\rho} \leq \rho \leq \overline{\rho} < \infty$, we can use the approximation (A.3) in (A.4) to establish

$$\alpha_{-1} = -\frac{a_0}{h}\frac{e^{\frac{\rho a_0}{2}}}{e^{\frac{\rho a_0}{2}} - e^{-\frac{\rho a_0}{2}}}(1 + O(h))$$

and

$$\alpha_1 = -\frac{a_0}{h}\frac{e^{-\frac{\rho a_0}{2}}}{e^{\frac{\rho a_0}{2}} - e^{-\frac{\rho a_0}{2}}}(1 + O(h)) \quad .$$

The inequality $\alpha_1 \leq 0$ is clearly valid for h sufficiently small. The inequalities $\alpha_{-1} \leq -\varepsilon/h^2 \leq \alpha_1$ are equivalent (to leading order) to

$$F_1(\sigma) \equiv \frac{\sigma e^{\sigma}}{e^{\sigma} - e^{-\sigma}} \geq \frac{1}{2} \geq \frac{\sigma e^{-\sigma}}{e^{\sigma} - e^{-\sigma}} \equiv F_2(\sigma) \quad ,$$

where $\sigma \equiv ha_0/2\varepsilon$. Now the function F_1 is increasing on $0 < \sigma < \infty$ with limit 1/2 as $\sigma \to 0$; while F_2 is decreasing on $0 < \sigma < \infty$, also with limit 1/2 as $\sigma \to 0$. It follows that given $\underline{\sigma} > 0$, there is a constant $\delta = \delta(\underline{\sigma}) > 0$ such that

$$F_1(\sigma) \geq \frac{1}{2} + \delta \quad \text{and} \quad \frac{1}{2} - \delta \geq F_2(\sigma) \quad , \qquad \underline{\sigma} \leq \sigma < \infty \quad .$$

This is sufficient to verify inequalities (3.4c) for h sufficiently small with $\underline{\rho} \leq \rho \leq \overline{\rho}$.

Lastly, in the limit as $\rho \to \infty$ we have

$$\alpha_{-1} = -\frac{a_0}{h}(1 + O(h))(1 + O(E(h)^{-1}))$$

and

$$\alpha_1 = -\frac{a_0}{h}E(h)^{-1}(1 + O(h))(1 + O(E(h)^{-1})) \quad .$$

So clearly, $\alpha_1 \leq 0$ for all h sufficiently small and h/ε sufficiently large. Furthermore,

$$-\frac{a_0}{h} \leq -\frac{\varepsilon}{h^2} \quad \text{if and only if} \quad \frac{ha_0}{\varepsilon} \geq 1$$

and

$$-\frac{a_0}{h}E^{-1}(h) \geq -\frac{\varepsilon}{h^2} \quad \text{if and only if} \quad \frac{ha_0}{\varepsilon} \leq e^{\frac{1}{\varepsilon}\int_0^h a} \quad .$$

It follows from above, then, that $\alpha_{-1} \leq -\varepsilon/h^2 \leq \alpha_1$ for all h sufficiently small and h/ε sufficiently large (independent of each other). Therefore, the inequalities (3.4c) are valid for all h sufficiently small uniformly in h/ε, and the theorem is proved. $\square$