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Supplement to The Discrete Galerkin Method for Integral Equations

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6. Bound on G_h^{-1}. In this section we adopt the approach of [17] in order to bound $\|G_h^{-1}\|$ in (4.20). This approach is quite general, and the reader may wish to refer ahead to Section 7, which contains more specific examples of subspaces S_h.

We distinguish between two cases:

(i) D is a region in $\mathbb{R}^n$,

(ii) D is a surface in $\mathbb{R}^n$.

In the first case, the modifications needed of Descloux's presentation are just notational; while in the second, they are minor in every respect. Thus we introduce the following notations and assumptions.

(a) $D \subset \mathbb{R}^n$ is either a region or a piecewise smooth hypersurface in $\mathbb{R}^n$. In the latter case, the surface D is assumed to be Lipschitz; and the general schema for D is that of application 2 in Section 3.

(b) m is either the Lebesgue measure in $\mathbb{R}^n$ or the measure over D induced by the latter.

(c) The closure $\bar{D}$ of D is subdivided into the union of closed sets e_i, $i = 1,\cdots,E_h = E$, each one of which is the closure of its interior in the natural topology of D. h is a characteristic parameter of the division. In the case D is a piecewise smooth surface, the interior of each e_i belongs to one and only one of the smooth sections that make up D.

(d) $m(e_i) > 0$, $i = 1,\cdots,E$;
$m(e_i \cap e_j) = 0$, $i,j = 1,\cdots,E$; $i \neq j$.

(e) $S_h = \text{span}\{\varphi_1, \cdots, \varphi_N\}$, $N = N_h$; and for all $i = 1, \cdots, N$, $C_i = \text{supp}\ \varphi_i$ is a union of a finite number of e_i's.

(f) For all i, there exists j such that $C_j \supset e_i$.

(g) Let $I_i = \{j : C_j \supset e_i\}$; and let $|Z|$ denote the number of elements in a finite set Z. Then

$$|I_i| \le M \ , \quad i = 1, \cdots, E$$

independent of h.

(h) For $Y \subset \mathbb{R}^n$, let $d(Y)$ denote the diameter of Y. Then with $\bar{n} = n$ or $\bar{n} = n-1$, depending on whether D is a region or a surface,

$$d^{\bar{n}}(C_i)/m(e_j) \le \gamma \ , \quad i = 1, \cdots, N, \quad j = 1, \cdots, E,$$

and

$$d^{\bar{n}}(e_i)/d^{\bar{n}}(e_j) \le \delta \ , \quad i, j = 1, \cdots, E,$$

both independent of h.

(i) On $S_h|_{e_i}$, let there be given a scalar product $(\cdot,\cdot)_{e_i}$. Assume also that for all $x \in \mathbb{R}^n$ and $i = 1, \cdots, E$,

$$\alpha \sum_{j \in I_i} x_j^2 \le \sum_{j,k \in I_i} (\varphi_j, \varphi_k)_{e_i} x_j x_k \le \omega \sum_{j \in I_i} x_j^2 \tag{6.1}$$

with $\alpha > 0$.

With this, we state the following theorem.

Theorem 6.1. Assume (a)-(i) *hold. Define*

$$(\cdot,\cdot)_h = \sum_{i=1}^{E} (\cdot,\cdot)_{e_i}. \tag{6.2}$$

Then $(\cdot,\cdot)_h$ *is an inner product on* S_h; *and for the Gram matrix* $G_h = ((\varphi_i, \varphi_j)_h)$, *one has*

$$\|G_h^{-1}\| \le C\ \alpha^{-1}, \tag{6.3}$$

where C *does not depend on* h.

<u>Proof</u>. The proof follows that of Theorem 1 in [17] very closely. For some $f \in S_h$, suppose $(f,f)_h = \sum_{i=1}^{E} (f,f)_{e_i} = 0$. This implies $(f,f)_{e_i} = 0$, $i = 1, \cdots, E$, and it is obvious that $f|_{e_i} = 0$, $i = 1, \cdots, E$. By (e), $\text{supp}\ f \subset \bigcup_{j=1}^{N} \text{supp}\ \varphi_j = \bigcup_{i=1}^{E} e_i$, by (f) and (e). Therefore, $f \equiv 0$.

Now in the case of D a subregion of $\mathbb{R}^n$,

$$C = s^{-1}(Mc_n p^n \gamma)^{\frac{1}{2}}, \tag{6.4}$$

where $0 < s < 1$, p is the smallest integer satisfying

$$\mu^{-2} Mc_n\gamma n(1-\mu)^{p-1}p^{n-1} \leq (1-s)^2, \tag{6.5}$$

$\mu = \alpha\omega^{-1} \leq 1$, and c_n is the volume of the unit ball in $\mathbb{R}^n$.

The relations (6.4)-(6.5) are found in [17]. These can be modified in different ways to the case of D being a surface in $\mathbb{R}^n$. The most immediate modification is as follows.

Let τ be a constant with the property

$$m(D \cap B_n(r)) \leq \tau \text{ area}(S_n(r)) \tag{6.6}$$

for all balls $B_n(r)$ of radius r and the corresponding spheres $S_n(r)$, about all points of D. Let $\bar{c}_n$ denote the area of the unit sphere. Then

$$C = s^{-1}(\tau M\bar{c}_n p^{n-1}\gamma)^{1/2}, \tag{6.7}$$

where $0 < s < 1$ and p is the smallest integer satisfying

$$\tau\mu^{-2}M\bar{c}_n\gamma(1-\mu)^{p-1}p^{n-1} \leq (1-s)^2. \tag{6.8}$$

The latter inequality (6.8) can be improved by making a more stringent assumption than (6.6) on the behavior of D. (6.6) in turn evidently holds for surfaces that are Lipschitz and piecewise smooth. ∎

Using Theorem 6.1, the problem of bounding $\|G_h^{-1}\|$ is thus reduced to evaluating (6.1). To see the connection with the previously discussed bound (4.19)-(4.20), note that one possible choice of the inner products $(\cdot,\cdot)_{e_i}$ in (6.1) is through the use of integration formulas on e_i ([17] uses exact integration). Therefore, the inner product (4.1) based on (4.4) and the integration (2.3) fits well into the above framework.

Observe also, that the set I_i that appears in (6.1) admits a very simple interpretation, as indicated by the following

Lemma 6.2. Assume (a)-(i) *hold. Then the set* $S^0_{e_i} = \{\varphi_k|_{e_i} : k \in I_i\}$ *forms a basis in* $S_h|_{e_i} = \{\varphi|_{e_i} : \varphi \in S_h\}$.

<u>Proof</u>. Since $\alpha > 0$, it follows from (6.1) that functions from $S^0_{e_i}$ are linearly independent.

On the other hand, if $\psi \in S_h|_{e_i}$, then there exists $\varphi \in S_h$ such that $\psi = \varphi|_{e_i}$. By (e), $\varphi = \sum_{k=1}^{N} c_k\varphi_k$, hence $\psi = \varphi|_{e_i} = \sum_{k=1}^{N} c_k\varphi_k|_{e_i} = \sum_{k\in I_i} c_k\varphi_k|_{e_i}$. ∎

Construction of S_h. Now let us be more specific about the construction of the finite-dimensional spaces S_h. As in

[11, pp. 88-90], we assume there is given a reference element $\hat{e}$, a finite-dimensional space $\hat{S}$ of functions over $\hat{e}$, and for each e_k, $k = 1,\cdots,E$, a 1-1 transformation F_k of $\hat{e}$ onto e_k. We shall write $\hat{s} \in \hat{e}$ and $s = F_k(\hat{s}) \in e_k$. Set

$$S_{e_k} = \{\varphi : \varphi(s) = \hat{\varphi}(\hat{s}) \text{ for some } \hat{\varphi} \in \hat{S}\}. \tag{6.9}$$

Finally, suppose on $\hat{S}$ there is defined an inner product $(\cdot,\cdot)^\wedge$. Then it induces inner products $(\cdot,\cdot)_{e_k}$ on S_{e_k} by means of

$$(\varphi(s),\psi(s))_{e_k} = (|DF_k|^{\frac{1}{2}}\hat{\varphi}, |DF_k|^{\frac{1}{2}}\hat{\psi})^\wedge, \tag{6.10}$$

where DF_k is the Jacobian matrix of F_k.

In the case when each F_k is an affine mapping,

$$s = F_k\hat{s} = B_k\hat{s} + b_k, \tag{6.11}$$

$DF_k = \det B_k$ and $|DF_k| = m(e_k)/m(\hat{e})$. Hence (6.10) becomes

$$(\varphi,\psi)_{e_k} = |\det B_k|(\hat{\varphi},\hat{\psi})^\wedge = \frac{m(e_k)}{m(\hat{e})}(\hat{\varphi},\hat{\psi})^\wedge. \tag{6.12}$$

Spaces S_h we consider all possess the property

$$S_h\big|_{e_k} = S_{e_k}, \quad k = 1,\cdots,E. \tag{6.13}$$

It is not difficult to see that all finite element spaces do satisfy (6.13). The proof for spaces of spline functions is much less immediate; see [27, Thm. 4.18].

In addition, we require existence of a basis $\hat{S}^0$ of $\hat{S}$ with the property

(j) for all $k = 1,\cdots,E$, $i = 1,\cdots,N$ and $j \in I_i$ there exists $\hat{\varphi} \in \hat{S}^0$ such that $\hat{\varphi} = \varphi_i\big|_{e_k} \circ F_k$ (6.14)

and write $\hat{S}^0 = \{\hat{\varphi}_i : i = 1,\cdots,\upsilon\}$. By Lemma 6.2 the function $\hat{\varphi}$ in (j) is unique.

The condition (j) is what singles out finite element spaces, and most of the spline spaces fail to satisfy it. For some that do, see Corollary 7.2.

Assuming (j), (6.1) is rewritten as

$$\alpha\sum_{i=1}^{\upsilon} X_i^2 \le \sum_{i,j=1}^{\upsilon} (|DF_k|^{\frac{1}{2}}\hat{\varphi}_i, |DF_k|^{\frac{1}{2}}\hat{\varphi})^\wedge X_iX_j \le \omega\sum_{i=1}^{\upsilon} X_i^2. \tag{6.15}$$

■

If (6.11) holds, (6.15) simplifies to

$$\alpha\sum_{i=1}^{\upsilon} X_i^2 \le |\det B_k| \sum_{i,j=1}^{\upsilon} (\hat{\varphi}_i,\hat{\varphi}_j)^\wedge X_iX_j \le \omega\sum_{i=1}^{\upsilon} X_i^2. \tag{6.16}$$

In this case, if α_0 and ω_0 are lower and upper bounds for the eigenvalues of $((\hat{\varphi}_i,\hat{\varphi}_j)^{\wedge})_{i,j=1,\cdots,\nu}$, then $\alpha \le |\det(B_k)|\alpha_0$ and $\omega \ge |\det(B_k)|\omega_0$, $k = 1,\cdots,E$.

In the more general case, assume

$$(|DF_k|^{\frac{1}{2}}\hat{\varphi},|DF_k|^{\frac{1}{2}}\hat{\varphi})^{\wedge} = (\hat{\varphi},|DF_k|\hat{\psi})^{\wedge} \tag{6.17}$$

for all $\hat{\varphi},\hat{\psi} \in \hat{S}$ and also that

$$(DF_k)(\hat{s}) = \det(B_k)(1+O_k(h,\hat{s})) \tag{6.18}$$

for some matrix B_k.

Then write

$$A = ((\hat{\varphi}_i,\hat{\varphi}_j)^{\wedge})$$

and

$$\Delta = [(\hat{\varphi}_i,O_k(h,\hat{s})\hat{\varphi}_j)^{\wedge})] = [O_k(h,i,j)]. \tag{6.19}$$

Lemma 6.3. *With the above notations, assume*

$$|O_k(h,i,j)| \le O(h)\ ,\quad i,j = 1,\cdots,\nu;\quad k = 1,\cdots,E. \tag{6.20}$$

Then in (6.15) *one can choose*

$$\alpha = \frac{\alpha_0}{2}\ \min_{1\le k\le E}|\det(B_k)|, \tag{6.21}$$

provided h *is sufficiently small.*

<u>Proof</u>. The proof uses a standard result from perturbation theory; see [6, p. 505]. If λ is an eigenvalue of $A+\Delta$ then

$$\min_{1\le j\le \nu}|\lambda-\lambda_j| \le \|\Delta\|_2,$$

where λ_i are eigenvalues of A. Then, since $\|\Delta\|_2 \le \|\Delta\|_F \le O(h)$, where $\|\cdot\|_F$ is the Frobenius matrix norm, the smallest eigenvalue λ of $A+\Delta$ satisfies

$$\lambda \ge \alpha_0 - |O(h)| > \frac{1}{2}\alpha_0,$$

provided h is sufficiently small. ∎