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A Method for Computing the Iwasawa λ-Invariant

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Abstract. We present a method for computing the minus-part of the Iwasawa λ-invariant of an Abelian field K. Applying this method, we have computed λ^- for several odd primes p when K runs through a large number of quadratic extensions of the pth cyclotomic field. A report on these computations and an analysis of the results is included.

1. Introduction. Let K be an Abelian field, i.e., a finite Abelian extension of $\mathbf{Q}$. For a prime $p > 2$, consider the cyclotomic $\mathbf{Z}_p$-extension K_∞ of K. Let K_n $(n \geqslant 0)$ denote the intermediate field of K_∞/K which is cyclic of degree p^n over K. The p-part of the class number of K_n equals $p^{\lambda n+\nu}$, for all sufficiently large n, where $\lambda = \lambda(p)$ and $\nu = \nu(p)$ are integral constants, $\lambda \geqslant 0$. Call λ the Iwasawa λ-invariant of K and write $\lambda = \lambda^+ + \lambda^-$, where λ^+ is the corresponding invariant of the maximal real subfield of K. In this paper we present a method for computing λ^-, developed by the second author, and report on computer calculations by the first author, performed by this method.

If the conductor f_K of the field K is divisible by p^2, then K has a subfield L such that $p^2 \nmid f_L$ and the cyclotomic $\mathbf{Z}_p$-extension of L equals K_∞. Hence we assume, without loss of generality, that $p^2 \nmid f_K$. Denote by $\mathrm{Ch}(K)$ the character group of K. It is known that λ^- decomposes as

$$\lambda^- = \sum_{\chi \in X} \lambda_\chi$$

with

$$X = X(K) = \{\chi \in \mathrm{Ch}(K) \colon \chi(-1) = -1,\ \chi \neq \omega^{-1}\},$$

where ω denotes the Teichmüller character mod p and λ_χ is the λ-invariant of the Iwasawa power series representing the p-adic L-function $L_p(s, \chi\omega)$.

Thus, the computation of λ^- is reduced to the determination of the components λ_χ. This will be done in two steps: We first relate λ_χ to the p-orders of certain generalized Bernoulli numbers and then show how to determine these p-orders by means of a series of character sum congruences. As an application we consider the fields $\mathbf{Q}(\sqrt{m}, \zeta_p)$, where m is an integer prime to p and ζ_p denotes a primitive pth root of 1. In this case the congruences in question are simply rational congruences mod p.

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The computational part of our work consists of the determination of λ^- for quite a large collection of fields $\mathbf{Q}(\sqrt{m}, \zeta_p)$, chosen so that either p or $|m|$ is small. More precisely, we computed for these fields the components λ_χ with $\chi = \theta_m \omega^t$, where θ_m is the quadratic character of the field $\mathbf{Q}(\sqrt{m})$; this is sufficient since the λ^--invariants of the cyclotomic subfields $\mathbf{Q}(\zeta_p)$ are known. Our results also give the λ-invariant of $\mathbf{Q}(\sqrt{m})$ for the negative m in the range under consideration.

There are previous numerical results about λ^- in [1], [2], [3], [4], [6]. These concern mainly quadratic fields and the fields $\mathbf{Q}(\sqrt{-1}, \zeta_p)$ and $\mathbf{Q}(\sqrt{-3}, \zeta_p)$, and in all cases the decomposition of λ^- is simple in the sense that either there is but one positive component λ_χ, or all the positive components are equal to 1. In the present results this is no longer the case.

A detailed description of our computations appears in Sections 7–9.

2. On p-Adic L-Functions. For the theory of this section, the reader is referred to Washington's book [11], in particular to Sections 5.2 and 7.2.

We fix an embedding of the field of algebraic numbers in an algebraic closure Ω_p of $\mathbf{Q}_p$, the field of p-adic numbers. Denote by ord_p the p-adic valuation on Ω_p, normalized so that $\operatorname{ord}_p(p) = 1$.

Let χ be a character in $X(K)$ (all characters are assumed primitive). Since the conductor f_χ of χ divides f_K, it is not divisible by p^2; we say that χ is of the "first kind". Put

$$f_\chi = d \text{ or } dp \quad \text{with } (d, p) = 1.$$

As in the introduction, let K_n denote the nth layer of the $\mathbf{Z}_p$-extension K_∞/K ($n \geq 0$). The character group of K_n is of the form $\operatorname{Ch}(K) \times \langle \pi_n \rangle$, where π_n is a character with order p^n and conductor p^{n+1} (or 1, if $n = 0$); π_n is called a character of the "second kind".

Now consider the p-adic L-function $L_p(s, \psi)$ for the (nonprincipal) character $\psi = \chi\omega\pi_n$. This function is defined in Ω_p in a neighborhood of 1 containing $\mathbf{Z}_p$, the p-adic integers, and it has the fundamental property that

$$(1) \qquad L_p(1-k, \psi) = -\left(1 - \psi_k(p) p^{k-1}\right) B^k(\psi_k)/k \qquad (k \geq 1),$$

where $\psi_k = \psi\omega^{-k}$ and $B^k(\psi_k)$ stands for the kth generalized Bernoulli number attached to the character ψ_k.

Denote by $\mathbf{Q}_p(\chi)$ the extension of $\mathbf{Q}_p$ generated by the values of χ. Iwasawa's theory of p-adic L-functions asserts that there exists a power series

$$(2) \qquad f(x, \chi\omega) = \sum_{j=0}^{\infty} a_j x^j$$

whose coefficients $a_j = a_j(\chi)$ are integers of $\mathbf{Q}_p(\chi)$, such that

$$(3) \qquad L_p(s, \chi\omega\pi_n) = f\left(\frac{(1+dp)^s}{\pi_n(1+dp)} - 1, \chi\omega\right).$$

According to the Ferrero-Washington theorem, the power series $f(x, \chi\omega)$ has $\mu = 0$, in other words, there is an index j for which $\operatorname{ord}_p(a_j) = 0$. The least such j is called the λ-invariant (or Weierstrass degree) of $f(x, \chi\omega)$. This is the number λ_χ introduced in Section 1.

3. The p-Orders of Generalized Bernoulli Numbers. Let us decompose χ as

$$\chi = \theta\omega^{t-1} \quad \text{with } f_\theta = d\ (\geqslant 1), \qquad 1 \leqslant t \leqslant p-1.$$

In this section we obtain a relation between λ_χ and the p-order of $B^t(\theta\pi_n)$.

For a fixed $n \geqslant 1$, put

$$\alpha_k = \frac{(1+dp)^{1-k}}{\pi_n(1+dp)} - 1 \qquad (k \geqslant 1).$$

It follows from (3) and (1) that, for all $t = 1, \dots, p-1$,

$$\begin{aligned} f(\alpha_t, \theta\omega^t) &= L_p(1-t, \theta\omega^t\pi_n) \\ &= -\left(1 - (\theta\pi_n)(p)p^{t-1}\right)B^t(\theta\pi_n)/t = -B^t(\theta\pi_n)/t. \end{aligned}$$

By using this result we prove the following proposition in which ϕ denotes Euler's totient function and e is the ramification index of $\mathbf{Q}_p(\theta)/\mathbf{Q}_p$.

Proposition 1. *Let $n \geqslant 1$ and $1 \leqslant t \leqslant p-1$. We have*

$$\begin{aligned} &\operatorname{ord}_p\left(B^t(\theta\pi_n)\right) = \lambda_\chi/\phi(p^n) < 1/e && \text{if } \lambda_\chi < \phi(p^n)/e, \\ &\operatorname{ord}_p\left(B^t(\theta\pi_n)\right) \geqslant 1/e && \text{if } \lambda_\chi \geqslant \phi(p^n)/e. \end{aligned}$$

Proof. We evaluate the p-order of $f(\alpha_t, \theta\omega^t) = \sum_{j=0}^{\infty} a_j\alpha_t^j$.

By the definition of π_n, the number $\pi_n(1+dp) = \zeta$ is a primitive p^nth root of 1. Since

$$\alpha_t = \frac{1 - \zeta(1+dp)^{t-1}}{\zeta(1+dp)^{t-1}},$$

we have $\operatorname{ord}_p(\alpha_t) = \operatorname{ord}_p(1-\zeta) = 1/\phi(p^n)$.

As to the p-orders of the coefficients a_j, observe that these are integers of $\mathbf{Q}_p(\chi) = \mathbf{Q}_p(\theta)$. Therefore, if $\operatorname{ord}_p(a_j) > 0$ then $\operatorname{ord}_p(a_j) \geqslant 1/e$.

Recalling the definition of λ_χ we now see that

$$\begin{aligned} \operatorname{ord}_p\left(a_j\alpha_t^j\right) &\geqslant 1/e && \text{for } 0 \leqslant j \leqslant \lambda_\chi - 1, \\ &= j\operatorname{ord}_p(\alpha_t) = \lambda_\chi/\phi(p^n) && \text{for } j = \lambda_\chi, \\ &\geqslant j\operatorname{ord}_p(\alpha_t) > \lambda_\chi/\phi(p^n) && \text{for } j > \lambda_\chi. \end{aligned}$$

Consequently, if $\lambda_\chi < \phi(p^n)/e$, then

$$\operatorname{ord}_p\left(f(\alpha_t, \theta\omega^t)\right) = \lambda_\chi/\phi(p^n),$$

while otherwise this p-order is at least $1/e$. Hence the result. □

Proposition 1 gives us the value of λ_χ, once we know $\operatorname{ord}_p(B^t(\theta\pi_n))$ for a sufficiently large n. For later purposes it is convenient to reformulate this proposition, actually in a bit weaker form, as follows.

Note that the congruence $\alpha \equiv \beta \pmod{p^r}$ in Ω_p means that $\operatorname{ord}_p(\alpha - \beta) \geqslant r$.

Proposition 2. *Let $n \geqslant 1$ and $1 \leqslant t \leqslant p-1$. Assume that $h \in \mathbf{Z}$, $1 \leqslant h \leqslant \phi(p^n)/e$. Then*

$$\lambda_\chi \geqslant h \quad \textit{if and only if} \quad B^t(\theta\pi_n) \equiv 0 \left(\operatorname{mod} p^{h/\phi(p^n)}\right).$$

Proof. Suppose that the above congruence holds. If $\lambda_\chi < \phi(p^n)/e$, then a comparison of this congruence with the first part of Proposition 1 shows that $\lambda_\chi \geqslant h$. If $\lambda_\chi \geqslant \phi(p^n)/e$, then the assertion follows directly from the assumption made about h.

To verify the converse, apply both parts of Proposition 1 separately. □

Remark. Proposition 2 is of the same kind as the main result in the second author's paper [8]. This relates λ_χ to certain Kummer type congruences of $B^k(\theta)$, provided $\lambda_\chi \leqslant p-1$. Proposition 2 would enable one to replace the proof presented in [8] by a somewhat simpler proof.

4. Bernoulli Numbers and Character Sums. We now express the residue of $B^t(\theta\pi_n)$ modulo p in terms of suitable character sums.

For any character ψ with conductor f we have, in the usual symbolic notation,

$$B^k(\psi) = \frac{1}{f}\sum_{a=1}^{f}\psi(a)(fB + a - f)^k \qquad (k \geqslant 0)$$

(e.g., [7, p. 134]), where the B^m denote ordinary Bernoulli numbers. On changing the summation variable a into $f-a$ we obtain

$$B^k(\psi) = \frac{(-1)^k\psi(-1)}{f}\sum_{a=1}^{f}\psi(a)(a - fB)^k.$$

Let $\psi = \theta\pi_n$ with $n \geqslant 1$. Then $\psi(-1) = (-1)^t$ since the character $\chi = \theta\omega^{t-1}$ is odd and π_n, being of p-power order, is even. Hence we find that

$$\begin{aligned} B^t(\theta\pi_n) &= \frac{1}{dp^{n+1}}\sum_{a=1}^{dp^{n+1}}(\theta\pi_n)(a)(a - dp^{n+1}B)^t \\ &\equiv \frac{1}{dp^{n+1}}\sum_{a=1}^{dp^{n+1}}(\theta\pi_n)(a)a^t - tB^1\sum_{a=1}^{dp^{n+1}}(\theta\pi_n)(a)a^{t-1} \pmod{p}. \end{aligned}$$

The second sum of the last expression vanishes mod p, as can be verified again by the transformation $a \to dp^{n+1} - a$. Therefore,

$$B^t(\theta\pi_n) \equiv \frac{1}{dp^{n+1}}\sum_{a=1}^{dp^{n+1}}(\theta\pi_n)(a)a^t \pmod{p}. \tag{4}$$

From this result we derive the following congruence which is of the same type as the classical Voronoĭ congruence for ordinary Bernoulli numbers. We point out that the congruence (in a sharper form) has also been proved by Slavutskiĭ [9, congr. (6)].

PROPOSITION 3. *Let b be a positive rational integer with $(b, dp) = 1$. Then*

$$\left(b^t - (\theta\pi_n)(b)^{-1}\right)B^t(\theta\pi_n) \equiv tb^{t-1}\sum_{a=1}^{dp^{n+1}}(\theta\pi_n)(a)a^{t-1}\left[\frac{ba}{dp^{n+1}}\right] \pmod{p},$$

where, as in the above, $n \geqslant 1$ and $1 \leqslant t \leqslant p-1$.

Proof. Put $\psi = \theta\pi_n$. Let a and b be positive rational integers prime to dp. Keeping b fixed, we write

$$ba = dp^{n+1}\left[\frac{ba}{dp^{n+1}}\right] + r_a, \qquad 0 < r_a < dp^{n+1}.$$

On raising this equation to the tth power and multiplying by $\psi(a) = \psi(b)^{-1}\psi(r_a)$, we get

$$\psi(a)b^t a^t \equiv \psi(b)^{-1}\psi(r_a)r_a^t + \psi(a)tr_a^{t-1}dp^{n+1}\left[\frac{ba}{dp^{n+1}}\right] \pmod{p^{2n+2}}.$$

If a runs through $1,\dots,dp^{n+1}$, excepting those numbers for which $(a, dp) > 1$, then so does r_a. Summing over a we find that (observe that $\psi(a) = 0$ if $(a, dp) > 1$)

$$\left(b^t - \psi(b)^{-1}\right)\sum_{a=1}^{dp^{n+1}} \psi(a)a^t \equiv tdp^{n+1}\sum_{a=1}^{dp^{n+1}} \psi(a)r_a^{t-1}\left[\frac{ba}{dp^{n+1}}\right] \pmod{p^{2n+2}}.$$

Since $r_a \equiv ba \pmod{p^{n+1}}$, this result together with (4) yields the claimed congruence. □

5. The Main Result. Every rational integer a prime to p has the following unique representation mod p^{n+1}:

(5) $$a \equiv \omega(a)(1+p)^{v(a)} \pmod{p^{n+1}}, \qquad 0 \leqslant v(a) < p^n.$$

For $b \in \mathbf{Z}$, $(b, dp) = 1$, put

(6) $$S_{nk} = S_{nk}(b) = \sum_{v(a)=k} \theta(a)a^{t-1}\left[\frac{ba}{dp^{n+1}}\right] \qquad (k = 0,\dots,p^n - 1),$$

where the sum is extended over those numbers a for which $1 \leqslant a < dp^{n+1}$, $(a, dp) = 1$ and $v(a) = k$. Moreover, set

(7) $$T_u = T_u^{(n)} = \sum_{k=u}^{p^n-1} \binom{k}{u} S_{nk} \qquad (u = 0,\dots,p^n - 1).$$

THEOREM. *Let $\chi = \theta\omega^{t-1} \in X(K)$, where $f_\theta = d$ is prime to p and $1 \leqslant t \leqslant p - 1$. Let b be a positive integer such that*

$$(b, dp) = 1, \qquad \theta(b)b^t \not\equiv 1 \pmod{\mathfrak{p}},$$

where $\mathfrak{p}$ is the maximal ideal of the ring of integers of $\mathbf{Q}_p(\theta)$. Denote by e the ramification index of $\mathbf{Q}_p(\theta)/\mathbf{Q}_p$. Let $n \geqslant 1$ and let $h \in \mathbf{Z}$, $1 \leqslant h \leqslant \phi(p^n)/e$. With the above notations,

$$\lambda_\chi \geqslant h \quad \textit{if and only if} \quad T_0^{(n)} \equiv T_1^{(n)} \equiv \cdots \equiv T_{h-1}^{(n)} \equiv 0 \pmod{\mathfrak{p}}.$$

Proof. Since the nonzero values of π_n are p^nth roots of 1, we have $\pi_n(b) \equiv 1 \pmod{\mathfrak{p}}$. Hence

$$b^t - (\theta\pi_n)(b)^{-1} \not\equiv 0 \pmod{\mathfrak{p}},$$

and it follows from Propositions 2 and 3 that

$$\lambda_\chi \geqslant h \quad \text{if and only if} \quad \sum_{a=1}^{dp^{n+1}} \theta(a)\pi_n(a)a^{t-1}\left[\frac{ba}{dp^{n+1}}\right] \equiv 0 \pmod{p^{h\kappa}},$$

where $\kappa = 1/\phi(p^n)$.

For a fixed $n \geqslant 1$, write

$$\pi_n(1+p) = 1 + \eta.$$

Then we have $\text{ord}_p(\eta) = \kappa$ and, by (5),

$$\pi_n(a) = (1+\eta)^{v(a)} \quad \text{for } p \nmid a.$$

Consequently,

$$\sum_{a=1}^{dp^{n+1}} \theta(a)\pi_n(a)a^{t-1}\left[\frac{ba}{dp^{n+1}}\right] = \sum_{k=0}^{p^n-1}(1+\eta)^k S_{nk} = \sum_{u=0}^{p^n-1} T_u\eta^u,$$

and we are done, once the congruence

$$\sum_{u=0}^{p^n-1} T_u\eta^u \equiv 0 \pmod{p^{h\kappa}} \tag{8}$$

is shown to be equivalent to

$$T_0 \equiv T_1 \equiv \cdots \equiv T_{h-1} \equiv 0 \pmod{\mathfrak{p}}. \tag{9}$$

Suppose that the congruences (9) hold true. Then these congruences are satisfied mod $p^{1/e}$ as well, and so mod $p^{h\kappa}$ since $1/e \geqslant h/\phi(p^n) = h\kappa$. Moreover, $\eta^u \equiv 0$ (mod $p^{h\kappa}$) whenever $u \geqslant h$. This proves (8). The converse implication is established with similar arguments by induction on h. □

The above theorem enables us to determine λ_χ, once the numbers $T_u^{(n)}$ modulo $\mathfrak{p}$ are known for a sufficiently large n. We state this more explicitly as follows.

COROLLARY. *Put* $z_n = [\phi(p^n)/e]$. *With the notations of the theorem,*

(i) *if* $T_0^{(n)} \equiv T_1^{(n)} \equiv \cdots \equiv T_{h-1}^{(n)} \equiv 0$ *and* $T_h^{(n)} \not\equiv 0 \pmod{\mathfrak{p}}$, *where* $0 \leqslant h \leqslant z_n - 1$, *then* $\lambda_\chi = h$;

(ii) *if* $T_0^{(n)} \equiv T_1^{(n)} \equiv \cdots \equiv T_{z_n-1}^{(n)} \equiv 0 \pmod{\mathfrak{p}}$, *then* $\lambda_\chi \geqslant z_n$.

6. A Special Case. Suppose that $\theta = \theta_m$ is the nontrivial character of the quadratic field $\mathbf{Q}(\sqrt{m})$, where m is prime to p. Then the character $\chi = \theta\omega^{t-1}$ dealt with in the previous sections belongs to the character group of the field $\mathbf{Q}(\sqrt{m}, \zeta_p)$. Note that $f_\theta = d$ equals the absolute value of the discriminant of $\mathbf{Q}(\sqrt{m})$.

In this case, $\mathbf{Q}_p(\theta_m) = \mathbf{Q}_p$, so that $e = 1$ and $\mathfrak{p} = p\mathbf{Z}_p$. Hence we can determine λ_χ, provided it does not exceed $p - 2$, through the numbers $T_u = T_u^{(1)}$ as follows (see the corollary):

If $T_0 \equiv T_1 \equiv \cdots \equiv T_{h-1} \equiv 0$, $T_h \not\equiv 0 \pmod{p}$, *where* $0 \leqslant h \leqslant p - 2$, *then* $\lambda_\chi = h$.

If this criterion fails, then the computation of λ_χ requires passing to a higher level, i.e., computing $T_u^{(n)}$ mod p for a higher value of n.

Remark. As is seen from (5), working on a level n involves computations with integers mod p^{n+1}. We point out that, for $n = 1$, the congruence (5) can be written as

$$a \equiv a^p(1 + v(a)p) \pmod{p^2}.$$

Thus, $v(a) \equiv -q_a \pmod{p}$, where q_a denotes the Fermat quotient for a, defined by $q_a \equiv (a^{p-1} - 1)/p \pmod{p}$, $0 \leqslant q_a < p$.

7. Numerical Results. Consider, for a moment, the case of the cyclotomic field $\mathbf{Q}(\zeta_p)$. Then $X = \{\omega, \omega^3, \ldots, \omega^{p-4}\}$ and it is known that

$$\lambda_\chi > 0 \quad \text{with } \chi = \omega^{t-1} \quad \text{if and only if} \quad B^t \equiv 0 \pmod{p}$$

$(t = 2, 4, \ldots, p - 3)$. The values of λ_χ have been computed for $p < 125000$ [10]; it has turned out that in this range every positive value of λ_χ equals 1. So the λ^--invariant of $\mathbf{Q}(\zeta_p)$, say λ_0^-, equals the *index of irregularity* of p, i.e., the number of *irregular pairs* (p, t). Tables of irregular pairs can be found in many books, e.g., [11].

Now let us enlarge the field to $K = \mathbf{Q}(\sqrt{m}, \zeta_p)$ with $p \nmid m$. Then the character set X is enlarged by the characters $\theta_m\omega^{t-1}$ discussed in Section 6. To be precise, we have

$$\lambda^- = \lambda_0^- + \sum_{\chi} \lambda_\chi,$$

where the sum is extended over the characters

$$(10) \qquad \chi = \theta\omega^{t-1} \quad \text{with} \begin{cases} t = 2, 4, \ldots, p-1 & \text{if } m > 0, \\ t = 1, 3, \ldots, p-2 & \text{if } m < 0, \end{cases}$$

$\theta = \theta_m$ being the quadratic character of $\mathbf{Q}(\sqrt{m})$. If $m < 0$, the component λ_θ is just the λ-invariant of this quadratic field.

The actual computations associated with the present work comprised the determination of λ_χ for the characters (10) when p and m range through the following values (m squarefree):

$$\begin{array}{lll} p = 3 & \text{and} & -3235 \leqslant m \leqslant 3454,^* \\ p = 5 & \text{and} & -5000 < m \leqslant 3147, \\ p = 7 & \text{and} & -3002 \leqslant m < 1000, \\ p = 11 & \text{and} & -1000 < m < 500, \\ 11 < p < 200 & \text{and} & m = -7, -3, -2, -1, 2, 5. \end{array}$$

The asterisk above indicates that for a few values of m the computation was stopped at the result $\lambda_\chi \geqslant 6$ (see below).

The numerical material thus obtained contains about 22000 values of λ_χ, some 6400 of them being positive. Samples from this material are exhibited in Tables 1 and 2 of the appendix. Table 1 presents the results for $p = 5$, $m > 0$, and Table 2 for $p < 200$, $m = -1, \pm 2, -3, 5, -7$. Note that every odd prime p below 200 really appears in Table 2, i.e., to every p there is at least one m and t such that $\lambda_\chi > 0$ for $\chi = \theta_m\omega^{t-1}$.

For $p > 3$, only few cases were found in which $\lambda_\chi > p - 2$. These cases, which had to be settled on the level $n = 2$, are listed here:

p	m	t	λ_χ	p	m	t	λ_χ
5	439	4	4	5	-3178	1	4
5	1427	4	4	5	-3471	1	4
5	-311	1	4	5	-3547	3	4
5	-761	1	4	5	-3923	3	4
5	-966	1	4	5	-4026	1	5
5	-2861	3	4	5	-4774	1	4
5	-3081	1	4	7	-1371	1	7

For $p = 7$ it in fact turned out that λ_χ varies between 0 and 4 (assuming all values $0, \ldots, 4$) except in the single case given above. For $p = 11$ we have the maximum $\lambda_\chi = 3$ for $m = -723$, $t = 1$.

If $p = 3$, then $\lambda_\chi > 1$ $(= p - 2)$ in about a third of the cases. These could be settled on the level $n = 2$ (i.e., $\lambda_\chi \leqslant 5$), except in six cases. In the latter cases the continuation of the procedure was given up since the values of λ_χ can be found in [6]; they are as follows:

$$\begin{array}{ll} \lambda_\chi = 6 & \text{for } m = -239, -1022, -1427, -1777; \\ \lambda_\chi = 7 & \text{for } m = -458, \\ \lambda_\chi = 8 & \text{for } m = -2789. \end{array}$$

An examination of the results shows that the values of λ_χ seem to be distributed in the expected way. For example, if we keep p and t fixed, $t \neq 1$, and let m vary, then the number of cases with $\lambda_\chi \geqslant k$ (for $\chi = \theta_m \omega^{t-1}$ and $k \geqslant 0$) should be about a p^kth part of the number of all λ_χ; this corresponds to the natural hypothesis that the coefficients of the power series $f(x, \chi\omega)$ are randomly distributed mod p. In the following table, N_k denotes the number of $\lambda_\chi \geqslant k$ in our range:

p	t	N_0	N_1	N_2	N_3	N_1/N_0	$1/p$	N_2/N_0	$1/p^2$	N_3/N_0	$1/p^3$
3	2	1577	553	172	50	0.35	0.33	0.11	0.11	0.032	0.037
5	2	1596	326	55	9	0.20	0.20	0.034	0.040	0.006	0.008
5	4	1596	329	68	15	0.21	0.20	0.043	0.040	0.009	0.008
5	3	2535	490	88	14	0.19	0.20	0.035	0.040	0.006	0.008
7	3	1599	221	29		0.14	0.14	0.018	0.020		
7	5	1599	256	39		0.16	0.14	0.024	0.020		

If $t = 1$, the situation is different. Indeed, by Eqs. (1)–(3) the constant term of $f(x, \chi\omega)$ equals

(11) $$a_0 = (\chi(p) - 1)B^1(\chi);$$

hence, in the present case λ_χ is positive whenever $\chi(p) = \theta_m(p) = +1$. We must therefore modify the above hypothesis so as to concern those $f(x, \theta_m\omega)$ only for which $\theta_m(p) = -1$. We tested this hypothesis for $p = 5$, $m > 0$, obtaining the following (N_k' denotes the number of $\lambda_\chi \geqslant k$ when $\theta_m(5) = -1$):

$$N_0' = 1268, \quad N_1' = 241, \quad N_2' = 36; \qquad N_1'/N_0' = 0.19, \quad N_2'/N_0' = 0.028.$$

We may also ask how often λ^- is, say, positive as p is fixed and $|m|$ increases. If $p \leqslant 11$, then $\lambda_0^- = 0$, and so $\lambda^- > 0$ exactly when at least one of the $s = (p-1)/2$ numbers T_0 corresponding to the characters $\theta_m \omega^{t-1}$ vanishes mod p. To avoid the exceptional case $t = 1$, consider positive m only. Then it is again natural to assume that the values of T_0 be randomly distributed mod p, and this implies that the proportion of the number of fields with $\lambda^- > 0$ to the number of all fields should be about $\rho_p = 1 - (1 - p^{-1})^s$. Below is a comparison between the observed and expected values of this proportion:

p	observed proportion	ρ_p
5	$587/1596 = 0.37$	0.36
7	$204/530 = 0.38$	0.37
11	$100/279 = 0.36$	0.38

A table including all the results of our computations has been deposited in the UMT file; see Review **29** in this issue.

8. Comparison with Previous Results. We next describe the contents of the previously published tables about λ^-. These tables were used by us to check our results.

Gold [3], [4] has computed, for $p = 3, 5, 7, 11$, the λ-invariant of the quadratic field with discriminant $-d < 0$. His results in [4, Table 2] cover the range $0 < d \leqslant 264$. They agree completely with ours, and so do also the additional results presented in [3, Tables 2 and 5] after the following apparent errors are corrected: In Table 2, the value 1253 for d should be 1263 (corresponding to the given class number 20); in

Table 5, lines 5 and 6, instead of $\lambda = 3$ and $\lambda = 4$ one should read $\lambda = 2$. The latter correction is confirmed not only by [6] quoted below, but also by Corollary 5 in [3]. The expressions for e_n in Table 5 should be correspondingly corrected.

Kobayashi [6] investigates, for $p = 3$, the power series $f(x, \chi\omega)$ with $\chi = \theta_m$ and $\chi = \theta_m\omega$. He has determined the coefficients $a_0, \ldots, a_8 \bmod 9$ of this power series for $-10^4 < m < 0$ and $0 < 3m < 10^4$. From his table one can read the value of λ_χ, since in all cases $\lambda_\chi \leqslant 8$. Note that for $\chi = \theta_m$ the table is far more extensive than ours, while for $\chi = \theta_m\omega$ our computations go a bit farther. The overlapping parts of both tables are in agreement, except that the table in [6] omits the first negative m with $\lambda_\chi > 0$, namely $m = -2$. The nonvanishing of λ_χ in this case follows, by (11), from the fact that $\chi(3) = \theta_{-2}(3) = +1$. Our computation indeed shows, in agreement with [4], that $\lambda_\chi = 1$.

The first author has determined, for $p < 10^4$, the components λ_χ with $\chi = \theta_m\omega^{t-1}$ for $m = -1$ and $m = -3$ (see [2] and [1], respectively). For $t = 3, 5, \ldots, p-2$, one has in this range $\lambda_\chi = 1$ if $(p, t-1)$ is an E-irregular or D-irregular pair, respectively, and $\lambda_\chi = 0$ otherwise. A comparison of the tables in [1] and [2] with the present Table 2 shows no discrepancies.

The paper [5] by Hao and Parry tabulates the "m-irregular" primes $p < 5025$ for the values of m that appear in our Table 2. For a fixed m, the prime p is m-irregular if and only if there is at least one $t > 1$ such that $\lambda_\chi > 0$ with $\chi = \theta_m\omega^{t-1}$. It is easily checked that, for $p < 200$, the lists given in [5] coincide with the corresponding lists extracted from Table 2. Our computations show the somewhat interesting fact that every positive value of λ_χ in this region in fact equals 1, except for a single value $\lambda_\chi = 2$ occurring for $p = 23$ and $\chi = \theta_{-2}\omega^{10}$.

Let us finally mention that if $m = -q$, with q a prime, and $\theta_m(p) = -1$, then it follows from (11) that $\lambda_{\theta_m} > 0$ exactly when the class number of the field $\mathbf{Q}(\sqrt{-q})$ is divisible by p. Thus a partial check of our results is also provided by the class number tables of imaginary quadratic fields.

9. The computations. The computations were run on the DEC-20 computer at the University of Turku. The programs, written in Fortran, used only integer arithmetic.

As is seen from Sections 5 and 6, the main task was the computing of the sums S_{nk} (mostly for $n = 1$). This was started by searching a primitive root $\bmod p$ and constructing the index table. After decomposing m into prime factors, the character values $\theta_m(a)$ were calculated via the Legendre symbol, using the congruence

$$\left(\frac{a}{q}\right) \equiv a^{(q-1)/2} \pmod q \qquad (q \text{ an odd prime factor of } m)$$

and then checking that $\theta_m(a)$ indeed equals ± 1 or 0. For a fixed t, we chose a minimal $b > 0$ such that $(b, dp) = 1$ and $\theta_m(b)b^t \not\equiv 1 \pmod p$. To find the value of $v(a)$ for $n = 1$ (see (6) and (5)), we computed $a^{p-1} \bmod p^2$ by employing the 2-adic expansion of $p - 1$ and the residues of $a^2, a^4, a^8, \ldots \bmod p^2$.

After computing the numbers $S_{1k} \bmod p$ we searched for the first nonvanishing number in the sequence $T_0^{(1)}, \ldots, T_{p-2}^{(1)} \bmod p$. The cases in which such a number did not exist were afterwards picked out by hand and dealt with on the level $n = 2$. The procedure on this level was similar, except that this time the determination of $v(a)$ required computations $\bmod p^3$.

APPENDIX

TABLE 1

The positive values of λ_χ *for* $p = 5$, $\chi = \theta_m \omega^{t-1}$ $(t = 2$ *or* $4)$ *and* $0 < m \leqslant 3147$.

m	t	λ_χ	m	t	λ_χ	m	t	λ_χ	m	t	λ_χ
14	2	1	267	2	2	509	2	2	734	2	1
23	2	2	271	4	1	509	4	1	734	4	2
26	2	1	278	4	3	514	4	1	741	2	1
31	2	1	281	2	1	519	4	1	743	2	1
37	2	2	282	2	1	523	4	1	752	2	1
38	4	1	287	4	1	526	4	1	753	4	2
39	4	1	293	2	1	534	4	1	754	2	1
42	4	1	298	2	1	537	4	1	758	2	1
51	4	1	298	4	1	541	4	1	759	2	1
53	4	1	307	2	1	543	4	1	761	4	1
59	2	1	313	2	2	554	2	1	763	2	1
62	4	1	314	4	2	559	2	1	766	2	1
69	4	1	326	4	1	574	2	1	767	4	1
73	4	1	347	2	1	574	4	1	781	2	1
82	4	1	353	2	1	577	2	1	789	4	1
86	2	1	366	4	1	581	2	1	791	4	1
89	4	1	382	4	1	581	4	2	794	2	1
107	4	1	391	2	1	587	2	1	798	4	1
109	4	2	398	2	2	587	4	2	809	2	1
114	4	3	401	4	1	589	4	1	814	2	1
123	2	1	407	4	1	591	2	1	817	4	1
127	2	2	422	4	1	597	4	2	822	2	1
127	4	2	426	2	1	602	4	1	839	2	1
129	2	1	426	4	1	606	4	1	842	2	1
134	4	1	427	2	1	611	2	1	851	2	1
139	4	1	427	4	1	617	4	1	857	4	1
143	4	1	433	4	2	622	4	1	861	2	2
149	2	2	434	4	2	623	4	2	869	4	2
159	2	1	438	2	1	626	2	2	874	4	1
161	4	1	439	4	4	626	4	1	881	2	1
183	4	1	446	2	2	627	4	1	881	4	1
186	4	1	453	4	2	629	2	2	887	2	1
187	2	1	457	2	3	629	4	1	889	2	1
191	2	1	457	4	1	631	2	1	893	4	1
191	4	1	458	2	1	633	2	3	903	2	1
199	2	1	466	4	2	634	4	1	911	2	1
202	4	1	467	2	1	643	2	1	917	2	1
211	4	1	467	4	1	654	2	1	921	2	1
213	2	1	469	2	1	662	4	1	922	2	1
214	4	1	471	2	2	673	4	1	922	4	1
217	4	1	473	4	1	674	4	1	923	4	1
222	2	1	479	2	1	678	2	1	926	2	1
223	2	1	489	4	1	679	2	1	933	4	1
227	2	1	497	4	1	681	2	1	937	2	1
237	2	1	498	2	2	683	2	1	939	2	1
238	4	1	499	2	1	687	4	1	943	4	1
241	4	1	499	4	1	689	2	1	946	2	1
253	2	1	501	2	1	695	4	1	947	4	1
257	2	1	501	4	1	717	2	1	949	2	1
257	4	1	502	2	1	719	4	2	957	2	1
259	4	1	502	4	1	727	4	1	966	4	1

TABLE 1 (*continued*)

m	t	λ_χ	m	t	λ_χ	m	t	λ_χ	m	t	λ_χ
973	2	1	1214	4	1	1483	2	1	1797	4	2
978	4	1	1217	2	1	1486	4	1	1798	4	1
982	4	1	1226	4	1	1493	2	1	1799	2	1
983	4	1	1231	2	3	1493	4	1	1803	4	1
986	2	1	1231	4	1	1506	2	1	1817	4	2
997	2	1	1238	2	1	1509	2	1	1829	2	1
1003	2	1	1238	4	1	1511	2	1	1829	4	1
1006	2	2	1243	4	1	1514	2	1	1834	2	1
1007	2	1	1247	4	2	1518	2	1	1834	4	1
1018	2	1	1253	2	1	1518	4	3	1837	2	2
1031	4	1	1254	2	1	1529	4	2	1838	4	1
1034	2	1	1261	4	1	1531	2	1	1846	2	1
1037	2	1	1262	4	1	1531	4	1	1847	2	1
1041	4	1	1267	4	1	1533	2	1	1851	4	1
1042	4	1	1273	2	1	1541	2	1	1853	2	1
1051	4	1	1279	2	1	1546	2	1	1853	4	1
1059	2	1	1279	4	1	1571	2	1	1861	4	1
1063	2	1	1281	4	1	1577	4	1	1874	4	1
1069	2	1	1289	4	1	1579	2	1	1882	2	1
1073	2	3	1291	2	1	1582	2	1	1891	4	1
1074	4	1	1293	4	1	1586	4	1	1897	4	3
1077	2	1	1294	2	1	1597	2	1	1898	4	1
1079	4	2	1301	4	1	1597	4	1	1907	2	1
1086	4	2	1313	4	2	1621	4	1	1907	4	1
1087	4	1	1317	4	3	1631	2	1	1913	2	1
1093	4	1	1321	4	2	1631	4	1	1913	4	1
1097	2	1	1327	2	1	1633	4	2	1914	2	1
1106	2	1	1327	4	1	1637	4	1	1914	4	1
1111	4	2	1338	4	1	1641	4	1	1921	4	1
1113	2	1	1339	4	1	1654	2	1	1923	2	1
1113	4	1	1342	2	1	1658	4	1	1934	2	1
1114	2	1	1351	4	1	1662	4	2	1937	2	1
1118	4	1	1354	2	1	1663	4	1	1938	2	1
1119	2	1	1366	2	1	1686	4	1	1938	4	1
1121	2	1	1366	4	1	1699	4	1	1941	2	1
1122	2	1	1379	2	1	1702	4	2	1943	4	3
1123	2	1	1382	4	1	1713	4	1	1949	2	2
1126	2	1	1389	2	1	1717	4	1	1954	2	1
1126	4	1	1389	4	1	1721	2	1	1957	2	1
1129	2	1	1393	4	2	1723	4	1	1959	4	1
1131	2	2	1398	4	1	1731	4	1	1966	4	1
1133	4	1	1401	2	1	1738	2	1	1969	4	1
1137	2	1	1402	4	2	1738	4	2	1973	2	1
1137	4	1	1406	4	1	1739	2	1	1977	2	1
1142	2	1	1407	4	1	1741	4	1	1979	2	1
1149	2	1	1426	4	1	1754	4	1	1982	4	1
1157	4	1	1427	4	4	1757	2	1	1986	4	1
1169	2	1	1429	4	1	1758	2	1	1999	2	1
1173	4	1	1434	2	2	1758	4	1	1999	4	1
1182	4	2	1434	4	1	1761	4	2	2001	2	1
1187	4	1	1441	4	1	1762	4	1	2002	2	1
1191	4	1	1443	4	3	1766	4	1	2002	4	1
1193	2	2	1451	2	1	1769	2	1	2003	2	1
1194	4	1	1461	4	1	1777	4	1	2003	4	3
1198	2	1	1466	2	2	1779	2	1	2014	4	1
1203	2	1	1478	2	3	1783	2	1	2027	2	1
1213	4	1	1479	4	2	1786	4	1	2027	4	1

TABLE 1 (*continued*)

m	t	λ_χ	m	t	λ_χ	m	t	λ_χ	m	t	λ_χ
2029	2	2	2307	2	1	2571	4	1	2878	4	1
2031	4	1	2307	4	1	2573	2	1	2882	2	2
2033	4	1	2314	2	1	2573	4	1	2886	4	1
2038	4	1	2317	2	1	2577	4	1	2887	2	1
2039	2	1	2323	4	3	2578	4	1	2911	4	1
2051	2	2	2326	4	1	2579	4	1	2914	2	2
2081	4	1	2329	2	1	2581	2	1	2923	4	1
2083	2	1	2333	2	1	2599	4	1	2927	2	1
2083	4	1	2334	4	1	2602	4	1	2931	4	1
2087	2	1	2341	2	1	2603	2	1	2946	2	1
2089	2	1	2342	4	1	2609	4	2	2949	2	1
2093	2	1	2347	2	1	2633	2	1	2959	2	1
2098	2	1	2353	2	2	2634	4	1	2963	2	2
2101	4	2	2354	2	1	2639	2	1	2966	4	1
2102	2	1	2359	4	1	2647	2	1	2967	2	2
2111	[illegible]	2	2362	4	1	2654	4	2	2971	2	1
2117	4	1	2373	2	2	2657	4	1	2973	4	1
2122	4	2	2381	2	1	2661	4	1	2974	2	2
2123	2	1	2386	2	1	2667	4	2	2983	4	2
2126	2	1	2386	4	1	2669	2	1	2986	4	2
2127	2	1	2391	2	1	2671	2	1	2991	2	1
2129	2	1	2391	4	2	2671	4	1	2991	4	1
2131	2	1	2397	4	1	2683	4	3	2993	4	1
2143	2	1	2399	4	1	2686	2	1	2994	4	1
2146	2	1	2406	4	1	2687	2	1	2998	2	1
2153	2	1	2411	4	1	2687	4	1	3013	2	1
2153	4	1	2433	2	2	2694	4	1	3014	2	2
2157	4	1	2438	2	2	2698	4	2	3023	2	1
2158	4	1	2438	4	1	2706	2	1	3039	4	1
2159	2	1	2446	2	1	2711	2	1	3041	4	1
2161	2	2	2449	2	1	2714	2	3	3053	2	1
2171	4	1	2459	4	2	2722	2	2	3054	4	1
2177	2	2	2462	2	2	2723	2	3	3059	4	2
2181	2	1	2471	4	2	2723	4	2	3077	4	1
2182	2	1	2481	2	2	2731	4	1	3082	4	1
2189	2	1	2482	2	1	2739	4	2	3098	2	1
2189	4	1	2486	2	1	2741	4	2	3101	4	2
2194	2	1	2487	2	1	2742	2	1	3102	2	1
2219	2	1	2489	2	3	2743	4	2	3103	2	1
2221	2	1	2498	4	3	2746	4	1	3106	2	2
2222	2	1	2501	4	2	2759	2	2	3107	2	1
2229	2	1	2503	2	1	2766	2	1	3111	4	1
2231	2	3	2503	4	1	2771	2	1	3113	2	1
2234	4	3	2509	2	1	2773	2	1	3121	2	1
2243	2	1	2513	4	1	2778	2	1	3121	4	1
2245	4	1	2519	2	1	2803	4	1	3126	2	1
2257	4	1	2522	4	2	2814	2	1	3127	2	1
2267	4	1	2526	2	1	2821	2	1	3129	2	1
2263	2	2	2531	2	1	2822	4	1	3129	4	1
2263	4	1	2533	4	1	2829	2	1	3134	4	1
2271	2	1	2534	2	2	2829	4	3	3138	4	1
2273	2	1	2546	4	1	2841	2	1	3147	2	1
2276	2	1	2551	2	1	2843	4	1			
2279	4	1	2557	4	1	2851	4	1			
2281	2	2	2558	4	1	2859	2	1			
2287	4	1	2563	2	1	2877	4	1			
2306	2	1	2566	4	1	2878	2	2			

TABLE 2

The values of t, $1 \leqslant t \leqslant p-1$, *for which* $\lambda_\chi > 0$ *with* $\chi = \theta_m \omega^{t-1}$, *in the region* $2 < p < 200$, $m = -7, -3, -2, -1, 2, 5$. *The dagger* (†) *indicates that* $\lambda_\chi = 2$; *in all other cases* $\lambda_\chi = 1$.

p \ m	-7	-3	-2	-1	2	5
3			1			
5				1		
7		1				
11	1		1		4	
13		1†		1	12	
17			1	1		14
19		1	1 5	11	6	8
23	1	17	11†			
29	1 19			1		
31		1		23	30	
37	1	1		1	34	
41			1	1		18
43	1	1	1	13		
47	25	13		15		
53	1 19 43	29 45		1		
59	33		1 19		34 36 50	
61		1		1 7		42
67	1	1 47	1	27		6
71	1			29	68	
73	7 11	1	1 31	1		70
79	1	1		19	16 30	
83	53 65		1 15			
89			1 33	1	32	
97		1	1	1		

p \ m	-7	-3	-2	-1	2	5
101				1 63		
103		1 93				
107	1		1		64 86	22 100
109	1 105	1		1		
113	1 65	55	1 13 109	1		
127	1	1			40 56	62
131			1 51 57			32 64
137	1 45 101		1 57	1 43		84
139	21	1 99	1 19	129		44 104
149	1 39 103		79	1 147	146	22
151	1	1			14	66
157	101	1		1		
163	1	1	1			144
167						66
173	13 97 153		121	1	74	
179	1		1 119		74	
181	35 177	1†		1		
191	1 31					10
193	1 59	1	1	1 75		
197	1	179 183	191	1		
199		1 161			186	

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